J. Math. Soc. Japan
Vol. 31, No. 1, 1979

Tight spherical designs, I

By E. BANNAI and R. M. DAMERELL

(Received Sept. 21, 1977)

§ 1. Introduction.

Let $\boldsymbol{R}^d$ be Euclidean space of dimension d and Ω_d the set of unit vectors in $\boldsymbol{R}^d$. A non-empty finite set $X \subseteqq \Omega_d$ is called a *spherical t-design* in Ω_d if

$$\sum_{\alpha \in X} W(\alpha)=0$$

for all homogeneous harmonic polynomials W on $\boldsymbol{R}^d$ of degree $1, 2, \cdots, t$. This is equivalent to the condition that the k-th moments of X are invariant under orthogonal transformations of $\boldsymbol{R}^d$ for $k=0, 1, 2, \cdots, t$. These designs were studied by Delsarte, Goethals and Seidel [4]. They proved that the cardinality of a design is bounded below;

$$|X| \geqq \binom{d+n-1}{d-1}+\binom{d+n-2}{d-1} \qquad \text{if} \quad t=2n,$$

$$|X| \geqq 2\binom{d+n-1}{d-1} \qquad \text{if} \quad t=2n+1.$$

They called a design *tight* if it attains this bound. They constructed examples of tight spherical t-designs for $t=2, 3, 4, 5, 7, 11$, and proved ([4], Theorem 7.7) that no such designs exist for $t=6$, except the regular heptagon in Ω_2. Bannai [1] proved that for given $t \geqq 8$, there exist tight spherical designs in Ω_d for only finitely many values of d.

In this paper we will prove

THEOREM 1. *Let $t=2n$ and $n \geqq 3$ and $d \geqq 3$. Then there exists no tight spherical t-design in Ω_d.*

In a subsequent paper we hope to prove a similar result when t is odd. Note that if $d=2$ the only tight spherical design is the regular $(t+1)$-gon.

The proof is similar to that of Theorem 7.7 in [4], which is the special case $t=6$. We first prove that if a design exists, then a certain polynomial (written $R_n(x)$, defined in § 2 below) has all its roots rational. By reducing $R_n(x)$ modulo various primes, we show that if its roots are all rational, then their reciprocals are all integers, and all of the same parity as d. We define $S_n(x)$ as the polynomial having these integers as its roots.

We now consider the two cases where n is even or odd. If n is even, say $n=2m$, the sum of the roots of $S_n(x)$ is $-2m$. Now $R_n(x)$ is the sum of two Gegenbauer polynomials whose roots interlace; using the interlacing we can divide the roots of $S_n(x)$ into pairs, say a and b such that $a>0$, $b<0$, $a>|b|$. Since these are integers of the same parity, we find $b=-a+2$. Therefore $S_n(x)$ is an even function of $(x-1)$. By expressing $S_n(x)$ as a polynomial in $(x-1)$ and finding a nonzero coefficient we obtain a contradiction. This proves the Theorem for even n.

If n is odd, say $n=2m+1$, then we pair off all but one of the roots in a similar way. As before, $a+b\geqq 2$; since the sum of the roots of $S_n(x)$ is $-(d+2m)$ the unpaired root is $\leqq -(d+4m)$. But we can show $S_n(x)\neq 0$ in this interval; this contradiction proves the Theorem for n odd.

§2. Notation.

Let λ be a real number and m a positive integer. Define

$$(\lambda)_m=\Gamma(\lambda+m)/\Gamma(\lambda)=\lambda(\lambda+1)\cdots(\lambda+m-1) \tag{2.1}$$

and

$$(2m-1)!!=1\cdot 3\cdot 5\cdot \cdots \cdot(2m-1)=2m!/2^m\cdot m!=2^m\cdot\left(\frac{1}{2}\right)_m. \tag{2.2}$$

The Gegenbauer polynomials $C_n^{(\lambda)}(x)$ are defined by the equations ([**5**], §10.9, (21) and (22)):

$$\frac{m!(-1)^m}{(\lambda)_m}C_{2m}^{(\lambda)}(x)=F\left(-m,\ m+\lambda,\ \frac{1}{2},\ x^2\right) \tag{2.3}$$

$$=1+\sum_{r=1}^{m}{}^mC_r(-1)^r\frac{(m+\lambda)_r}{(1/2)_r}x^{2r} \tag{2.4}$$

and

$$\frac{m!(-1)^m}{(\lambda)_{m+1}}C_{2m+1}^{(\lambda)}(x)=2xF\left(-m,\ m+\lambda+1,\ \frac{3}{2},\ x^2\right) \tag{2.5}$$

$$=2\sum_{r=0}^{m}{}^mC_r(-1)^r\frac{(m+\lambda+1)_r}{(3/2)_r}x^{2r+1} \tag{2.6}$$

where F is Gauss' hypergeometric function. From now on, λ will always have the value $\lambda=(1/2)d$ and will be omitted where possible. We define the polynomial $R_n(x)$ by

$$R_n(x)=C_n(x)+C_{n-1}(x). \tag{2.7}$$

Apart from constant factors, R_n and C_n have the same meaning as in §2 of [4]. From these definitions we note that the leading coefficients of R_n, C_n and C_{n-1} are all positive, also that C_n is even if n is even and odd if n is odd.

Define $S_n(x)$ as the monic polynomial whose roots are the reciprocals of those of $R_n(x)$.

$$\left.\begin{aligned} \text{If} \qquad & S_n(x)=x^n+\sum_{k=1}^{n} u_k x^{n-k} \\ \text{then} \qquad & c\cdot R_n(x)=1+\sum_{k=1}^{n} u_k x^k \end{aligned}\right\} \tag{2.8}$$

for a suitable constant c. We now derive some information about the u_k.

First suppose $n=2m$ is even. Replace m by $m-1$ in (2.6), multiply by $-m$ and add (2.4). This gives

$$\begin{aligned} & R_{2m}(x)(-1)^m m!/(\lambda)_m \\ & \quad =1+\sum_{r=1}^{m} {}^mC_r(-1)^r(m+\lambda)_r x^{2r}\Big/\left(\frac{1}{2}\right)_r \\ & \qquad -2m\sum_{r=0}^{m-1} {}^{m-1}C_r(-1)^r(m+\lambda)_r x^{2r+1}\Big/\left(\frac{3}{2}\right)_r. \end{aligned} \tag{2.9}$$

Now suppose $n=2m+1$ is odd. Then $(\lambda+m)\cdot(2.6)+(2.4)$ gives

$$\begin{aligned} & R_{2m+1}(x)(-1)^m m!/(\lambda)_m \\ & \quad =1+\sum_{r=1}^{m} {}^mC_r(-1)^r(m+\lambda)_r x^{2r}\Big/\left(\frac{1}{2}\right)_r \\ & \qquad +2(\lambda+m)\sum_{r=0}^{m} {}^mC_r(-1)^r(m+\lambda+1)_r x^{2r+1}\Big/\left(\frac{3}{2}\right)_r. \end{aligned} \tag{2.10}$$

Now define

$$h=(2m+2\lambda)=2m+d. \tag{2.11}$$

Then we have

$$\frac{(m+\lambda)_r}{(1/2)_r}=\frac{h(h+2)\cdots(h+2r-2)}{(2r-1)!!} \tag{2.12}$$

and similar formulae for $(m+\lambda)_r/(3/2)_r$ and $(m+\lambda+1)_r/(3/2)_r$. By inspection we have the following results:

LEMMA 2.1. *Let u_r be defined for $1\leqq r\leqq n$ by (2.8) above. Then*

(1) *the denominator of u_{2r} divides $(2r-1)!!$,*

(2) *the denominator of u_{2r+1} divides $(2r+1)!!$,*

(3) *if d is even all the u_r are even (because $2|h$),*

(4) *if d is odd, the constant term of $S_n(x)$ is odd (because it equals*

$$u_n=\pm h(h+2)\cdots(h+2m-2)/(2m-1)!!),$$

(5) *the sum of the roots of $S_n(x)$ is*

$$\left.\begin{array}{ll} +2m & \text{if } n=2m \\ -h & \text{if } n=2m+1, \end{array}\right\} \tag{2.13}$$

(because this sum$=-u_1=-1\times$coefficient of x in (2.9) or (2.10).)

§ 3. Lloyd type theorem.

The following result is implicit in Theorem 7.7 of [4].

THEOREM 2. *Suppose there exists a tight spherical t-design in Ω_d with $d\geqq 3$. If $t=2n$ then all n zeros of the polynomial $R_n(x)$ are rational. If $t=2n+1$, then all n zeros of the polynomial $C_n(x)$ are rational.*

PROOF. By [4] Theorem 7.5 the design induces an s-class association scheme (in the sense of [2]) with $s=\left[\frac{t+1}{2}\right]$. The Bose-Mesner algebra $\boldsymbol{A}$ of this scheme is as described in [3] Chapter 2. The notation agrees except that $i=0$ in [3] corresponds to the relation R_0=identity, which corresponds to $\alpha=1$ in [4]. By comparing Theorem 3.6 of [4] with (2.16) in [3] we see that $Q_k(\alpha)$ and $Q_k(i)$ have the same meaning.

By [4] Theorem 2.4 the $Q_k(1)$ are all distinct for $d\geqq 3$ and $k\geqq 1$, because $Q_{k+1}(1)>Q_k(1)$. (If $d=2$ the proof breaks down here because then $Q_k(1)=2$ for all k). So by [3] (2.18) the matrices J_k have distinct ranks.

Let σ be any field automorphism of the complex numbers. Since the algebra $\boldsymbol{A}$ has only the unique set

$$\{J_0, J_1, \cdots, J_s\} \qquad ([4], 7.6) \tag{3.1}$$

of orthogonal idempotents, σ permutes them. Since the J_i's have distinct ranks, σ fixes all of them, so each J_i is rational. By Theorem 3.6 of [4], the number $Q_k(\alpha)$ is rational for all α in $A(X)$ and $1 \leqq k \leqq s$. So all elements of $A(X)$ are rational.

If $t=2n$ then by [4] Theorem 5.11, $A(X)$ consists of the zeros of $R_n(x)$ so $R_n(x)$ has all its roots rational. Similarly if $t=2n+1$, by Theorem 5.12 of [4], $C_n(x)$ has all its roots rational. This proves Theorem 2.

LEMMA 3.1. *Suppose there exists a tight $(2n)$-design in Ω_d with $d \geqq 3$. Then $S_n(x)$ has all its roots integers and these integers all have the same parity as d.*

PROOF. We have to show that the u_k in (2.8) are all integral. Let a be the least integer >0 such that $acR_n(x)$ has all coefficients integral. By (2.8)

$$acR_n(x) = a + \sum_{k=1}^{n} au_k x^k. \tag{3.2}$$

If $a \neq 1$, let p be a prime factor. By the minimality of a there exits a k such that p does not divide au_k: let $k=r$ be the least. Then

$$acR_n(x) \equiv \sum_{k=r}^{n} au_k x^k \quad \mod p. \tag{3.3}$$

Therefore r of the roots of $R_n(x)$ are multiples of p, so p^r divides a. Since p does not divide au_r, p^r divides the denominator of u_r. By Lemma 2.1 this is a factor of either $r!!$ (if r is odd) or $(r-1)!!$ (if r is even). This is impossible because the largest power of p dividing $r!!$ is $<p^r$. So all the u_k are integers. By Lemma 2.1, if d is odd, the constant term of $S_n(x)$ and hence all the roots is odd. If d is even all the u_k are even, so that $S_n(x) \equiv x^n \mod 2$. Therefore all roots are even. Q.E.D.

For future use, we give the corresponding result when t is odd.

LEMMA 3.2. *Suppose there exists a tight $(2n+1)$-design in Ω_d with $d \geqq 3$. Then the reciprocals of the nonzero roots of $C_n(x)$ are all integers, of the same parity as d.*

This is proved by the same method; details are left to the reader.

§ 4. Interlacing roots.

We now apply the theory of orthogonal polynomials to prove an inequality for the roots of $S_n(x)$. Put $m=[(1/2)n]$, recall $R_n(x)=C_n(x)+C_{n-1}(x)$.

LEMMA 4.1. *The roots of $R_n(x)$ are real and distinct and nonzero. Exactly m of them are positive.*

PROOF. For fixed λ and varying n the $C_n^{\lambda}(x)$ form a system of orthogonal polynomials ([5], §10.9). By standard theory ([5], §10.3) the zeros of C_n are real and distinct; between any two there lies a zero of C_{n-1}. Accordingly we write

$$z_1>y_1>z_2>y_2>\cdots>z_{n-1}>y_{n-1}>z_n \tag{4.1}$$

where $\{z_1, \cdots, z_n\}$ are the zeros of C_n and $\{y_1, \cdots, y_{n-1}\}$ those of C_{n-1}. From (2.4) and (2.6), the leading coefficients of C_n and C_{n-1} are both positive. Therefore

$$\mathrm{sign}(R_n(z_i))=\mathrm{sign}(C_{n-1}(z_i))=(-1)^{i+1}, \tag{4.2}$$

$$\mathrm{sign}(R_n(y_i))=\mathrm{sign}(C_n(y_i))=(-1)^i. \tag{4.3}$$

Therefore $R_n(x)$ has a zero in each of the intervals

$$z_i>x>y_i, \quad i=1, 2, \cdots, (n-1). \tag{4.4}$$

Also

$$\mathrm{sign}(R_n(z_n))=(-1)^{n-1} \tag{4.5}$$

and if X is very large, then

$$\mathrm{sign}(R_n(-X))=\mathrm{sign}(-X)^n=(-1)^n. \tag{4.6}$$

So the last root of $R_n(x)$ lies in the interval

$$z_n>x>-\infty. \tag{4.7}$$

Now if $n=2m$, the middle root of $C_{n-1}(x)$ is $y_m=0$ (because $C_{n-1}(x)$ is odd). Hence $R_n(x)$ has m positive roots (in the intervals (4.4) for $i=1, 2, \cdots, m$). If $n=2m+1$ then the middle root of $C_n(x)$ is $z_{m+1}=0$; so $R_n(x)$ again has m positive roots. Thus the Lemma is proved.

Accordingly we label the roots of $R_n(x)$ as follows:

$$p_1>p_2>\cdots>p_m(>0>)q_{n-m}>q_{n-m-1}>\cdots>q_1. \tag{4.8}$$

Define $a_i=1/p_i$ and $b_i=1/q_i$; then the numbers

$$\{a_1, \cdots, a_m, b_1, \cdots, b_{n-m}\} \tag{4.9}$$

are the roots of $S_n(x)$.

LEMMA 4.2. *With this notation, $a_r+b_r>0$ for $1 \leqq r \leqq m$.*

PROOF. In the scheme (4.8) q_r is the $(n+1-r)$-th root of $R_n(x)$ (in decreasing order). Therefore q_r lies in the $(n+1-r)$-th interval (4.4), so

$$q_r<z_{n+1-r}. \tag{4.10}$$

Similarly

$$p_r < z_r \tag{4.11}$$

Thus,

$$p_r + q_r < z_r + z_{n+1-r} = 0 \tag{4.12}$$

because $C_n(x)$ is either even or odd, so its roots are symmetrical about $x=0$. Since $p_r > 0$ and $q_r < 0$ we have

$$a_r + b_r = (p_r + q_r)/p_r q_r > 0. \tag{4.13}$$

Q. E. D.

§ 5. Proof of Theorem 1.

We suppose that a tight spherical $2n$-design exists, with $n \geqq 3$ and $d \geqq 3$, and deduce a contradiction. First suppose n is even. Then by Lemma 4.2 we can pair off all the roots of $S_n(x)$ so that the sum of any pair is positive. But by Lemma 3.1 these roots are integers of the same parity, so

$$a_r + b_r \geqq 2 \qquad \text{for} \quad 1 \leqq r \leqq \frac{1}{2} n. \tag{5.1}$$

But by Lemma 2.1 the sum of all the roots is n, so we must have

$$a_r + b_r = 2. \tag{5.2}$$

Therefore $S_n(x)$ is an even polynomial in $x-1=w$, say.

Take the formula (2.9) for $R_n(x)$, apply the transformation (2.8); this gives

$$\begin{aligned} S_n(x) = x^{2m} - \frac{m(m+\lambda)}{(1/2)} x^{2m-2} - 2m\, x^{2m-1} + \frac{2m(m-1)(m+\lambda)}{(3/2)} x^{2m-3} \\ + \text{terms of degree} < (2m-3). \end{aligned} \tag{5.3}$$

In this we put $x = w+1$ and extract the coefficient of w^{2m-3}. This equals

$$\begin{aligned} &{}^{2m}C_3 - m h \cdot {}^{2m-2}C_1 - 2m \cdot {}^{2m-1}C_2 + \frac{2}{3} m(m-1) h \\ &= -\frac{4}{3} m(m-1)(2m-1+h) < 0. \end{aligned} \tag{5.4}$$

Since this coefficient is nonzero (for $m > 1$), $S_{2m}(w)$ is not an even function of w. This proves the Theorem for even n.

Now suppose n is odd. As before, we can divide all but one of the roots of $S_n(x)$ into pairs satisfying (5.1). Since the sum of all the roots is $-h$, the only unpaired root (called b_{m+1} in §4) satisfies

$$-h=b_{m+1}+(\text{other roots})\geqq b_{m+1}+2m. \tag{5.5}$$

Therefore

$$b_{m+1}\leqq -h-2m. \tag{5.6}$$

Now consider $S_n(x)$. Applying (2.8) to (2.10) we have

$$S_{2m+1}(x)=\sum_{r=0}^{m}(-1)^r\,{}^mC_r\frac{h\cdot(h+2)\cdots(h+2r-2)}{(2r-1)!!}x^{2m+1-2r}$$

$$+\sum_{r=0}^{m}(-1)^r\,{}^mC_r\frac{h\cdot(h+2)\cdots(h+2r)}{(2r+1)!!}x^{2m-2r} \tag{5.7}$$

$$=T_0(x)+T_1(x)+\cdots+T_m(x) \tag{5.8}$$

where $T_r(x)$ is the sum of the terms in $x^{2m+1-2r}$ and x^{2m-2r}, so that

$$T_r(x)=(-1)^r\,{}^mC_r\frac{h\cdot(h+2)\cdots(h+2r-2)}{(2r-1)!!}x^{2m-2r}\left\{x+\frac{h+2r}{1+2r}\right\}. \tag{5.9}$$

LEMMA 5.1. *Let* $x\leqq -h-2m$. *Then* (*for* $m\geqq 1$, $h>2m$), $S_n(x)<0$.

PROOF. Put

$$T=T_0(x)=x^{2m}(x+h)<0. \tag{5.10}$$

For all $r\geqq 0$, it follows from the given inequalities that

$$\frac{h+2r}{1+2r}\leqq h\leqq |x|-2m. \tag{5.11}$$

Therefore

$$\left|\frac{T_{r+1}}{T_r}\right|=\frac{m-r}{r+1}\times\frac{h+2r}{2r+1}\times|x|^{-2}\times\left\{|x|-\frac{h+2r+2}{2r+3}\right\}\Big/\left\{|x|-\frac{h+2r}{2r+1}\right\}$$

$$\leqq m(|x|-2m)|x|^{-2}\cdot|x|/2m<\frac{1}{2}. \tag{5.12}$$

Thus,

$$\sum_{r=1}^{m}|T_r|<|T|\sum_{r=1}^{m}2^{-r}<|T|, \tag{5.13}$$

so that

$$S_n=T+\sum_{r=1}^{m}T_r<0. \tag{5.14}$$

This inequality proves the Lemma, and the contradiction with (5.6) completes the proof of the Theorem.

References

[1] E. Bannai, On tight spherical designs, J. Combinatorial Theory, Ser. A, (to appear).
[2] R.C. Bose and D.M. Mesner, On linear associative algebras corresponding to association schemes of partially balanced designs, Ann. Math. Statist., **30** (1959), 21-38.
[3] P. Delsarte, An algebraic approach to the association schemes of coding theory, Philips Res. Rep. Suppl., **10**, 1973.
[4] P. Delsarte, J.M. Goethals and J.J. Seidel, Spherical codes and designs, Geometriae Dedicata, **6** (1977), 363-388.
[5] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, Higher Transcendental Functions (vol. 2), Bateman Manuscript Project, McGraw-Hill, 1953.

Eiichi BANNAI
Department of Mathematics
Gakushuin University
Toshima-ku, Tokyo
171 Japan

Robert Mark DAMERELL
Department of Mathematics
Royal Holloway College
Egham, Surrey
England

Present address:
Department of Mathematics
The Ohio State University
Columbus, Ohio
43210 U.S.A.