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On relations between conformal mappings and isomorphisms of spaces of analytic functions on Riemann surfaces

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§ 1. Introduction.

Let $\mathfrak{S}$ be the set consisting of all compact bordered Riemann surfaces. For $\bar{S}$ in $\mathfrak{S}$, we denote its interior and its border by S and ∂S, respectively. Let p ($\geqq 0$) be the genus of $\bar{S}$ and q ($\geqq 1$) be the number of boundary components of $\bar{S}$. We set

$$N=2p+q-1 .$$

Furthermore we denote by $A(S)$ the set of all functions which are analytic in S and continuous on $\bar{S}$. It forms a Banach algebra with the supremum norm

$$\|f\|=\sup_{z\in S}|f(z)| .$$

For $\bar{S}$ and $\bar{S}'$ in $\mathfrak{S}$, let $L(A(S), A(S'))$ denote the set of all continuous invertible linear mappings of $A(S)$ onto $A(S')$. It is shown by Rochberg [4] that $L(A(S), A(S'))$ is nonvoid if S and S' are homeomorphic. We set

$$c(T)=\|T\| \, \|T^{-1}\|$$

for T in $L(A(S), A(S'))$. We have always

$$c(T)\geqq 1 ,$$

and we can easily see that $T/\|T\|$ is an isometry if and only if $c(T)=1$. If $T1=1$, then

$$1\leqq\|T\|\leqq c(T) , \quad 1\leqq\|T^{-1}\|\leqq c(T) .$$

Let z and z' be points of S and S', respectively. If there exist a positive number ε and an element T of $L(A(S), A(S'))$ such that

$$|f(z)-(Tf)(z')|\leqq\varepsilon \min(\|f\|, \|Tf\|)$$

for all f in $A(S)$, then we say that z and z' are *ε-related with respect to T*, or z and z' satisfy an *ε-relation with respect to T*.

The purpose of the present paper is to prove the following theorems:

THEOREM 1. *For $\bar{S}$ and $\bar{S}' \in \mathfrak{S}$, suppose that there exists a $T \in L(A(S), A(S'))$ which is an isometry and satisfies $T1=1$. Then there exists a conformal mapping w of S onto S' such that*

$$Tf = f \circ w^{-1}$$

for all f in $A(S)$.

This result is not new. According to a result of Nagasawa [2; Theorem 3], a T satisfying the above assumption is an algebraic isomorphism. Then, as is well known, T induces a natural mapping of the maximal ideal space of $\bar{S}$ onto that of $\bar{S}'$, which determines a conformal mapping w.

In § 4, we shall give a more direct proof of Theorem 1.

THEOREM 2. *If $\bar{S}$ and $\bar{S}' \in \mathfrak{S}$ satisfy*

$$\inf\{c(T) \mid T \in L(A(S), A(S'))\} = 1,$$

then S and S' are conformally equivalent.

This result has been obtained by Rochberg [4]. In § 5, we shall give an alternative proof by constructing a conformal mapping directly.

THEOREM 3. *Let $\bar{S} \in \mathfrak{S}$ be such that $N=2p+q-1 \geqq 2$. For every sufficiently small $\varepsilon > 0$ and every relatively compact subdomain D of S, there exists a constant $d > 1$ having the following property:*

If $T \in L(A(S), A(S))$ satisfies $c(T) < d$ and $T1=1$, then there exists a unique conformal automorphism w of S such that, for every $z \in D$, z and $w(z)$ are ε-related with respect to T.

To state the following theorem we need a notation: For a subdomain D of S and an analytic function f in D, we mean by $N_f(D)$ the set of zeros of f in D.

THEOREM 4. *Let $\bar{S} \in \mathfrak{S}$ be such that $N=2p+q-1 \geqq 2$. Consider an arbitrary $f_0 \in A(S)$. For every sufficiently small $\varepsilon > 0$ and every relatively compact subdomain D of S such that f_0 does not vanish on the boundary of D, there exists a constant $d > 1$ having the following property:*

If $T \in L(A(S), A(S))$ satisfies $c(T) < d$ and $T1=1$, then the number of zeros of f_0 in D is equal to that of Tf_0 in $w(D)$, where w is the conformal automorphism of S determined by Theorem 3; and furthermore there exists a unique mapping θ of $N_{Tf_0}(w(D))$ onto $N_{f_0}(D)$ such that, for every $\zeta \in N_{Tf_0}(w(D))$, $\theta(\zeta)$ and ζ are ε-related with respect to T.

§ 2. The construction of the function ϕ_ζ.

For $\bar{S} \in \mathfrak{S}$, we denote its boundary components by $\Gamma_1, \cdots, \Gamma_q$. Let $\alpha_1, \beta_1, \cdots, \alpha_p, \beta_p$ be simple loops on S which are homologically independent modulo ∂S such that

$$\alpha_i \cap \alpha_j = \emptyset, \quad \beta_i \cap \beta_j = \emptyset, \quad \alpha_i \cap \beta_j = \emptyset$$

for $i \neq j$, and α_i intersects β_i at exactly one point. By removing $\alpha_1, \beta_1, \cdots, \alpha_p, \beta_p$ from S, we obtain a planar domain S_0. If we set

$$\gamma_{2i-1}=\alpha_i, \quad \gamma_{2i}=\beta_i \quad (i=1, \cdots, p),$$
$$\gamma_{2p+j}=\Gamma_j \quad (j=1, \cdots, q-1),$$

$\gamma_1, \cdots, \gamma_N$ form a canonical homology basis of $\bar{S}$. By Ahlfors [1], there exists a basis $\omega_1, \cdots, \omega_N$ of the space of analytic Schottky differentials satisfying

$$\int_{\gamma_i} \omega_j = \delta_{ij} \quad (i, j=1, \cdots, N).$$

For ζ in S we denote by $G_\zeta(z)$ the Green function on $\bar{S}$ with pole at ζ. For each j $(j=1, \cdots, N)$ and every point ζ in $S-\gamma_j$ we set

$$\pi_j(\zeta)=\int_{\gamma_j} *dG_\zeta. \tag{1}$$

Evidently, $\pi_j(\zeta)$ is a continuous function of ζ on $S-\gamma_j$. If ζ is a point of γ_j $(1 \leqq j \leqq 2p)$, it defines two distinct accessible boundary points ζ_1 and ζ_2 of $S-\gamma_j$. Let C_1 be a curve in $S-\gamma_j$ which ends at ζ and defines ζ_1. If we modify γ_j in a parametric disk about ζ by making a detour along a circular arc which does not meet C_1, then another loop γ'_j is obtained. Let $\pi_j(\zeta_1)$ denote the value obtained by using γ'_j in place of γ_j in (1). Similarly, we can also define $\pi_j(\zeta_2)$. Clearly,

$$\pi_j(\zeta_2)-\pi_j(\zeta_1)=\pm 2\pi. \tag{2}$$

Furthermore, $\pi_j(\zeta)$ is continuous on the set S_0^* obtained by adding to $S_0=S-\bigcup_{j=1}^{2p}\gamma_j$ all accessible boundary points which are defined by points of $\bigcup_{j=1}^{2p}\gamma_j$. We note that a natural topology can be defined on S_0^*.

Now we fix a point z_0 in S_0. For each point ζ in $S_0^*-\{z_0\}$ we define a function

$$f_\zeta(z)=\exp\left[-\int_{z_0}^{z}(dG_\zeta+i*dG-i\sum_{j=1}^{N}\pi_j(\zeta)\omega_j)\right]. \tag{3}$$

For a fixed ζ, $f_\zeta(z)$ is single-valued and analytic on $\bar{S}$, consequently, it is in $A(S)$. Moreover, $f_\zeta(z)$ is continuous on $S_0^*-\{z_0\}$ with respect to ζ for a fixed z in $\bar{S}$. We choose another point $\tilde{z}_0$ $(\neq z_0)$ in S_0 and denote by $\tilde{f}_\zeta(z)$ the function defined by using $\tilde{z}_0$ in place of z_0 in (3). Then, there exists the limit

$$g(z)=\lim_{\zeta \to z_0} \tilde{f}_{z_0}(\zeta) f_\zeta(z)$$

for each z in $\bar{S}$, and g is an analytic function on $\bar{S}$. Now we set

$$\phi_\zeta(z)=\begin{cases} \tilde{f}_{z_0}(\zeta)f_\zeta(z) & \text{for } \zeta \neq z_0 \\ g(z) & \text{for } \zeta = z_0. \end{cases}$$

The function $\phi_\zeta(z)$ has the following properties.

(i) For a fixed ζ in S_0^*, ϕ_ζ is analytic on $\bar{S}$, consequently, it is in $A(S)$. Moreover, for a fixed z in $\bar{S}$, $\phi_\zeta(z)$ is continuous on S_0^* with respect to ζ.

(ii) For each ζ in S_0^*, ζ is a simple zero of ϕ_ζ, and it is the only zero of ϕ_ζ.

(iii) For every compact subset K^* of S_0^* there is a constant m (>1) such that

$$\frac{1}{m}\leqq|\phi_\zeta(z)|\leqq m \tag{4}$$

for all z on ∂S and all ζ in K^*.

(iv) For each ζ_0 in S_0^*

$$\lim_{\zeta\to\zeta_0}\|\phi_\zeta-\phi_{\zeta_0}\|=0\,. \tag{5}$$

(v) If a point ζ on γ_j defines two distinct accessible boundary points ζ_1 and ζ_2 of S_0, (2) implies that

$$\phi_{\zeta_2}(z)=g_j(z)\phi_{\zeta_1}(z) \quad\text{or}\quad \phi_{\zeta_2}(z)=g_j(z)^{-1}\phi_{\zeta_1}(z)\,, \tag{6}$$

where g_j is a function in $A(S)$ defined by

$$g_j(z)=\exp\left(2\pi i\int_{z_0}^{z}\omega_j\right).$$

If ζ defines four distinct accessible boundary points ζ_1, ζ_2, ζ_3 and ζ_4 of S_0, we obtain, for example, the following relations;

$$\phi_{\zeta_2}(z)=g_{2j-1}(z)\phi_{\zeta_1}(z)\,,\quad \phi_{\zeta_3}(z)=g_{2j}(z)\phi_{\zeta_1}(z)\,,$$

$$\phi_{\zeta_4}(z)=g_{2j-1}(z)g_{2j}(z)\phi_{\zeta_1}(z)\,.$$

§ 3. Lemmas.

The following lemmas are due to Rochberg [3], [4] and [5].

LEMMA 1. *For $\bar{S}$, $\bar{S}'\in\mathfrak{S}$ and for every $\varepsilon>0$, there exists a constant $d>1$ such that*

$$\|T(fg)-(Tf)(Tg)\|\leqq\varepsilon\|f\|\,\|g\|$$

for all $T\in L(A(S), A(S'))$ with $c(T)<d$ and $T1=1$ and for all $f, g\in A(S)$ (cf. [3]).

LEMMA 2. *For every $\varepsilon>0$, there exists a constant $d>1$ having the following property:*

For $\bar{S}$, $\bar{S}'\in\mathfrak{S}$ and every $T\in L(A(S), A(S'))$ with $c(T)<d$ and $T1=1$, there exists a homeomorphism h of ∂S onto $\partial S'$ such that

$$|f(z)-(Tf)(h(z))|\leqq\varepsilon\|f\|$$

for all z on ∂S and all $f\in A(S)$ (cf. [3], [5]).

LEMMA 3. *If* $\bar{S}$ *and* $\bar{S}'\in\mathfrak{S}$ *satisfy*

$$\inf\{c(T)\mid T\in L(A(S),\ A(S'))\}=1\,,$$

then there exists a sequence $\{T_n\}$ *in* $L(A(S), A(S'))$ *such that* $c(T_n)\to 1$ *and* $T_n 1=1$ (cf. [3]).

LEMMA 4. *Under the same assumption of Lemma* 3, *there exist a subsequence* $\{T_{n_j}\}$ *of* $\{T_n\}$, *an analytic mapping* τ *of* S' *into* S *and an analytic mapping* σ *of* S *into* S' *such that*

$$\lim_{j\to\infty}(T_{n_j}f)(w)=f(\tau(w))$$

uniformly on every compact subset of S' *for all* $f\in A(S)$, *and*

$$\lim_{j\to\infty}(T_{n_j}^{-1}g)(z)=g(\sigma(z))$$

uniformly on every compact subset of S *for all* $g\in A(S')$ (cf. [4], [5]).

Let D be a relatively compact subdomain of S. If $\bar{D}\cap(\bigcup_{j=1}^{2p}\gamma_j)$ is nonvoid, we denote by D^* the set obtained by adding to $\bar{D}-\bigcup_{j=1}^{2p}\gamma_j$ all accessible boundary points which are defined by points of $\bar{D}\cap(\bigcup_{j=1}^{2p}\gamma_j)$. The set D^* is a subset of S_0^*. If $\bar{D}\cap(\bigcup_{j=1}^{2p}\gamma_j)$ is void, D^* is equal to $\bar{D}$.

LEMMA 5. *Let* $\bar{S}$ *be an element of* $\mathfrak{S}$. *For every* $\varepsilon>0$ *and every relatively compact subdomain* D *of* S, *there exists a constant* $d>1$ *having the following property:*

For $\bar{S}'\in\mathfrak{S}$ *and* $T\in L(A(S), A(S'))$ *with* $c(T)<d$ *and* $T1=1$, *there exists a continuous mapping* w_T *of* D^* *into* S' *such that, for every* $\zeta\in D^*$, ζ *and* $w_T(\zeta)$ *are* ε-*related with respect to* T (cf. [5]).

PROOF. By property (iii) there is a constant $m>1$ for D^* such that (4) holds for all z on ∂S and all ζ in D^*. We set $\varepsilon_1=1/(2m^2)$. By Lemma 2 there is a constant $d_1>1$ as follows. If $c(T)<d_1$, there exists a homeomorphism h of ∂S onto $\partial S'$ such that

$$|\phi_\zeta(z)-(T\phi_\zeta)(h(z))|\leqq\varepsilon_1\|\phi_\zeta\|$$

for all z on ∂S and all ζ in D^*. Combining (4) and the above inequality it follows that

$$|(T\phi_\zeta)(h(z))|>\frac{1}{2m}$$

for all z on ∂S and all ζ in D^*. Hence the change of argument of $T\phi_\zeta$ around $\partial S'$ is equal to the change of argument of ϕ_ζ around ∂S. Therefore, by the argument principle, $T\phi_\zeta$ has the same number of zeros as ϕ_ζ, that is, exactly one. We denote this zero by $w_T(\zeta)$. It follows from (5) that the mapping

$w_T(\zeta)$ of D^* into S' is continuous. Furthermore, using Lemma 1, we can show by the same argument as Proof of Proposition 1 in [5] that if $c(T)$ is sufficiently close to 1, ζ in D^* and $w_T(\zeta)$ are ε-related with respect to T.

§4. Proof of Theorem 1.

Let $\{S_n\}$ be an exhaustion of S. Since T is an isometry by assumption, $c(T)=1$. Consequently, by Lemma 5 there exists a continuous mapping $w_T^{(n)}$ of S_n^* into S' for each n such that every point ζ in S_n^* and $w_T^{(n)}(\zeta)$ satisfy an ε-relation with respect to T for every $\varepsilon>0$. By the definition of $w_T^{(n)}$, $w_T^{(n)}=w_T^{(n+1)}=\cdots=w_T^{(n+k)}=\cdots$ in S_n^*. Thus we obtain a continuous mapping w_T of S_0^* into S' such that

$$|f(\zeta)-(Tf)(w_T(\zeta))|\leqq\varepsilon\|f\|$$

for all ζ in S_0^* and all f in $A(S)$. Since $\varepsilon>0$ is arbitrary, we have a relation

$$(Tf)(w_T(\zeta))=f(\zeta) \tag{7}$$

for every ζ in S_0^* and every f in $A(S)$. If a point ζ on $\bigcup_{j=1}^{2p}\gamma_j$ defines two distinct accessible boundary points ζ_1 and ζ_2 of S_0^*, it follows from (7) that

$$g(w_T(\zeta_1))=g(w_T(\zeta_2))$$

for all g in $A(S')$. Hence

$$w_T(\zeta_1)=w_T(\zeta_2)\,,$$

for $A(S')$ separates points on S'. If a ζ on $\bigcup_{j=1}^{2p}\gamma_j$ defines four distinct accessible boundary points ζ_1, ζ_2, ζ_3 and ζ_4 of S_0, we similarly obtain

$$w_T(\zeta_1)=w_T(\zeta_2)=w_T(\zeta_3)=w_T(\zeta_4)\,.$$

Therefore w_T is a continuous mapping of S into S', and (7) holds for all ζ in S and all f in $A(S)$.

On the other hand, since $c(T^{-1})=1$, we can use the same method as above to T^{-1}. Hence we obtain a continuous mapping $w_{T^{-1}}$ of S' into S such that

$$(T^{-1}g)(w_{T^{-1}}(\zeta'))=g(\zeta') \tag{8}$$

for all ζ' in S' and all g in $A(S')$.

Now, for each fixed ζ in S we set $\zeta'=w_T(\zeta)$. If we set $f=T^{-1}\phi_{\zeta'}$ in (7),

$$(T^{-1}\phi_{\zeta'})(\zeta)=\phi_{\zeta'}(\zeta')=0\,.$$

Since $T^{-1}\phi_{\zeta'}$ has the only zero $w_{T^{-1}}(\zeta')$,

$$w_{T^{-1}}(\zeta')=\zeta\,.$$

Thus

$$w_{T^{-1}}(w_T(\zeta))=\zeta$$

for all ζ in S. Similarly, it follows from (8) that

$$w_T(w_{T^{-1}}(\zeta'))=\zeta'$$

for all ζ' in S'. Therefore $w=w_T$ is a homeomorphism of S onto S' and $w^{-1}=w_{T^{-1}}$. We know by (7) that $w=w_T$ is conformal and

$$Tf=f\circ w^{-1}$$

for all f in $A(S)$.

§5. Proof of Theorem 2.

1. By Lemmas 3 and 4 there exists a sequence $\{T_j\}$ in $L(A(S), A(S'))$ with

$$\lim_{j\to\infty} c(T_j)=1\,,\quad T_j1=1\,, \tag{9}$$

and there exist an analytic mapping τ of S' into S and an analytic mapping σ of S into S' such that

$$\lim_{j\to\infty}(T_jf)(w)=f(\tau(w)) \tag{10}$$

uniformly on every compact subset of S' for all f in $A(S)$, and

$$\lim_{j\to\infty}(T_j^{-1}g)(z)=g(\sigma(z)) \tag{11}$$

uniformly on every compact subset of S for all g in $A(S')$.

Let $\{S_n\}$ be an exhaustion of S and $\{\varepsilon_n\}$ be a sequence of positive numbers which tends to zero. In Lemma 5 we set $\varepsilon=\varepsilon_n$ and $D=S_n$ for each n, and we denote the corresponding constant by d_n (>1). By (9), $c(T_{j_n})<d_n$ for a sufficiently large j_n. We may assume $j_n<j_{n+1}$. By Lemma 5 there exists a continuous mapping $w_{T_{j_n}}$ of S_n^* into S' for each n such that ζ and $w_{T_{j_n}}(\zeta)$ are ε_n-related with respect to T_{j_n} for all ζ in S_n^*. For simplicity, we shall use the notation T_n in place of T_{j_n}. By ε_n-relation

$$|f(\zeta)-(T_nf)(w_{T_n}(\zeta))|\leqq\varepsilon_n\|f\| \tag{12}$$

for all ζ in S_n^* and all f in $A(S)$.

2. Now we take distances $d(\cdot,\cdot)$ and $d'(\cdot,\cdot)$ on S_0^* and $\bar{S}'$, respectively, which induce the original topologies of S_0^* and $\bar{S}'$, respectively. We shall verify that for every compact subset K^* of S_0^* the mappings w_{T_n} for sufficiently large n are equicontinuous on K^*. For this purpose we show that the set of the zeros $w_{T_n}(\zeta)$ of $T_n\phi_\zeta$ for all n and all ζ in K^* is apart from $\partial S'$. If it were not, then there is a sequence $\{\zeta_n\}$ in K^* such as $\{w_{T_n}(\zeta_n)\}$ has an accumulating point z_0' on $\partial S'$. By choosing a subsequence if necessary, we may assume that $\zeta_n \to \zeta_0$ for some ζ_0 on K^* and $w_{T_n}(\zeta_n) \to z_0'$. Let g be a nonconstant function in $A(S')$ satisfying $|g|=1$ on $\partial S'$. (This is a so-called inner function.) By ε_n-relation

$$|(T_n^{-1}g)(\zeta_n)-g(w_{T_n}(\zeta_n))| \leqq \varepsilon_n ,$$

consequently,

$$|g(w_{T_n}(\zeta_n))| \leqq |(T_n^{-1}g)(\zeta_n)|+\varepsilon_n$$

for all n. Hence it follows from (11) that

$$1=|g(z_0')| \leqq |g(\sigma(\zeta_0))|<1 .$$

This is a contradiction.

So, there is a domain D' whose boundary consists of a finite number of analytic closed curves such that

$$w_{T_n}(K^*) \subset D' \subset \bar{D}' \subset S'$$

for all n. If we apply the residue theorem to every function ϕ in $A(S)$, then

$$\phi(w_{T_n}(\zeta))=\frac{1}{2\pi i}\int_{\partial D'} \phi(z') \frac{(T_n\phi_\zeta)'(z')}{(T_n\phi_\zeta)(z')} dz'$$

for all n and all ζ in K^*. Hence, using (5) we can show that if an $\varepsilon>0$ is given there is a $\delta>0$ such that

$$|\phi(w_{T_n}(\zeta_1))-\phi(w_{T_n}(\zeta_2))|<\varepsilon \tag{13}$$

for all ζ_1 and ζ_2 in K^* with $d(\zeta_1,\zeta_2)<\delta$ and for all n.

The above implies that if an $\varepsilon>0$ is given there is a $\delta>0$ such as

$$d'(w_{T_n}(\zeta_1),\ w_{T_n}(\zeta_2))<\varepsilon$$

for all ζ_1 and ζ_2 in K^* with $d(\zeta_1,\zeta_2)<\delta$ and for all n. If it were not, then there are an $\varepsilon>0$ and points ζ_{1n}, ζ_{2n} in K^* with $d(\zeta_{1n},\zeta_{2n}) \to 0$ such that

$$d'(w_{T_n}(\zeta_{1n}), w_{T_n}(\zeta_{2n})) \geqq \varepsilon .$$

We may assume that $\zeta_{1n} \to \zeta_0$, $\zeta_{2n} \to \zeta_0$ for some ζ_0 in K^*, $w_{T_n}(\zeta_{1n}) \to w_1$ for some w_1 in S' and $w_{T_n}(\zeta_{2n}) \to w_2$ for some w_2 in S'. The above inequality implies $d'(w_1, w_2) \geqq \varepsilon>0$. Hence $w_1 \neq w_2$. Since the space $A(S')$ separates points on S', there is a function ϕ in $A(S')$ such as

$$\phi(w_1) \neq \phi(w_2) .$$

If an $\varepsilon>0$ is given, it follows from (13) that

$$|\phi(w_{T_n}(\zeta_{1n}))-\phi(w_{T_n}(\zeta_{2n}))|<\varepsilon$$

for sufficiently large n. Letting n go to infinity, we obtain

$$|\phi(w_1)-\phi(w_2)|\leqq\varepsilon\,.$$

Since $\varepsilon>0$ is arbitrary, we have

$$\phi(w_1)=\phi(w_2)\,.$$

This is a contradiction. Thus the mappings w_{T_n} are equicontinuous on K^*. From this and the fact that $\{w_{T_n}(\zeta)\}$ has a limit point in S' for every ζ, we conclude that $\{w_{T_n}(\zeta)\}$ is a normal family.

3. By choosing a subsequence if necessary, we may assume that there is a continuous mapping w of S_0^* into S' such that

$$\lim_{n\to\infty} w_{T_n}(\zeta)=w(\zeta) \tag{14}$$

uniformly on every compact subset of S_0^*. In order to show that the mapping $w(\zeta)$ can be defined on S and it is continuous on S, we must show that it assumes the same value at distinct accessible boundary points of S_0^* defined by each point ζ on $\bigcup_{j=1}^{2p}\gamma_j$. Suppose that a point ζ on γ_j defines two distinct boundary points ζ_1 and ζ_2. By (6) and Lemma 1, if an $\varepsilon>0$ is given, we have

$$\begin{aligned}\|T_n\phi_{\zeta_2}-(T_ng_j)(T_n\phi_{\zeta_1})\|&=\|T_n(g_j\phi_{\zeta_1})-(T_ng_j)(T_n\phi_{\zeta_1})\|\\&\leqq\varepsilon\|g_j\|\,\|\phi_{\zeta_1}\|\end{aligned} \tag{15}$$

for all sufficiently large n. We know from (10) that the sequences $\{T_n\phi_{\zeta_2}\}$ and $\{(T_ng_j)(T_n\phi_{\zeta_1})\}$ converge uniformly on every compact subset of S'. By (15) they have the same limit function

$$h=\lim_{n\to\infty} T_n\phi_{\zeta_2}=\lim_{n\to\infty}(T_ng_j)(T_n\phi_{\zeta_1})\,.$$

By (12) and (14) we have

$$h(w(\zeta))=\phi_{\zeta_2}(\zeta)$$

for all ζ in S_0^*. Consequently, neither h nor w is a constant.

Now $T_n\phi_{\zeta_2}$ has the only zero $w_{T_n}(\zeta_2)$. Since g_j has no zeros, by the same argument as the proof of Lemma 5 we can see that T_ng_j also has no zeros for all sufficiently large n. Hence $(T_ng_j)(T_n\phi_{\zeta_1})$ has the only zero $w_{T_n}(\zeta_1)$ for all sufficiently large n. By Hurwitz' theorem, h has only one zero, and the zeros of $T_n\phi_{\zeta_2}$ and $(T_ng_j)(T_n\phi_{\zeta_1})$ converge to it as $n\to\infty$. Therefore

$$\lim_{n\to\infty} w_{T_n}(\zeta_1)=\lim_{n\to\infty} w_{T_n}(\zeta_2)\,,$$

that is,

$$w(\zeta_1)=w(\zeta_2)\,.$$

If a point ζ on γ_j defines four distinct accessible boundary points $\zeta_1, \zeta_2, \zeta_3$ and

ζ_4 of S_0, we can similarly show that

$$w(\zeta_1)=w(\zeta_2)=w(\zeta_3)=w(\zeta_4).$$

Thus w is a continuous mapping of S into S'.

4. Let p' be the genus of $\bar{S}'$ and q' be the number of its boundary components. We set $N'=2p'+q'-1$ and denote by $\gamma'_1, \cdots, \gamma'_{N'}$ the canonical homology basis of $\bar{S}'$ as before mentioned. We set $S'_0=S'-\bigcup_{j=1}^{2p'}\gamma'_j$ and denote by $(S'_0)^*$ the set obtained by adding to S'_0 all accessible boundary points which are defined by points of $\bigcup_{j=1}^{2p'}\gamma'_j$.

Since $c(T_j^{-1})=c(T_j)$, (9) implies that

$$\lim_{j\to\infty} c(T_j^{-1})=1, \quad T_j^{-1}1=1.$$

Let $\{S'_n\}$ be an exhaustion of S'. By the same argument as before there exists a continuous mapping $w_{T_n^{-1}}$ of $(S'_n)^*$ into S for each n such that

$$\lim_{n\to\infty} w_{T_n^{-1}}(\zeta')=w'(\zeta') \tag{16}$$

uniformly on every compact subset of $(S'_0)^*$, where w' is a continuous mapping of S' into S, and moreover ζ' and $w_{T_n^{-1}}(\zeta')$ are ε_n-related with respect to T_n^{-1} for all ζ' in $(S'_0)^*$. It follows from ε_n-relation that

$$|(T_n f)(\zeta')-f(w_{T_n^{-1}}(\zeta'))|\leqq\varepsilon_n\|f\| \tag{17}$$

for all ζ' in $(S'_n)^*$ and all f in $A(S)$.

5. Now we set $\zeta'=w(\zeta)$ for each fixed ζ in S. If n is sufficiently large, ζ is in S_n and ζ' is in S'_n. Then, for every f in $A(S)$

$$\begin{aligned} &|f(\zeta)-f(w'(\zeta'))| \\ &\leqq|f(\zeta)-(T_n f)(w_{T_n}(\zeta))|+|(T_n f)(w_{T_n}(\zeta))-(T_n f)(w(\zeta))| \\ &\quad+|(T_n f)(\zeta')-f(w_{T_n^{-1}}(\zeta'))|+|f(w_{T_n^{-1}}(\zeta'))-f(w'(\zeta'))|. \end{aligned} \tag{18}$$

By (12), (17) and (16), the first term, the third term and the last term of the right side of (18) converge to 0 as $n\to\infty$. The functions $T_n f$ are equicontinuous on every compact subset of S', for they are uniformly bounded on S'. Hence, by (14) the second term of the right side of (18) converges to 0 as $n\to\infty$. Thus it follows from (18) that

$$f(w'(\zeta'))=f(\zeta).$$

If we set $f=\phi_\zeta$ particularly, we obtain

$$\phi_\zeta(w'(\zeta'))=\phi_\zeta(\zeta)=0.$$

Since ζ is the only zero of ϕ_ζ, we have

$$w'(\zeta')=\zeta .$$

Thus

$$w'(w(\zeta))=\zeta$$

for all ζ in S. We also obtain by the same argument as above

$$w(w'(\zeta'))=\zeta'$$

for all ζ' in S'. Therefore w is a homeomorphism of S onto S'.

By (10) a sequence $\{T_n f\}$ converges uniformly on every compact subset of S' for every f in $A(S)$. We set

$$g=\lim_{n\to\infty} T_n f ,$$

where g is an analytic function on S'. It follows from (12) that

$$g(w(\zeta))=f(\zeta) \tag{19}$$

for all ζ in S. If f is not a constant, g is not one. Since f and g are analytic, (19) implies that $w(\zeta)$ is analytic on S. Therefore w is a conformal mapping of S onto S'.

§6. Proof of Theorem 3.

Since $N\geqq 2$, there are only a finite number of conformal automorphisms of S. We denote them by $w_1, \cdots, w_M$. For every relatively compact subdomain D of S and for every $\varepsilon>0$, we want to show the existence of a $d>1$ such that if $c(T)<d$ and $T1=1$, then

$$|f(z)-(Tf)(w_j(z))|\leqq\varepsilon \min(\|f\|, \|Tf\|)$$

for a certain j $(1\leqq j\leqq M)$, for all f in $A(S)$ and for all z in D. If it is not true, then for some relatively compact subdomain D of S and for some $\varepsilon>0$ there is a sequence d_n (>1) which converges to 1 as follows. For each n there is a T_n in $L(A(S), A(S))$ with $c(T_n)<d_n$ and $T_n 1=1$ such that for each j $(1\leqq j\leqq M)$ there are an f_{jn} in $A(S)$ and a z_{jn} in D satisfying

$$|f_{jn}(z_{jn})-(T_n f_{jn})(w_j(z_{jn}))|>\varepsilon\|f_{jn}\| \tag{20}$$

or

$$|f_{jn}(z_{jn})-(T_n f_{jn})(w_j(z_{jn}))|>\varepsilon\|T_n f_{jn}\| . \tag{21}$$

Since $c(T_n)\to 1$ and $T_n 1=1$, we can use the same arguments as the proof of Theorem 2. Hence, by choosing a subsequence if necessary, we may assume that for a certain j $(1\leqq j\leqq M)$

$$\lim_{n\to\infty} w_{T_n}(z)=w_j(z) \tag{22}$$

uniformly on every compact subset of S_0^*, where w_{T_n} is the mapping in Lemma 5. By using Lemma 5, we may simultaneously assume that

$$|f_{jn}(z_{jn})-(T_n f_{jn})(w_{T_n}(z_{jn}))|\leqq\frac{\varepsilon}{2}\min(\|f_{jn}\|,\|T_n f_{jn}\|). \tag{23}$$

In addition, we may assume that with respect to the j we have selected, either (20) or (21) holds for every n. Furthermore, we may assume that $z_{jn}\rightarrow z_0$ as $n\rightarrow\infty$ for some z_0 in $\bar{D}$.

Since the functions $f_{jn}/\|f_{jn}\|$ and $(T_n f_{jn})/\|f_{jn}\|$ are uniformly bounded on S, we can assume that for certain analytic functions f_0 and g_0 on S

$$\lim_{n\to\infty}\frac{f_{jn}}{\|f_{jn}\|}=f_0, \quad \lim_{n\to\infty}\frac{T_n f_{jn}}{\|f_{jn}\|}=g_0$$

uniformly on every compact subset of S. If (20) holds for every n, then

$$\left|\frac{f_{jn}(z_{jn})}{\|f_{jn}\|}-\frac{(T_n f_{jn})(w_j(z_{jn}))}{\|f_{jn}\|}\right|>\varepsilon.$$

Letting n go to infinity, we obtain

$$|f_0(z_0)-g_0(w_j(z_0))|\geqq\varepsilon.$$

On the other hand, it follows from (22) and (23) that

$$|f_0(z_0)-g_0(w_j(z_0))|\leqq\frac{\varepsilon}{2},$$

which is a contradiction. If (21) holds for every n, the similar argument yields a contradiction.

Finally we must prove the uniqueness of w for every sufficiently small $\varepsilon>0$. If it is not true, there are positive sequence $\{\varepsilon_n\}$ with $\varepsilon_n\rightarrow 0$, a sequence $\{T_n\}$ in $L(A(S), A(S))$ and distinct conformal automorphisms w_j, w_k $(j\neq k)$ such that

$$|f(z)-(T_n f)(w_j(z))|\leqq\varepsilon_n\min(\|f\|,\|T_n f\|) \tag{24}$$

and

$$|f(z)-(T_n f)(w_k(z))|\leqq\varepsilon_n\min(\|f\|,\|T_n f\|) \tag{25}$$

for all f in $A(S)$ and all z in D. We choose a point z_1 in D such that

$$w_j(z_1)\neq w_k(z_1).$$

It follows from (24) and (25) that

$$\begin{aligned}&|f(w_j(z_1))-f(w_k(z_1))|\\ &\leqq|f(w_j(z_1))-(T_n^{-1}f)(z_1)|+|f(w_k(z_1))-(T_n^{-1}f)(z_1)|\\ &\leqq 2\varepsilon_n\|f\|\end{aligned}$$

for all f in $A(S)$. Hence

$$f(w_j(z_1))=f(w_k(z_1))$$

for all f in $A(S)$. This is a contradiction, for the space $A(S)$ separates points on S. Thus the uniqueness has been proved.

§ 7. Proof of Theorem 4.

1. Let D be a relatively compact subdomain of S. We may assume that the boundary C of D consists of a finite number of contours and f_0 does not vanish on C. Let m be the minimum of $|f_0|$ on C. For every ε with $0<\varepsilon<m$, we set

$$\varepsilon_1=\min\left(\frac{\varepsilon}{2}, \frac{\varepsilon}{2\|f_0\|}\right).$$

By Theorem 3 there is a constant $d_1>1$ as follows. If a T in $L(A(S), A(S))$ satisfies $c(T)<d_1$ and $T1=1$, then there is a unique automorphism w of S such that

$$\begin{aligned}(26)\qquad |f(z)-(Tf)(w(z))| &\leqq \varepsilon_1 \min(\|f\|, \|Tf\|)\\ &\leqq \frac{\varepsilon}{2}\min(\|f\|, \|Tf\|)\end{aligned}$$

for all f in $A(S)$ and all z in $\bar{D}$. Particularly,

$$\begin{aligned}(27)\qquad |f_0(z)-(Tf_0)(w(z))| &\leqq \varepsilon_1\|f_0\|<\varepsilon\\ &<m\leqq|f_0(z)|\end{aligned}$$

for all z on C. Hence, by the theorem of Rouché, $f_0(z)$ and $(Tf_0)(w(z))$ have the same number of zeros in D.

2. Now we take a distance $d(\cdot,\cdot)$ on S which induces the original topology of S. All functions $f/\|f\|$ and $f/\|Tf\|$ for $f\in A(S)$ and for T with $c(T)<d_1$ are equicontinuous on D, consequently, if $\delta>0$ is sufficiently small and $c(T)<d_1$, then

$$(28)\qquad |f(z_1)-f(z_2)|\leqq\frac{\varepsilon}{2}\min(\|f\|, \|Tf\|)$$

for all z_1, z_2 in D with $d(z_1, z_2)<\delta$ and for all f in $A(S)$.

3. Let $a_1,\cdots,a_l$ be the elements of $N_{f_0}(D)$. We may assume that the neighborhoods $U_\delta(a_j)=\{z|d(z, a_j)<\delta\}$ $(j=1,\cdots,l)$ are contained in D and mutually disjoint. We want to show that for every sufficiently small $\varepsilon>0$ there is a $d>1$ such that, if $c(T)<d$ and w is the conformal automorphism corresponding to T in the sense of Theorem 3, then for every ζ in $N_{Tf_0}(w(D))$, $w^{-1}(\zeta)$ is contained in $U_\delta(a_j)$ for some j with $1\leqq j\leqq l$. If it is not true, then there are

a positive sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$, a sequence $\{T_n\}$ in $L(A(S), A(S))$ satisfying $c(T_n) \to 1$ and $T_n 1 = 1$, and a point ζ_n in $N_{T_n f_0}(w(D))$, where w is a fixed conformal automorphism corresponding to T_n in the sense of Theorem 3 such that $w^{-1}(\zeta_n)$ is not in $U_\delta(a_j)$ for every n and every j with $1 \leqq j \leqq l$. Since we may use (27) for $\varepsilon = \varepsilon_n$ and $T = T_n$, we obtain

$$|f_0(z) - (T_n f_0)(w(z))| < \varepsilon_n \tag{29}$$

for all z in D. Hence

$$|f_0(w^{-1}(\zeta_n))| < \varepsilon_n ,$$

consequently,

$$\lim_{n\to\infty} f_0(w^{-1}(\zeta_n)) = 0 .$$

We may assume that $\zeta_n \to \zeta_0$ for some ζ_0 in S. Then, $f_0(w^{-1}(\zeta_0)) = 0$, so, $w^{-1}(\zeta_0) = a_j$ for some j with $1 \leqq j \leqq l$. Hence

$$\lim_{n\to\infty} w^{-1}(\zeta_n) = a_j ,$$

which is a contradiction.

4. In the previous section we have shown that for every ζ in $N_{Tf_0}(w(D))$ there is an a_j in $N_{f_0}(D)$ whose δ-neighborhood contains $w^{-1}(\zeta)$ if $\varepsilon > 0$ is sufficiently small and $c(T)$ is sufficiently close to 1. Then, it follows from (26) and (28) that if $c(T)$ is sufficiently close to 1 and $T1 = 1$,

$$\begin{aligned} &|f(a_j) - (Tf)(\zeta)| \\ &\leqq |f(a_j) - f(w^{-1}(\zeta))| + |f(w^{-1}(\zeta)) - (Tf)(\zeta)| \\ &\leqq \varepsilon \min(\|f\|, \|Tf\|) \end{aligned}$$

for all f in $A(S)$. Namely, a_j and ζ are ε-related with respect to T. Thus, if $\varepsilon > 0$ is sufficiently small and $c(T)$ is sufficiently close to 1, we can define a mapping θ of $N_{Tf_0}(w(D))$ into $N_{f_0}(D)$ by setting, for every ζ in $N_{Tf_0}(w(D))$, $\theta(\zeta) = a_j$. Observe that $\theta(\zeta)$ and ζ are ε-related with respect to T.

5. Next, we shall prove that θ is characterized as the mapping of $N_{Tf_0}(w(D))$ into $N_{f_0}(D)$ such that $\theta(\zeta)$ and ζ are ε-related with respect to T. We may show that if $\varepsilon > 0$ is sufficiently small and $c(T)$ is sufficiently close to 1, then a point a_j in $N_{f_0}(D)$ is uniquely determined for a given ζ in $N_{Tf_0}(w(D))$ by the condition that a_j and ζ are ε-related with respect to T. If it were not, then there are a positive sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$, a sequence $\{T_n\}$ with $c(T_n) \to 1$, distinct points a_j, a_k in $N_{f_0}(D)$ $(j \neq k)$ and a point ζ_n in $N_{T_n f_0}(w(D))$, where w is a fixed conformal automorphism corresponding to T_n, such that a_j and ζ_n, a_k and ζ_n are ε_n-related with respect to T_n. Hence

$$\begin{aligned}&|f(a_j)-f(a_k)|\\ &\leqq|f(a_j)-(T_nf)(\zeta_n)|+|f(a_k)-(T_nf)(\zeta_n)|\\ &\leqq 2\varepsilon_n\|f\|\end{aligned}$$

for all f in $A(S)$. Therefore

$$f(a_j)=f(a_k)$$

for all f in $A(S)$, which is a contradiction.

6. Let $\varepsilon>0$ be a sufficiently small number. We denote by D_0 the union of $w(D)$ for all conformal automorphisms w of S. It is a relatively compact subdomain of S. Since all functions $(Tf)/\|f\|$ and $(Tf)/\|Tf\|$ for $f\in A(S)$ and for T with $c(T)$ close to 1 are equicontinuous on D_0, we can choose a $\delta>0$ such that

$$|(Tf)(z_1)-(Tf)(z_2)|\leqq\frac{\varepsilon}{2}\min(\|f\|,\|Tf\|) \tag{30}$$

for all z_1, z_2 in D_0 with $d(z_1,z_2)<\delta$, for all f in $A(S)$ and for all T with $c(T)$ sufficiently close to 1.

7. To continue, we need the following proposition:

For every sufficiently small $\varepsilon>0$, there exists a $d>1$ such that, if $c(T)<d$ and $T1=1$, then there exists a point $\zeta\in N_{Tf_0}(w(D))$ whose δ-neighborhood contains $w(a)$, where a is an arbitrary point of $N_{f_0}(D)$ and w is the conformal automorphism of S corresponding to T in the sense of Theorem 3.

Suppose that this proposition does not hold. Then there are a positive sequence $\{\varepsilon_n\}$ with $\varepsilon_n\to 0$ and a sequence $\{T_n\}$ in $L(A(S),A(S))$ with $c(T_n)\to 1$ and $T_n1=1$ satisfying the following property; there is a conformal automorphism w of S independent of n such that (29) is satisfied and $w(a)$ is not in $U_\delta(\zeta)$ for any ζ in $N_{T_nf_0}(w(D))$. It follows from (29) that

$$|(T_nf_0)(w(a))|<\varepsilon_n,$$

consequently

$$\lim_{n\to\infty}(T_nf_0)(w(a))=0. \tag{31}$$

We may assume that $\{T_nf_0\}$ converges uniformly on every compact subset of S. We set

$$g_0=\lim_{n\to\infty}T_nf_0 \tag{32}$$

The inequality (29) implies that

$$f_0(z)=g_0(w(z))$$

in D. Hence the zeros of g_0 in $w(D)$ are

$$\zeta_j = w(a_j) \quad (j=1, \cdots, l).$$

If we choose a sufficiently small δ_1 with $0<\delta_1<\delta/2$, the neighborhoods $U_{\delta_1}(\zeta_j)$ $(j=1, \cdots, l)$ are contained in $w(D)$ and mutually disjoint. There is a constant $\eta>0$ such that

$$|g_0(z)|>\eta$$

for all z in $w(D)-\bigcup_{j=1}^{l} U_{\delta_1}(\zeta_j)$. Since the convergence in (32) is uniform on $w(D)$,

$$|(T_n f_0)(z)|>\eta$$

for all z in $w(D)-\bigcup_{j=1}^{l} U_{\delta_1}(\zeta_j)$ and for all sufficiently large n. By Hurwitz' theorem, each $U_{\delta_1}(\zeta_j)$ contains a point ζ in $N_{T_n f_0}(w(D))$ if n is sufficiently large. Then, $\delta>2\delta_1$ implies $U_\delta(\zeta)\supset U_{\delta_1}(\zeta_j)$. Hence $w(a)$ is not in $U_{\delta_1}(\zeta_j)$ $(j=1, \cdots, l)$, for $w(a)$ is not in $U_\delta(\zeta)$ for any ζ in $N_{T_n f_0}(w(D))$. Therefore we can conclude that

$$|(T_n f_0)(w(a))|>\eta$$

for all sufficiently large n. This contradicts (31).

8. Now, let us prove that the mapping θ is onto. Let a be an arbitrary point in $N_{f_0}(D)$. For every sufficiently small $\varepsilon>0$, we take a T with $T1=1$ and $c(T)$ sufficiently close to 1 so that (26) and (30) are satisfied, and so that there exists the ζ satisfying the proposition in the previous section. Then we have

$$\begin{aligned} &|f(a)-(Tf)(\zeta)| \\ &\leqq |f(a)-(Tf)(w(a))|+|(Tf)(w(a))-(Tf)(\zeta)| \\ &\leqq \varepsilon \min(\|f\|, \|Tf\|) \end{aligned}$$

for all f in $A(S)$. Namely, a and ζ are ε-related with respect to T. Remember that $\theta(\zeta)$ is characterized by the condition that $\theta(\zeta)$ and ζ are ε-related with respect to T. Hence $a=\theta(\zeta)$, that is, the mapping θ is onto. Thus the proof has been completed.

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