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Asymptotic behavior of some oscillatory integrals

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The purpose of the present note is to prove an asymptotic expansion theorem for a certain kind of oscillatory integrals. Our theorem is a generalization of Theorem 3.2.4 of Hörmander [1] in the sense that we allow the phase function to contain certain inhomogeneous terms. Our motivation which leads us to considering such a kind of generalization lies in long-range scattering (cf. [3], [4]) as against Hörmander's purpose in [1] was to consider Fourier integral operators. Using our main result, we can give a proof of Theorem 3.1 of [4] which played a crucial role in the proof of the completeness of the modified wave operator for long-range scattering. We should note that Hörmander's Theorem 3.2.4 also can be used to prove the completeness when long-range potential V satisfies $V(x)=O(|x|^{-1/2-\varepsilon})$, $\varepsilon>0$, but for V which has longer range, we must use our generalized theorem proved in this paper.

The crucial tool we shall use is the method of stationary phase (see e.g. Hörmander [1] and [2]). Moreover, a method similar to Hörmander's proof of Theorem 3.2.4 will be used to estimate the integral on the region bounded away from the critical point of the phase function.

We shall first summarize our main result in §1 and then prove it in §2.

§1. Main result.

We shall consider the distribution $A_{\omega,\varepsilon}$ defined by

$$(1.1)\qquad \langle A_{\omega,\varepsilon}, u\rangle = \int_{R^n}\int_{R^N} e^{i(\varphi(\omega;x,\theta)-X(x,\theta))} a(x,\theta)u(x)\chi(\varepsilon\theta)\,d\theta dx,$$

$$\varepsilon\neq 0,\quad u\in C_0^\infty(R^n),$$

where ω is a parameter; functions φ, X, a are C^∞; and χ is a rapidly decreasing function on R^N with $\chi(0)=1$. The precise conditions imposed on those functions will be given below. Under those conditions, we shall prove the existence of the limit $A_{\omega,0}\equiv\lim_{\varepsilon\to 0} A_{\omega,\varepsilon}$ (which is called an "oscillatory integral" by Hörmander [1]) and investigate the asymptotic behavior of

$$(1.2)\qquad \langle A_{\omega,\varepsilon},\ ue^{-it\psi(\omega;\cdot)-iY(t,\omega;\cdot)}\rangle$$

when $t\to\infty$ for an arbitrary $\varepsilon\in R^1$. For the case when X and Y are identically zero, an asymptotic estimate of (1.2) has been obtained by Hörmander (cf. Theorem 3.2.4 of [1]).

We first state several conditions imposed on the functions φ, X, a, ψ, Y. Those conditions will be assumed throughout the paper. Let Ω be a compact set in R^m ($m\geqq 1$) and let Γ be an open conic set contained in $R^n\times(R^N-\{0\})$, $n\geqq 0$, $N\geqq 1$, that is, assume that $(x,\theta)\in\Gamma$, $t>0$ implies $(x,t\theta)\in\Gamma$. The first conditions are concerned with φ and ψ:

($C\varphi$) 1) There exists a bounded open neighborhood Ω' of Ω in R^m such that $\varphi(\omega;x,\theta)$ is a real-valued C^∞ function on $\Omega'\times\Gamma$.

2)[1)] For any $\omega\in\Omega'$, the function $\varphi(\omega;\cdot,\cdot)$ is positively homogeneous of degree 1 with respect to θ and has no critical point in Γ, in other words,

a) $\varphi(\omega;x,t\theta)=t\varphi(\omega;x,\theta)$, $\forall t>0$, $\forall(x,\theta)\in\Gamma$,

b) $(\partial_x\varphi(\omega;x,\theta),\partial_\theta\varphi(\omega;x,\theta))\neq 0$, $\forall(x,\theta)\in\Gamma$[2)].

($C\psi$) 1) ψ is a real-valued C^∞ function on $\Omega'\times R^n$.

2) For any $\omega\in\Omega'$ and $x\in\operatorname{supp}u$, $\partial_x\psi(\omega;x)\neq 0$.

Let Π_x and Π_θ be the projection mapping from $R^n\times R^N$ onto R^n and R^N, respectively, and let real numbers ρ, δ, h_1, h_2, ε_0 and h' be fixed as

$$(1.3)\qquad \begin{gathered}1>\rho>1/2>\delta>0,\quad h_1,\ h_2\in R^1,\\ 0\leqq\varepsilon_0<\min(\rho-1/2,\ 1-\rho),\quad h'\leqq 3\rho-2+\varepsilon_0.\end{gathered}$$

The next three conditions are concerned with the functions X, Y, and a:

(CX) 1) X is a real-valued C^∞ function on $R^n\times R^N$.

2) For any compact set L of $\Pi_x(\Gamma)$ and any multi-indices α, β, there exists a positive constant C such that for any $(x,\theta)\in L\times\Pi_\theta(\Gamma)$

$$(1.4)\qquad \begin{cases}|(\partial_\theta^\alpha\partial_x^\beta X)(x,\theta)|\leqq C(1+|\theta|)^{1-|\alpha|-\delta}, & \text{when } |\alpha|+|\beta|\leqq 2,\\ |(\partial_\theta^\alpha\partial_x^\beta X)(x,\theta)|\leqq C(1+|\theta|)^{h'-\rho|\alpha|+(1-\rho)|\beta|}, & \text{when } |\alpha|+|\beta|\geqq 3.\end{cases}$$

1) The real-valued C^∞ function φ satisfying this condition is called a phase function by Hörmander [1].

2) Here and hereafter, ∂_x and ∂_θ denote $(\partial/\partial x_1,\cdots,\partial/\partial x_n)$ and $(\partial/\partial\theta_1,\cdots,\partial/\partial\theta_N)$, respectively.

(CY) 1) Y is a real-valued C^∞ function on $(0, \infty)\times\Omega'\times R^n$.

2) For any compact set L of R^n and any multi-index α, there exists a positive constant C such that

$$|(\partial_x^\alpha Y)(t, \omega; x)|\leqq C(1+t)^{1-\delta}, \quad \forall x\in L, \quad \forall\omega\in\Omega', \quad \forall t>0. \tag{1.5}$$

(Ca) 1) a is a C^∞ function on $R^n\times R^N$.

2)[3] For any compact set L of R^n and any multi-indices α, β, there exists a positive constant C such that for any $(x, \theta)\in L\times R^N$

$$\begin{cases} |(\partial_\theta^\alpha\partial_x^\beta a)(x, \theta)|\leqq C(1+|\theta|)^{h_1-|\alpha|}, & \text{when } |\alpha|+|\beta|\leqq 1, \\ |(\partial_\theta^\alpha\partial_x^\beta a)(x, \theta)|\leqq C(1+|\theta|)^{h_2-\rho|\alpha|+(1-\rho)|\beta|}, & \text{when } |\alpha|+|\beta|\geqq 2. \end{cases} \tag{1.6}$$

3) There exists a compact set K of Γ such that $a(x, \theta)=0$ for any $(x, \theta)\in R^n\times R^N-\{(x, t\theta)\mid t\geqq 1, (x, \theta)\in K\}$.

Before stating our main result, we prepare the following proposition.

PROPOSITION 1.1. *Suppose that the functions φ, ψ, X, and Y satisfy conditions ($C\varphi$), ($C\psi$), (CX), and (CY), respectively. Suppose further that there exists a compact set W of Γ satisfying the following two conditions:*

1° *For any $\omega\in\Omega'$ there exists a unique point $(x_\infty(\omega), \theta_\infty(\omega))\in W^i$[4] such that*

$$\begin{cases} \partial_\theta\varphi(\omega; x_\infty(\omega), \theta_\infty(\omega))=0, \\ \partial_x\varphi(\omega; x_\infty(\omega), \theta_\infty(\omega))=\partial_x\psi(\omega; x_\infty(\omega)). \end{cases} \tag{1.7}$$

2° *For any $\omega\in\Omega'$ and any $(x, \theta)\in W$,*

$$\det\begin{pmatrix} \partial_\theta\partial_\theta\varphi & \partial_\theta\partial_x\varphi \\ \partial_x\partial_\theta\varphi & \partial_x\partial_x\varphi-\partial_x\partial_x\psi \end{pmatrix}(\omega; x, \theta)\neq 0. \tag{1.8}$$

Then there exist a positive constant $T>1$ and a bounded open neighborhood U of Ω with its closure contained in Ω' such that the following three assertions hold:

i) *For any $t>T$, $\omega\in U$, and $(x, \theta)\in W$, the real symmetric matrix*

$$J(t, \omega; x, \theta)\equiv\begin{pmatrix} \partial_\theta\partial_\theta f & \partial_x\partial_\theta f \\ \partial_\theta\partial_x f & \partial_x\partial_x f \end{pmatrix}(t, \omega; x, \theta) \tag{1.9}$$

is non-singular, where f is defined by

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3) The class of the functions satisfying the lower estimate of (1.6) for all α and β coincides with $S^{h_2}_{\rho,1-\rho}(R^n\times R^N)$ of Hörmander (cf. Definition 1.1.1 of [1]).

4) W^i denotes the interior of the set W.

(1.10) $$f(t, \omega; x, \theta)=\varphi(\omega; x, \theta)-\phi(\omega; x)-X(x, t\theta)/t-Y(t, \omega; x)/t.$$

Moreover there exists a positive constant C such that

(1.11) $$\left|J(t, \omega; x, \theta)-\begin{pmatrix} \partial_\theta\partial_\theta\varphi & \partial_x\partial_\theta\varphi \\ \partial_\theta\partial_x\varphi & \partial_x\partial_x\varphi-\partial_x\partial_x\phi \end{pmatrix}(\omega; x, \theta)\right|<Ct^{-\delta}$$

for any $t>T$, $\omega\in U$, *and* $(x, \theta)\in W$.

ii) *There exists a uniquely determined function* $(x_c, \theta_c):(T, \infty)\times U\rightarrow W$ *such that:*

a) *for any* $t>T$ *and* $\omega\in U$,

(1.12) $$\begin{cases} \partial_x f(t, \omega; x_c(t, \omega), \theta_c(t, \omega))=0, \\ \partial_\theta f(t, \omega; x_c(t, \omega), \theta_c(t, \omega))=0, \end{cases} \quad \textit{and}$$

b) *for any* $\omega\in U$,

(1.13) $$\lim_{t\to\infty}(x_c(t, \omega), \theta_c(t, \omega))=(x_\infty(\omega), \theta_\infty(\omega)).$$

iii) *The function* (x_c, θ_c) *defined in* ii) *satisfy the following three assertions:*

a) (x_c, θ_c) *is a* C^∞ *function on* $(T, \infty)\times U$.

b) *The convergence of* (1.13) *is uniform in* $\omega\in U$.

c) *For any* $t>T$ *and* $\omega\in U$, *the matrix* $J(t, \omega; x_c(t, \omega), \theta_c(t, \omega))$ *is non-singular.*

The proof is similar to that of Proposition 2.2 of [4] hence is omitted.

Now we can state our main result.

THEOREM 1.2. *Let* ν *and* k *be integers such that* $\nu\geqq 1$, $k>((h_1\vee h_2)+N)/\delta_1$, *where* $h_1\vee h_2=\max(h_1, h_2)$, $\delta_1=\min(\rho-h', \delta)>0$. *Suppose that all assumptions of Proposition* 1.1 *and condition* (*Ca*) *are satisfied. Then:*

i) *For any* $t>0$ *and* $\omega\in\Omega$ *the following limit exists:*

(1.14) $$\lim_{\varepsilon\to 0}\langle A_{\omega,\varepsilon}, ue^{-it\phi(\omega;\cdot)-iY(t,\omega;\cdot)}\rangle.$$

This defines a distribution $A_{\omega,0}\in\mathcal{D}'(R^n)$. *Moreover, for any* $T'>0$ *there exists a positive constant* $C=C_{T'}$ *such that*

(1.15) $$|\langle A_{\omega,\varepsilon}, ue^{-it\phi(\omega;\cdot)-iY(t,\omega;\cdot)}\rangle|\leqq C$$

holds for any $\varepsilon\in R^1$, $\omega\in\Omega$ *and* $0<t<T'$.

ii) *There exist positive constants* $T>1$ *and* $C>0$ *such that for any* $t>T$, $\varepsilon\in R^1$, *and* $\omega\in\Omega$ *the following estimate holds:*

$$
\begin{aligned}
(1.16)\qquad & \Big|\langle A_{\omega,\varepsilon},\, ue^{-it\phi(\omega;\cdot)-iY(t,\omega;\cdot)}\rangle \\
& -(2\pi)^{(N+n)/2} e^{\pi i\sigma/4} t^{(N-n)/2} e^{itf(t,\omega;x_c(t,\omega),\theta_c(t,\omega))} \\
& \times|\det J|^{-1/2}\sum_{j=0}^{\nu-1}(\langle J^{-1}D,D\rangle^j u^{\varepsilon}_{t,\omega})(0)((2i)^j/j!)t^{-j}\Big| \\
& \leqq C(2^\nu\nu!)^{-1}(N+n)^2|J|^{\nu(N+n-1)}|\det J|^{-(\nu+1/2)} \\
& \quad\times t^{(N-n)/2+h_1\vee h_2+\varepsilon_0+(N+n+1)(1-\rho+\varepsilon_0)+(1-2(\rho-\varepsilon_0))\nu} \\
& \quad+Ct^{N+h_1\vee h_2-k\delta_1}.
\end{aligned}
$$

Here f *is the function defined by* (1.10); (x_c,θ_c) *is the critical point of* f *defined in Proposition* 1.1; $J=J(t,\omega;x_c(t,\omega),\theta_c(t,\omega))$; σ *denotes the signature of the real symmetric matrix* J; $|J|=(\sum_{i,j=1}^{N+n}|J_{ij}|^2)^{1/2}$ *where* J_{ij} *denotes the* (i,j) *component of* J; $D=-i\partial_y$; $u^{\varepsilon}_{t,\omega}$ *is defined by* (2.16) *of* §2 *below; and* C *depends only on* ν, k, φ, ϕ, X, Y, *and* a *(see* §2.2, *2nd, 4th and 5th steps).*

iii) *In particular there exist positive constants* $T>1$ *and* $C>0$ *such that the following two estimates hold for* $t>T$, $\varepsilon\in R^1$, *and* $\omega\in\Omega$:

$$
\begin{aligned}
(1.17)\qquad & \Big|\langle A_{\omega,\varepsilon},\, ue^{-it\phi(\omega;\cdot)-iY(t,\omega;\cdot)}\rangle \\
& -(2\pi)^{(N+n)/2} e^{\pi i\sigma/4} t^{(N-n)/2} e^{itf(t,\omega;x_c(t,\omega),\theta_c(t,\omega))} \\
& \times|\det J|^{-1/2} a(x_c(t,\omega),t\theta_c(t,\omega)) \\
& \times\sum_{j=0}^{\nu-1}(\langle J^{-1}D,D\rangle^j v^{\varepsilon}_{t,\omega})(0)t^{-j}(2i)^j/j!\Big| \\
& \leqq C(t^{(N-n)/2+h_1\vee h_2+\varepsilon_0+(N+n+1)(1-\rho+\varepsilon_0)+(1-2(\rho-\varepsilon_0))\nu} \\
& \quad+t^{(N-n)/2-1+\max(h_1,h_1+h'+2-3\rho,h_2+2-2\rho)}+t^{N+h_1\vee h_2-k\delta_1}).
\end{aligned}
$$

$$
\begin{aligned}
(1.18)\qquad & |\langle A_{\omega,\varepsilon},\, ue^{-it\phi(\omega;\cdot)-iY(t,\omega;\cdot)}\rangle \\
& -(2\pi)^{(N+n)/2} e^{\pi i\sigma/4} t^{(N-n)/2} e^{itf(t,\omega;x_c(t,\omega),\theta_c(t,\omega))} \\
& \times|\det J|^{-1/2} a(x_c(t,\omega),t\theta_c(t,\omega))u(x_c(t,\omega)) \\
& \times\chi(\varepsilon t\theta_c(t,\omega))| \\
& \leqq C(t^{(N-n)/2+h_1\vee h_2+\varepsilon_0+(N+n+1)(1-\rho+\varepsilon_0)+(1-2(\rho-\varepsilon_0))\nu} \\
& \quad+t^{(N-n)/2-1+\max(h_1,h_1+h'+2-3\rho,h_2+2-2\rho)} \\
& \quad+t^{N+h_1\vee h_2-k\delta_1}+t^{(N-n)/2+h_1\vee h_2+\min(-1/2,1-2\rho)}).
\end{aligned}
$$

Here $v^{\varepsilon}_{t,\omega}$ *is defined by* (2.16) *in* §2 *below.*

iv) *When the function a satisfies the lower estimate of* (1.6) *for all* α *and* β, *that is, when* $a\in S^{h_2}_{\rho,1-\rho}(R^n\times R^N)$[5], *then* i)$\sim$iii) *above remain valid with* $h_1\vee h_2$ *and* $\max(h_1, h_1+h'+2-3\rho, h_2+2-2\rho)$ *in the powers replaced by* h_2 *and* $h_2+2-2\rho$, *respectively.*

§2. Proof of Theorem 1.2.

2.1. PROOF OF i). Since $\varphi(\omega;\cdot,\cdot)$ has no critical point in Γ for any $\omega\in\Omega'$, we can prove the existence of a first order differential operator

$$P=\sum a_j\partial/\partial\theta_j+\sum b_j\partial/\partial x_j+c$$

satisfying ${}^tPe^{i\varphi}=e^{i\varphi}$ in a way similar to Lemma 1.2.1 of Hörmander [**1**]. Here ${}^tP$ is the adjoint of P, and $a_j(\omega;x,\theta)$, $b_j(\omega;x,\theta)$, and $c(\omega;x,\theta)$ are C^∞ functions on $\Omega'\times\Gamma$ satisfying the following estimate: For any compact set L of $\Pi_x(\Gamma)$ and any multi-indices α, β, there exists a positive constant $C_{\alpha,\beta}$ such that for any $x\in L$, $\theta\in\Pi_\theta(\Gamma)$ and $\omega\in\Omega$,

$$\text{(2.1)}\qquad \begin{cases} |\partial_\theta^\alpha\partial_x^\beta a_j(\omega;x,\theta)|\leqq C_{\alpha,\beta}(1+|\theta|)^{-|\alpha|}, \\ |\partial_\theta^\alpha\partial_x^\beta b_j(\omega;x,\theta)|\leqq C_{\alpha,\beta}(1+|\theta|)^{-1-|\alpha|}, \\ |\partial_\theta^\alpha\partial_x^\beta c(\omega;x,\theta)|\leqq C_{\alpha,\beta}(1+|\theta|)^{-1-|\alpha|}. \end{cases}$$

Therefore by integration by parts, we get for any integer k

$$\text{(2.2)}\qquad \langle A_{\omega,\varepsilon}, ue^{-it\psi-iY}\rangle$$
$$=\int_{R^N}\int_{R^n}e^{i\varphi}P^k[a(x,\theta)e^{-ig(t,\omega;x,\theta)}u(x)\chi(\varepsilon\theta)]\,dxd\theta,$$

where $g(t,\omega;x,\theta)=t\psi(\omega;x)+X(x,\theta)+Y(t,\omega;x)$. On the other hand, if we put $\tilde a_{t,\omega}(x,\theta)=a(x,\theta)e^{-ig(t,\omega;x,\theta)}$, then one can prove the following estimate by direct computation: For any compact set L of $\Pi_x(\Gamma)$ and any multi-indices α, β, there exists a positive constant $C_{\alpha,\beta}$ such that

$$\text{(2.3)}\qquad |\partial_\theta^\alpha\partial_x^\beta\tilde a_{t,\omega}(x,\theta)|\leqq C_{\alpha,\beta}(1+|\theta|)^{h_1\vee h_2-\delta_1|\alpha|+(1-\delta_1)|\beta|}$$

holds for any $x\in L$, $\theta\in\Pi_\theta(\Gamma)$, $\omega\in\Omega$ and $0<t<T'$, where $h_1\vee h_2=\max(h_1, h_2)$ and $\delta_1=\min(\rho-h',\delta)>0$. From (2.1) and (2.3), we have for any k

$$\text{(2.4)}\qquad |P^k[\tilde a_{t,\omega}(x,\theta)u(x)\chi(\varepsilon\theta)]|\leqq C_k(1+|\theta|)^{h_1\vee h_2-k\delta_1}$$

for some positive constant C_k independent of $x\in L$, $\theta\in\Pi_\theta(\Gamma)$, $0<t<T'$, and

5) cf. footnote 3).

$\omega\in\Omega$. Thus taking $k>(h_1\vee h_2+N)/\delta_1$, we get from (2.2) and (2.4)

$$|\langle A_{\omega,\varepsilon},\ ue^{-it\psi-iY}\rangle|\leqq C_{T'}$$

for some positive constant $C_{T'}$ independent of $\varepsilon\neq 0$, $\omega\in\Omega$, and $0<t<T'$. The existence of the limit (1.14) can also be proved by using (2.4).

2.2. PROOF OF ii). We divide the proof into 5 steps.

1st step: We shall divide the integral (1.2) into the sum of two integrals, that is, into the one near the critical point (x_c, θ_c) of f defined in Proposition 1.1 and the one on the remainder region. For this purpose we first make a change of variable $\theta\rightarrow t\theta$ $(t>0)$ in (1.2). Then we obtain

$$(2.5)\qquad \langle A_{\omega,\varepsilon},\ ue^{-it\psi-iY}\rangle=t^N\int_{R^n}\int_{R^N}e^{itf(t,\omega;x,\theta)}\,a(x,\ t\theta)\,u(x)\chi(\varepsilon t\theta)\,d\theta dx.$$

For $T_1>T$ and $\omega\in\Omega$ put

$$A(\omega)=\{(x,\ \theta)\mid(x_c(t,\ \omega)+x,\ \theta_c(t,\ \omega)+\theta)\in\tilde{B},\ \forall t>T_1\},$$

where $\tilde{B}$ $(\subset W)$ is a bounded open neighborhood of $B\equiv(x_\infty, \theta_\infty)(\Omega)$ [6]. Then by iii), b) of Proposition 1.1, $A(\omega)$ is a neighborhood of $(0, 0)\in R^n\times R^N$ and satisfies

$$\inf_{\omega\in\Omega}\operatorname{dist}((0,\ 0),\ A(\omega)^c)>0\ ^{7)}$$

if T_1 is sufficiently large. Thus there exists an open ball $\tilde{A}$ in $R^n\times R^N$ with center (0, 0) such that $\tilde{A}\subset A(\omega)$ for any $\omega\in\Omega$. Now define

$$(2.6)\qquad \tilde{f}(x,\ \theta\,;\tau,\ \omega)=\begin{cases} f(\tau^{-1},\ \omega\,;\ x+x_c(\tau^{-1},\ \omega),\ \theta+\theta_c(\tau^{-1},\ \omega)), & 0<\tau<T_1^{-1},\\ \varphi(\omega\,;\ x+x_\infty(\omega),\ \theta+\theta_\infty(\omega))-\psi(\omega\,;\ x+x_\infty(\omega)), & \tau=0.\end{cases}$$

To this function we shall apply the following Morse lemma.

LEMMA 2.1. *Let B be an open ball in R^k $(k\geqq 1)$ with center $0\in R^k$, Ω be a compact space, and $J=[0, R]$ $(R>0)$. Let $f(\cdot\,;\tau,\omega)$ be a real-valued C^∞ function on B for an arbitrarily fixed $(\tau, \omega)\in J\times\Omega$, and suppose that f satisfies the following conditions:*

a) $\partial_x f(0\,;\tau,\ \omega)=0,\quad \forall(\tau,\ \omega)\in J\times\Omega$;

b) *The matrix $A(\tau, \omega)\equiv\partial_x\partial_x f(0\,;\tau, \omega)$ is non-singular for $\forall(\tau, \omega)\in J\times\Omega$; and*

6) $(x_\infty, \theta_\infty)(\Omega)$ denotes the image of Ω by the mapping $(x_\infty, \theta_\infty)$. Notice that $(x_\infty, \theta_\infty)(\Omega)$ is compact because Ω is compact and the mapping $(x_\infty, \theta_\infty)$ is continuous by iii), b) of Proposition 1.1.

7) $\operatorname{dist}(A, B)$ denotes the distance between the sets A and B, and A^c denotes the complement of the set A.

c) $\partial_x \partial_x f(x;\tau,\omega)$ *is continuous in* $(x,\tau,\omega)\in B\times J\times\Omega$.

Suppose furthermore that the following estimate holds for some convex sequence $\{a(k)\}_{k=1}^{\infty}$ *of real numbers satisfying* $a(1)=a(2)=0$: *For any multi-index* α, *there exists a positive constant* C_α *such that*

$$|\partial_x^\alpha f(x;\tau,\omega)|\leqq C_\alpha \tau^{-a(|\alpha|)} \tag{2.7}$$

holds for any $(x,\tau,\omega)\in B\times J^i\times\Omega$.

Then we can find an open ball $B'\subset B$ *in* R^k *with center* 0 *such that for any* $(\tau,\omega)\in J\times\Omega$ *there exist an open neighborhood* $V_{\tau,\omega}$ *of* 0 *in* R^k *and a* C^∞ *diffeomorphism* $\varphi_{\tau,\omega}: V_{\tau,\omega}\to B'$ *satisfying the following properties:*

i) $\forall(\tau,\omega)\in J\times\Omega,\quad \varphi_{\tau,\omega}(0)=0$;

ii) $\forall(\tau,\omega)\in J\times\Omega,\quad \forall y\in V_{\tau,\omega}$,

$$f(\varphi_{\tau,\omega}(y);\tau,\omega)=f(0;\tau,\omega)+\langle A(\tau,\omega)y,y\rangle/2; \tag{2.8}$$

iii) $\forall(\tau,\omega)\in J\times\Omega$,

$$\begin{cases} \operatorname{sgn}(\det\partial_y\varphi_{\tau,\omega}(y))=constant\ for\quad y\in V_{\tau,\omega}, \\ |\det\partial_y\varphi_{\tau,\omega}(0)|=1; \end{cases} \tag{2.9}$$

iv) $\forall\alpha,\quad \exists C_\alpha>0,\quad \forall(\tau,\omega)\in J^i\times\Omega,\quad \forall y\in V_{\tau,\omega}$,

$$\begin{cases} |\partial_y^\alpha\varphi_{\tau,\omega}(y)|\leqq C_\alpha\tau^{-a(|\alpha|+2)}, \\ |\partial_y^\alpha\varphi_{\tau,\omega}(0)|\leqq C_\alpha\tau^{-a(|\alpha|+1)};\ and \end{cases} \tag{2.10}$$

v)

$$\begin{cases} \inf_{(\tau,\omega)\in J\times\Omega}\operatorname{dist}(0,V_{\tau,\omega}{}^c)>0, \\ \sup_{(\tau,\omega)\in J\times\Omega}\operatorname{diam}V_{\tau,\omega}<\infty^{8)}. \end{cases} \tag{2.11}$$

The proof of this lemma is similar to that of Lemma A. 6 of Hörmander [2]. But our lemma is somewhat different from Hörmander's in the sense that our estimate (2. 10) includes the one on a neighborhood of the origin, while Hörmander's estimate (A. 10) is only concerned with the one restricted to the origin. Consequently we must somewhat modify the proof of Hörmander. However the proof does not change essentially and is easily reconstructed from that of Lemma A. 6 of [2] hence is omitted.

If we put $k=n+N$, $B=\tilde{A}$, $\Omega=\Omega$, $R=T_1^{-1}$, $f=\tilde{f}$, and $a(k)=\max(0,(1-\rho)k+h'-1)$ in this lemma, then all assumptions of the lemma are satisfied. Thus the assertions i)$\sim$v) of the lemma hold with

8) diam A denotes the diameter of the set A.

$$A(\tau, \omega)=\begin{cases} J(\tau^{-1}, \omega ; x_c(\tau^{-1}, \omega), \theta_c(\tau^{-1}, \omega)) & \text{for} \quad 0<\tau<T_1^{-1}, \\ \begin{pmatrix} \partial_\theta \partial_\theta \varphi & \partial_x \partial_\theta \varphi \\ \partial_\theta \partial_x \varphi & \partial_x \partial_x \varphi - \partial_x \partial_x \psi \end{pmatrix} (\omega ; x_\infty(\omega), \theta_\infty(\omega)) & \text{for} \quad \tau=0. \end{cases}$$

Let $r=\operatorname{diam} B'/4$ and choose $\tilde{\chi} \in C_0^\infty(R^n \times R^N)$ so that $\operatorname{supp} \tilde{\chi} \subset B'$ and $\tilde{\chi}(x, \theta)=1$ for $|(x, \theta)| \leqq r$. Now introducing the partition of unity $1=\tilde{\chi}+(1-\tilde{\chi})$, we write (2.5) in the form

$$\langle A_{\omega, \varepsilon}, u e^{-it\psi - iY} \rangle = I_1(\varepsilon, \omega ; t) + I_2(\varepsilon, \omega ; t), \tag{2.12}$$

where

$$\begin{aligned} I_1(\varepsilon, \omega ; t) & \\ = t^N \int_{R^n} \int_{R^N} & e^{itf(t, \omega ; x, \theta)} a(x, t\theta) u(x) \chi(\varepsilon t\theta) \\ & \times \tilde{\chi}(x - x_c(t, \omega), \theta - \theta_c(t, \omega)) \, d\theta dx, \end{aligned} \tag{2.13}$$

$$\begin{aligned} I_2(\varepsilon, \omega ; t) & \\ = t^N \int_{R^n} \int_{R^N} & e^{itf(t, \omega ; x, \theta)} a(x, t\theta) u(x) \chi(\varepsilon t\theta) \\ & \times (1 - \tilde{\chi}(x - x_c(t, \omega), \theta - \theta_c(t, \omega))) d\theta dx. \end{aligned} \tag{2.14}$$

2nd step: We shall estimate $I_1(\varepsilon, \omega ; t)$. Making a change of variable $x \to x + x_c(t, \omega)$, $\theta \to \theta_c(t, \omega)$ in (2.13), recalling the definition of $\tilde{f}$, and using the result of Lemma 2.1, we can rewrite I_1 as follows:

$$\begin{aligned} I_1(\varepsilon, \omega ; t) & \\ = t^N e^{itf(t, \omega ; x_c(t, \omega), \theta_c(t, \omega))} & \int_{R^{n+N}} e^{it\langle A(t^{-1}, \omega) y, y \rangle / 2} u_{t, \omega}^\varepsilon (y) \, dy, \end{aligned} \tag{2.15}$$

where

$$\begin{cases} u_{t, \omega}^\varepsilon(y) = a(x + x_c(t, \omega), t(\theta + \theta_c(t, \omega)))|_{(x, \theta) = \varphi_{t^{-1}, \omega}(y)} v_{t, \omega}^\varepsilon(y), \\ v_{t, \omega}^\varepsilon(y) = w_{t, \omega}^\varepsilon(x + x_c(t, \omega), t(\theta + \theta_c(t, \omega))) |_{(x, \theta) = \varphi_{t^{-1}, \omega}(y)} |\det \partial_y \varphi_{t^{-1}, \omega}(y)|, \\ w_{t, \omega}^\varepsilon(x, \theta) = u(x) \chi(\varepsilon \theta) \tilde{\chi}(x - x_c(t, \omega), \theta / t - \theta_c(t, \omega)). \end{cases} \tag{2.16}$$

Therefore by Lemma A.2 of Hörmander [2], we obtain

$$(2.17)\quad \Big| I_1(\varepsilon, \omega; t) - (2\pi)^{(N+n)/2} e^{\pi i\sigma/4} t^{(N-n)/2} e^{itf(t,\omega; x_c(t,\omega), \theta_c(t,\omega))} \times |\det J|^{-1/2} \sum_{j=0}^{\nu-1} (\langle J^{-1}D, D\rangle^j u^{\varepsilon}_{t,\omega})(0)((2i)^j/j!)t^{-j} \Big|$$
$$\leqq (2^\nu \nu!)^{-1}(N+n)^2 |J|^{\nu(N+n-1)} |\det J|^{-(\nu+1/2)} t^{(N-n)/2-\nu} \times \sum_{|\alpha|\leq 2\nu+N+n+1} \int_{R^{n+N}} |\partial_y^\alpha u^{\varepsilon}_{t,\omega}(y)| dy,$$

where σ and J are the same as in Theorem 1.2. On the other hand, direct computation gives for any α

$$|\partial_y^\alpha u^{\varepsilon}_{t,\omega}(y)| \leqq C_\alpha t^{h_1 \vee h_2 + \varepsilon_0 + (1-(\rho-\varepsilon_0))|\alpha|}$$

for some constant $C_\alpha > 0$ independent of $y \in R^{n+N}$, $\varepsilon \neq 0$, $\omega \in \Omega$, and $t > T_1$, where we have used $h' \leqq 3\rho - 2 + \varepsilon_0$[9]. Therefore the left-hand side of (2.17) is bounded by a constant times

$$(2.18)\quad (2^\nu \nu!)^{-1}(N+n)^2 |J|^{\nu(N+n-1)} |\det J|^{-(\nu+1/2)} \times t^{(N-n)/2-\nu+h_1\vee h_2+\varepsilon_0+(2\nu+N+n+1)(1-(\rho-\varepsilon_0))}.$$

3rd step: We shall next consider I_2. By (1.13) and iii), b) of Proposition 1.1 we can find a positive constant $T_2 > T_1$ such that for any $t > T_2$ and $\omega \in \Omega$,

$$|(x_c(t, \omega), \theta_c(t, \omega)) - (x_\infty(\omega), \theta_\infty(\omega))| < r/2.$$

Therefore if we put

$$E_{t,\omega} = \operatorname{supp}\{(1-\tilde{\chi}(x-x_c(t,\omega), \theta-\theta_c(t,\omega)))\, a(x, t\theta)\, u(x)\},$$

then we obtain

$$|(x, \theta) - (x_\infty(\omega), \theta_\infty(\omega))| > r/2$$

for any $\omega \in \Omega$, $t > T_2$ and $(x, \theta) \in E_{t,\omega}$. Then in the same way as in the proof of Theorem 3.2.4 of [1] (cf. page 152 of [1]), we can prove the existence of a positive number κ such that for any $t > T_2$, $\omega \in \Omega$, and $(x, \theta) \in E_{t,\omega}$ either

$$|\partial_\theta \varphi(\omega; x, \theta)| > \kappa \quad \text{or} \quad |\partial_x \varphi(\omega; x, \theta) - \partial_x \phi(\omega; x)| \geqq \kappa(1+|\theta|)$$

holds. Now we can choose $\hat{\chi}_\omega \in C^\infty(R^n \times R^N)$ so that any derivative of $\hat{\chi}_\omega$ is continuous in $\omega \in \Omega$; $\hat{\chi}_\omega$ does not depend on θ if $|\theta| > 1$; and $\hat{\chi}_\omega(x, \theta) = 1$ when

9) In all the other parts of this paper, we only need assume $h' \leqq 2\rho - 1$ essentially.

$|\partial_\theta\varphi(\omega;x,\theta)|<\kappa/2$ and $=0$ when $|\partial_\theta\varphi(\omega;x,\theta)|>\kappa$. Using this $\hat{\chi}_\omega$ we divide I_2 in the form

$$I_2(\varepsilon,\omega;t)=J_1(\varepsilon,\omega;t)+J_2(\varepsilon,\omega;t), \tag{2.19}$$

where

$$\begin{aligned} &J_1(\varepsilon,\omega;t)\\ &=t^N\int_{R^n}\int_{R^N}e^{it(\varphi(\omega;x,\theta)-\psi(\omega;x))}\{a(x,t\theta)e^{-iX(x,t\theta)-iY(t,\omega;x)}\\ &\qquad\times u(x)(1-\tilde{\chi}(x-x_c,\theta-\theta_c))\hat{\chi}_\omega(x,\theta)\}\chi(\varepsilon t\theta)\,d\theta dx,\end{aligned}$$

and

$$\begin{aligned} &J_2(\varepsilon,\omega;t)\\ &=t^N\int_{R^n}\int_{R^N}e^{it\varphi(\omega;x,\theta)}\{a(x,t\theta)e^{-iX(x,t\theta)}(1-\tilde{\chi}(x-x_c,\theta-\theta_c))\\ &\qquad\times(1-\hat{\chi}_\omega(x,\theta))\}\chi(\varepsilon t\theta)\,\varepsilon^{-it\psi(\omega;x)-iY(t,\omega;x)}u(x)\,d\theta dx.\end{aligned}$$

4th step: We shall first estimate J_1. As in page 152 of [1], we put

$$P=-\sum_{j=1}^n a_j\partial/\partial x_j+a_0,$$

$$\begin{cases} a_j=-i\partial(\varphi-\psi)/\partial x_j\Big(\sum_{k=1}^n(\partial(\varphi-\psi)/\partial x_k)^2\Big)^{-1},\\ a_0=-\sum_{j=1}^n\partial a_j/\partial x_j.\end{cases}$$

Then we have ${}^tPe^{it(\varphi-\psi)}=te^{it(\varphi-\psi)}$ and for any α

$$|\partial_x^\alpha a_j(\omega;x,\theta)|\leqq C_\alpha(1+|\theta|)^{-1},\quad j=0,1,\cdots,n, \tag{2.20}$$

where $C_\alpha>0$ is a constant independent of $\omega\in\Omega$, $(x,\theta)\in E_{t,\omega}$. Therefore by integration by parts we obtain for any k

$$\begin{aligned} &J_1(\varepsilon,\omega;t)\\ &=t^N\int_{R^n}\int_{R^N}e^{it(\varphi(\omega;x,\theta)-\psi(\omega;x))}t^{-k}P^k\{a(x,t\theta)\\ &\qquad\times e^{-iX(x,t\theta)-iY(t,\omega;x)}u(x)(1-\tilde{\chi}(x-x_c,\theta-\theta_c))\hat{\chi}_\omega(x,\theta)\}\chi(\varepsilon t\theta)\,d\theta dx.\end{aligned}$$

The integrand of this integral can be estimated by using (2.20), (Ca), (CX), and (CY) as follows.

$$|\text{integrand}| \leqq C_k t^{h_1 \vee h_2 - k\delta_1}(1+|\theta|)^{h_1 \vee h_2 - k\delta_1}$$

for any k where constant $C_k>0$ is independent of $\varepsilon \neq 0$, $t>T_2$, $\omega \in \Omega$, and $(x, \theta) \in E_{t,\omega}$. Thus taking $k>(N+h_1 \vee h_2)/\delta_1$, we get

$$|J_1(\varepsilon, \omega; t)| \leqq C t^{N+h_1 \vee h_2 - k\delta_1}. \tag{2.21}$$

5th step: Finally we shall estimate J_2. Putting

$$\begin{cases} A(\varepsilon, t, \omega; x) = \int_{R^N} e^{it\varphi(\omega; x, \theta)} b_{t,\omega}(x, \theta) \chi(\varepsilon t\theta)\, d\theta, \\ b_{t,\omega}(x, \theta) = a(x, t\theta) e^{-iX(x, t\theta)} (1-\tilde{\chi}(x-x_c, \theta-\theta_c))(1-\hat{\chi}_\omega(x, \theta)), \end{cases}$$

we can rewrite J_2 in the form

$$J_2(\varepsilon, \omega; t) = t^N \int_{R^N} A(\varepsilon, t, \omega; x) u(x) e^{-it\phi(\omega; x) - iY(t, \omega; x)}\, dx.$$

Define P by

$$P = \sum_{j=1}^{N} a_j \partial/\partial\theta_j + c,$$

$$\begin{cases} a_j = i(1-\tilde{\tilde{\chi}}) \partial\varphi/\partial\theta_j \left(\sum_{j=1}^{N} (\partial\varphi/\partial\theta_j)^2 \right)^{-1}, \\ c = \tilde{\tilde{\chi}} + \sum_{j=1}^{N} \partial a_j/\partial\theta_j, \end{cases}$$

where $\tilde{\tilde{\chi}} \in C_0^\infty(R^N)$ is taken as $\tilde{\tilde{\chi}}(\theta)=1$ near 0. Then P satisfies ${}^tP e^{it\varphi} = t e^{it\varphi}$ hence we have for any k

$$A(\varepsilon, t, \omega; x) = t^{-k} \int_{R^N} e^{it\varphi} P^k \{b_{t,\omega}(x, \theta) \chi(\varepsilon t\theta)\}\, d\theta. \tag{2.22}$$

On the other hand, a_j and c satisfy the following estimate: For any α we can find a constant $C_\alpha>0$ such that for any $t>T_2$, $\omega \in \Omega$ and $(x, \theta) \in E_{t,\omega}$,

$$\begin{cases} |\partial_\theta^\alpha a_j(\omega; x, \theta)| \leqq C_\alpha (1+|\theta|)^{-|\alpha|}, \\ |\partial_\theta^\alpha c(\omega; x, \theta)| \leqq C_\alpha (1+|\theta|)^{-1-|\alpha|}. \end{cases} \tag{2.23}$$

Furthermore direct computation gives for any α

$$\begin{aligned} &|\partial_\theta^\alpha b_{t,\omega}(x, \theta)| \\ &\quad \leqq C t^{h_1 \vee h_2 + (1-\delta_1)|\alpha|} (1+|\theta|)^{h_1 \vee h_2 - \delta_1 |\alpha|} \\ &\quad\quad + C \sum_{\substack{\alpha'+\alpha''=\alpha \\ |\alpha''| \geq 1}} t^{h_1 \vee h_2 + (1-\delta_1)|\alpha'|} (1+|\theta|)^{h_1 \vee h_2 - \delta_1 |\alpha'|} \Psi(\theta), \end{aligned} \tag{2.24}$$

where $\Psi(\theta)=1$ when $|\theta|\leqq A$ and $=0$ when $|\theta|>A$ for some constant $A>0$, and $C>0$ is a constant independent of $t>T_2$, $\omega\in\Omega$, $(x, \theta)\in E_{t,\omega}$. Therefore by (2.23) and (2.24), the integrand of (2.22) is bounded by a constant times

$$t^{h_1\vee h_2+k(1-\delta_1)}(1+|\theta|)^{h_1\vee h_2-k\delta_1}.$$

Thus taking $k>(h_1\vee h_2+N)/\delta_1$, we get

$$|J_2(\varepsilon, \omega; t)|\leqq Ct^{N+h_1\vee h_2-k\delta_1}. \tag{2.25}$$

Therefore we have by (2.19), (2.21), and (2.25)

$$|I_2(\varepsilon, \omega; t)|\leqq Ct^{N+h_1\vee h_2-k\delta_1},$$

where $C>0$ is independent of $t>T_2$, $\varepsilon\neq 0$, and $\omega\in\Omega$. Combining this with (2.18) proves ii) of Theorem 1.2.

2.3. PROOF OF iii). By virtue of ii), (1.17) follows from the following estimate: For any integer $j\geqq 1$, $t>T$, $\varepsilon\in R^1$, and $\omega\in\Omega$,

$$\Big|t^{-j}\sum_{\substack{|\beta|+|\gamma|=2j\\ |\beta|\geqq 1}}[\partial_y^\beta\{a(x+x_c, t(\theta+\theta_c))|_{(x,\theta)=\varphi_{t^{-1},\omega}(y)}\}]_{y=0}(\partial_y^\gamma v_{t,\omega}^\varepsilon)(0)\Big| \tag{2.26}$$
$$\leqq C_j t^{-1+\max(h_1, h_1+h'+2-3\rho, h_2+2-2\rho)},$$

where C_j is a positive constant independent of t, ε, and ω. But this can be proved by direct calculation using the estimate (2.10) of Lemma 2.1 and the assumption (Ca).

Thus to prove (1.18) it now suffices to note the next estimate which can also be proved by using (2.10):

$$\Big|\sum_{j=1}^{\nu-1}(\langle J^{-1}D, D\rangle^j v_{t,\omega}^\varepsilon)(0)t^{-j}(2i)^j/j!\Big|\leqq C_\nu t^{\min(-1/2, 1-2\rho)}, \quad \nu\geqq 2, \tag{2.27}$$

where constant C_ν is independent of t, ε, and ω.

2.4. PROOF OF iv). We have only to check that all the estimates stated in §§2.1~2.2 hold with $h_1\vee h_2$ replaced by h_2, and that (2.26) holds with the right-hand side replaced by $C_j t^{h_2+1-2\rho}$. But this has been essentially done already. This completes the proof of Theorem 1.2.

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