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Zero-divisors of character rings of finite groups

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Introduction.

In [9] Roquette gives a decomposition of 1 in $R_\lambda(G)$ into a sum of primitive idempotents, where G is a finite group and $R_\lambda(G)$ denotes the character ring of G with coefficients in $\mathfrak{p}$-adic integers λ. On the other hand, Serre [10] has shown that the prime spectrum $\mathrm{Spec}(R(G))$ of the character ring $R(G)$ is connected with respect to the Zariski topology, that is, $R(G)$ has no non-trivial idempotent.

This paper aims at extending these results to the case where the coefficient ring λ is a Dedekind domain in the complex number field. It is shown in the section 3 that if every non-zero prime ideal contains a prime number, then it is necessary and sufficient for $\mathrm{Spec}(R_\lambda(G))$ to be connected that no prime divisor of the order of G is a unit in λ (Corollary 1 to Proposition 6). From this result a characterization of finite p-groups is given in Theorem 3. In particular, G is a p-group if and only if $\mathrm{Spec}(R_\lambda(G))$ is connected when λ is a discrete valuation ring in which p is a non-unit.

The main step in the proofs of these results is to find special zero-divisors of $R_\lambda(G)$ (Theorem 2), and this is done by using the ideas of [9] and [11]. These zero-divisors are also used to prove a converse of some result due to Atiyah [1].

The above results contain the corresponding results for a finite abelian group ring, since in this case the group ring is isomorphic to the character ring. Swan [13, Corollary 8.1] has shown that the group ring $\lambda[G]$ has no non-trivial idempotent if λ is a Dedekind domain of characteristic 0 and no prime divisor of the order of G is a unit in λ. If G is abelian, then it follows from Proposition 5 that G is a p-group if $R(G)$ is Hausdorff with respect to the augmentation topology (see §3.1). This is a special case of Sinha [12, Corollary].

The section 1 of this paper deals with the prime ideals of $R_\lambda(G)$ for an arbitrary ring λ contained in the complex number field. As an analogue of [6, §2, h) and i)], Proposition 1 gives a necessary and sufficient condition for $R_\lambda(G)$ to be a local ring. Moreover some zero-divisors of $R_\lambda(G)$ are constructed and applied to an isomorphism problem of character rings (Theorem 1 and Proposition 3). The section 2 contains the proofs of Theorem 2 and Brauer's theorem on induced characters.

1. Prime ideals of $R_\lambda(G)$.

Let G be a finite group of order $|G|$, and let $R(G)$ be its character ring (for character rings we refer to [**1**, §6] and [**10**, §§9-11]). For a subring λ (with identity) of the complex number field $\boldsymbol{C}$ we define the ring

$$R_\lambda(G)=\lambda\otimes_{\boldsymbol{Z}} R(G),$$

where $\boldsymbol{Z}$ denotes the ring of rational integers. This is a commutative λ-algebra and its identity is the principal character 1_G of G. The elements of $R_\lambda(G)$ are λ-linear combinations of the complex irreducible characters χ of G, and the χ's form a free basis of $R_\lambda(G)$ as a λ-module.

We denote by A the subring of $\boldsymbol{C}$ generated by all $|G|$-th roots of unity over λ. Then $R_\lambda(G)$ is regarded as a subring of the ring A^G of all A-valued functions on G. The ring λ can be embedded in $R_\lambda(G)$ by $\lambda\cdot 1_G$. Thus we have inclusions

$$\lambda\subseteqq R_\lambda(G)\subseteqq A^G.$$

We note that A^G is integral over $R_\lambda(G)$, since A is integral over λ. Therefore every prime ideal of $R_\lambda(G)$ is the contraction of some prime ideal of A^G [**2**, Theorem 5.10]. In other words, it is of the form

$$P_{\mathfrak{p},x}=\{f\in R_\lambda(G)\,|\,f(x)\in\mathfrak{p}\}$$

for some $x\in G$ and some prime ideal $\mathfrak{p}$ of A. In particular, the minimal prime ideals $P_{0,x}$ are obtained by putting $\mathfrak{p}=0$. Since $f(e)\in\lambda$ for $f\in R_\lambda(G)$, we have $P_{\mathfrak{p},e}=P_{\mathfrak{m},e}$ where $\mathfrak{m}=\mathfrak{p}\cap\lambda$ and e denotes the identity of G.

For a prime number p every element x of G is uniquely expressed as $x=x_p\cdot y$, where x_p and y commute, the order of x_p is prime to p, and the order of y is a power of p. We call x_p the p-regular factor of x.

LEMMA 1. *If $p\in\mathfrak{p}$, then for any $f\in R_\lambda(G)$*

$$f(x)\equiv f(x_p)\qquad\mathrm{mod}\ \mathfrak{p}.$$

PROOF. See the proof of [**1**, Lemma (6.3)].

If G is a p-group and λ is a (Noetherian) local ring with maximal ideal $\mathfrak{m}$ such that the residue field $\lambda/\mathfrak{m}$ has characteristic p, then it follows from Lemma 1 that $R_\lambda(G)$ is a (Noetherian) local ring (cf. [8, §2]). We can also prove the converse.

PROPOSITION 1. *Suppose* $G\neq\{e\}$. *If* $R_\lambda(G)$ *is a (Noetherian) local ring, then* G *is a* p*-group and* λ *is a (Noetherian) local ring whose residue field has characteristic* p.

PROOF. Using an augmentation $\varepsilon: R_\lambda(G)\rightarrow\lambda$ defined by $\varepsilon(f)=f(e)$, we observe that λ is a (Noetherian) local ring. Let $\mathfrak{m}$ be the maximal ideal of λ. Then $P_{\mathfrak{m},e}=\varepsilon^{-1}(\mathfrak{m})$ is a unique maximal ideal of $R_\lambda(G)$. From the assumption $G\neq\{e\}$ the character r_G afforded by the regular representation of G is a non-unit in $R_\lambda(G)$, hence $r_G\in P_{\mathfrak{m},e}$. Since $r_G(e)=|G|$, it follows that $\mathfrak{m}\cap\boldsymbol{Z}=p\boldsymbol{Z}$ for some prime number p dividing $|G|$.

To prove that G is a p-group, let P be a Sylow p-subgroup of G. If f is the character of G induced from the principal character 1_P of P, then $f\notin P_{\mathfrak{m},e}$, hence f is a unit in $R_\lambda(G)$ and therefore $f(x)\neq 0$ for all $x\in G$. Consequently every element of G has for its order a power of p.

Now let K and L denote the quotient fields of λ and A, respectively. Then L is a finite normal extension of K, and each automorphism σ of the Galois group $Gal(L/K)$ is given by

$$\sigma(w)=w^t$$

for all $|G|$-th roots w of unity, where t is an integer prime to $|G|$. We denote by Γ_K the image of the homomorphism from $Gal(L/K)$ into the group of units of $\boldsymbol{Z}/|G|\boldsymbol{Z}$, and by σ_t the automorphism of $Gal(L/K)$ corresponding to t mod$|G|$ in Γ_K. For simplicity we shall write t instead of t mod$|G|$.

Two elements x, y of G are said to be K-conjugate (notation: $x\underset{K}{\sim}y$) if x^t, y are conjugate in G for some $t\in\Gamma_K$. By a K-class function we mean a function f on G such that $f(x)=f(y)$ if $x\underset{K}{\sim}y$.

LEMMA 2. *Every function* $f\in R_\lambda(G)$ *satisfies the equations*

$$\sigma_t(f(x))=f(x^t),$$

where $x\in G$ *and* $t\in\Gamma_K$. *If* λ *is integrally closed, then* $R_\lambda(G)$ *contains all* λ*-valued* K*-class functions of* $R_A(G)$ $(=A\underset{\lambda}{\otimes}R_\lambda(G))$.

PROOF. See [**10**, Theorem 26].

We shall frequently use the following orthogonality relations:

$$\sum_{\chi}\chi(x^{-1})\chi(y)=\begin{cases}|Z(x)| & \text{if } x \text{ and } y \text{ are conjugate,}\\ 0 & \text{otherwise,}\end{cases}$$

where $Z(x)$ is the centralizer of x in G.

THEOREM 1. *If* λ *is integrally closed, then for each* $x\in G$ *there exists a function* ξ_x *of* $R_\lambda(G)$ *such that* $\xi_x(y)\neq 0$ *if* $x\underset{K}{\sim}y$; *otherwise* $\xi_x(y)=0$.

PROOF. For each χ, define $a_\chi \in A$ by

$$a_\chi = \sum_{t \in \Gamma_K} \chi(x^{-t}).$$

Then Lemma 2 implies that $\sigma_t(a_\chi) = a_\chi$ for all $t \in \Gamma_K$, hence $a_\chi \in A \cap K = \lambda$, since λ is integrally closed. If we set

$$\xi_x = \sum_\chi a_\chi \chi,$$

then $\xi_x \in R_\lambda(G)$, and the result follows from the above orthogonality relations.

It is clear that $\xi_x \notin P_{0,x}$, but $\xi_x \in P_{0,y}$ if x, y are not K-conjugate. Therefore $P_{0,x} \neq P_{0,y}$. On the other hand, if $x \underset{K}{\sim} y$, then $P_{0,x} = P_{0,y}$ by Lemma 2. Thus we obtain a generalization of [**7**, Theorem 1] as follows:

PROPOSITION 2. *If λ is integrally closed, then the number of the minimal prime ideals of $R_\lambda(G)$ is equal to the number of the K-conjugate classes of G.*

As another application of Theorem 1 we consider an isomorphism problem of character rings. Let $\alpha: G \to G'$ be a homomorphism of groups. Then we have a canonical ring-homomorphism $\alpha^*: R_\lambda(G') \to R_\lambda(G)$ defined by $\alpha^*(f) = f \circ \alpha$. It is easy to see that α^* is injective if α is surjective.

PROPOSITION 3. *If α^* is an isomorphism, then so is α.*

PROOF. Choose elements $f_\chi \in R_\lambda(G')$ so that $\alpha^*(f_\chi) = \chi$. If $x \in \operatorname{Ker} \alpha$, then $\chi(x) = \chi(e)$ for all χ, hence $x = e$ by the orthogonality relations.

Suppose α is not surjective. Then there exists $y \in G'$ such that $y \notin t^{-1}\alpha(G)t$ for all $t \in G'$. Let ξ_y be the function as in Theorem 1. Then $\xi_y(\alpha(x)) = 0$ for $x \in G$, hence $\alpha^*(\xi_y) = 0$. Since α^* is injective, we have $\xi_y = 0$, which is contrary to $\xi_y(y) \neq 0$.

2. The main theorem.

We shall prove our main theorem. Suppose λ is Noetherian and of (Krull) dimension $\leqq 1$. Let $\mathfrak{m}$ be a maximal ideal of λ, and let $B = S^{-1}A$ where $S = \lambda - \mathfrak{m}$. Then B is a Noetherian semi-local domain of dimension $\leqq 1$ (cf. [**4**, Chap. 4, §2, Corollary 3 to Proposition 9]). We denote by $\mathfrak{a}$ the set J^G of all functions on G which take their values in the Jacobson radical J of B. Obviously $\mathfrak{a}$ is an ideal of B^G. We note that every non-zero ideal of B contains a power of J (cf. [**2**, Proposition 9. 1]).

LEMMA 3. *$R_B(G)$ is a closed (and open) subset of B^G with respect to the $\mathfrak{a}$-adic topology.*

PROOF. It suffices to show that $\mathfrak{a}^k \subseteqq R_B(G)$ for some k. Let $T = B - \{0\}$. Then $T^{-1}B$ is the quotient field L of A. The orthogonality relations yield

$T^{-1}R_B(G)=L^G$. Since $L^G=T^{-1}(B^G)$ and B^G is a finitely generated B-module, it follows that $t(B^G)(=(tB)^G)\subseteqq R_B(G)$ for some $t\in T$. If we choose k so that $J^k\subseteqq tB$, then we have $\mathfrak{a}^k\subseteqq R_B(G)$.

THEOREM 2. *Let λ be a Dedekind domain, and $\mathfrak{m}$ a maximal ideal of λ containing p. Then for any p-regular element $a\in G$ there exists a λ-valued K-class function $\phi_a\in R_\lambda(G)$ such that $\phi_a(a)\notin\mathfrak{m}$ and $\phi_a(x)=0$ if x_p is not K-conjugate to a.*

PROOF. Let $b\in G$ be K-conjugate to a. It follows from [**5**, Lemma (40.7)] that there exists a λ-valued class function $\eta\in R_B(G)$ such that $\eta(b)=1$ and $\eta(x)=0$ if x_p is not conjugate to b. (Note that η lies in $\operatorname{Ind} R_B(H)$, an ideal of $R_B(G)$ induced from the ring $R_B(H)$ of a p-elementary subgroup H of G. This fact is used in the proof of Lemma 4.)

Now let $\mathfrak{p}'$ be a maximal ideal of B, then $\mathfrak{m}\subseteqq\mathfrak{p}'$. If x_p is conjugate to b, then it follows from Lemma 1 that $\eta(x)\equiv 1 \mod \mathfrak{p}'$. Therefore we have $\eta(x)\equiv 1 \mod J$. Noting $p\in J$, one can show by induction on n that

$$\eta(x)^{p^n}\equiv 1 \mod J^{n+1}$$

for all n.

Define a sequence $\{\alpha_n\}$ of $R_B(G)$ by $\alpha_n=\eta^{p^n}$. Then for $n\geqq k$ we have, taking congruences modulo J^k,

$$\alpha_n(x)\equiv\begin{cases}1 & \text{if } x_p \text{ is conjugate to } b,\\ 0 & \text{otherwise.}\end{cases}$$

Let θ_b be a function of B^G such that $\theta_b(x)=1$ if x_p is a conjugate of b; otherwise $\theta_b(x)=0$. Then θ_b is a limit of $\{\alpha_n\}$ in B^G with respect to the $\mathfrak{a}$-adic topology, since $\theta_b-\alpha_n\in\mathfrak{a}^k$ for $n\geqq k$. Therefore Lemma 3 implies $\theta_b\in R_B(G)$.

We denote by C_a a full set of representatives of the conjugate classes in all elements of G which are K-conjugates of a, and define a function ϕ' of $R_B(G)$ by

$$\phi'=\sum_{b\in C_a}\theta_b.$$

Then $\phi'(x)=1$ if $x_p\underset{K}{\sim}a$; otherwise $\phi'(x)=0$. Choose $s\in S$ so that $s\phi'\in R_A(G)$, and set $\phi_a=s\phi'$. From Lemma 2 it follows that $\phi_a\in R_\lambda(G)$. Clearly ϕ_a has the properties asserted in this theorem.

We now give a proof of Brauer's theorem on induced characters. Let C_p denote a full set of representatives of the K-conjugate classes in the p-regular elements of G, and let

$$\Phi=\bigcup_p\{\phi_a | a\in C_p\},$$

where ϕ_a is the function for a as in Theorem 2.

PROPOSITION 4. *If every maximal ideal of λ contains a prime number, then Φ is contained in no proper ideals of $R_\lambda(G)$.*

PROOF. It is sufficient to prove that Φ is not contained in any maximal ideal $P_{\mathfrak{p},x}$. By assumption $\mathfrak{p}$ contains a prime number p. If x_p is K-conjugate to $a\in C_p$, then by Lemma 1 we have $\phi_a(x)\notin\mathfrak{p}$ showing $\phi_a\notin P_{\mathfrak{p},x}$.

Let us denote by X_p the set of all p-elementary subgroups of G, and define an ideal V_p of $R_\lambda(G)$ by

$$V_p=\sum_{H\in X_p}\operatorname{Ind} R_\lambda(H),$$

where Ind is a λ-homomorphism $R_\lambda(H)\to R_\lambda(G)$ defined by means of induced characters.

LEMMA 4 (Shiratani [**11**]). *If λ is a principal ideal domain in which p is a non-unit, then $\phi_a\in V_p$ for all $a\in C_p$.*

PROOF. We adopt the notation in the proof of Theorem 2. Let $\{c_i\}$ be a full set of representatives of the conjugate classes of G where the p-regular factor of c_i is conjugate to b. Define $n_i=\eta(c_i)\cdot|Z(c_i)|$. We can choose k so that $J^k\subseteqq n_iB$ for all i. Then the elements

$$a_{\chi,i}=\frac{1}{n_i}\{\theta_b(c_i)-\alpha_k(c_i)\}\chi(c_i^{-1})$$

are contained in B, since $\theta_b(c_i)-\alpha_k(c_i)\in J^k$. Using the orthogonality relations one can show that

$$\theta_b=\alpha_k+\sum_{\chi,i}a_{\chi,i}\chi\eta.$$

Since α_k, $\eta\in\operatorname{Ind} R_B(H)$, it follows that $\theta_b\in\operatorname{Ind} R_B(H)$, hence $\phi_a\in A\otimes_\lambda V_p$. However, by the same argument as [5, pp. 285-286] we have $V_p=R_\lambda(G)\cap A\otimes_\lambda V_p$ and therefore $\phi_a\in V_p$.

Taking $\lambda=\boldsymbol{Z}$, Proposition 4 together with Lemma 4 gives

BRAUER'S THEOREM ON INDUCED CHARACTERS. $R(G)=\sum_{p,H\in X_p}\operatorname{Ind} R(H)$.

3. Applications.

3.1. Augmentation topology. We shall extend a result of Atiyah [**1**, Proposition (6. 10)] to the case where λ is a Dedekind domain. To do this we need

LEMMA 5. *Suppose λ is a Dedekind domain. If $p\in\mathfrak{p}$, then $P_{0,x}\subseteqq P_{\mathfrak{p},y}$ implies $x_p\underset{K}{\sim}y_p$.*

PROOF. Let $a=y_p$, and ϕ_a the function as in Theorem 2. Then by Lemma 1 we have $\phi_a \notin P_{\mathfrak{p},y}$, hence $\phi_a \notin P_{0,x}$, which shows $x_p \underset{K}{\sim} a$.

If we set $I_\lambda(G)=P_{0,e}$, then the same argument as [1] yields

$$\bigcap_{n=1}^{\infty} I_\lambda(G)^n=\{f\in R_\lambda(G) \mid f(x)=0 \text{ for all } x \text{ having prime power order}\}.$$

It is clear that $\bigcap_n I_\lambda(G)^n=0$, i.e. $R_\lambda(G)$ is Hausdorff with respect to the augmentation topology if G consists of elements having prime power order. We shall prove a converse.

PROPOSITION 5. *Suppose λ is a Dedekind domain and contains a prime number p which is a non-unit in λ, and which does not divide the order of G. If $\bigcap_{n=1}^{\infty} I_\lambda(G)^n=0$, then G consists of elements having prime power order.*

PROOF. If there exists $a\in G$ whose order has two distinct prime divisors, then a is a p-regular element, and the function ϕ_a as in Theorem 2 lies in $\bigcap I_\lambda(G)^n$, for if x has a prime power order, then x_p is not K-conjugate to a, hence $\phi_a(x)=0$.

3.2. Connectedness of $\mathrm{Spec}(R_\lambda(G))$. We shall give a more precise result than [**10**, Proposition 31]. Let $\{P_{0,c_i}\}_{1\leq i\leq r}$ be the set of distinct minimal prime ideals of $R_\lambda(G)$. We denote by V_i the set of all prime ideals of $R_\lambda(G)$ containing P_{0,c_i}. Then the V_i are connected closed sets whose union is $\mathrm{Spec}(R_\lambda(G))$ (for prime spectrum see [**2**], [**4**]).

PROPOSITION 6. *Two distinct minimal prime ideals $P_{0,x}$ and $P_{0,y}$ are contained in the same connected component of* $\mathrm{Spec}(R_\lambda(G))$ *if there exist elements $x_0, x_1, \cdots, x_n$ of G and prime numbers $p_1, \cdots, p_n$ such that*

1) *$x_0=x$ and $x_n=y$,*

2) *the p_α are non-units in λ, and*

3) *the p_α-regular factors of $x_{\alpha-1}$ and x_α are K-conjugate ($1\leqq\alpha\leqq n$).*

Furthermore, the converse is true when λ is a Dedekind domain such that every non-zero prime ideal contains a prime number.

PROOF. We may assume that $P_{0,x_\alpha}=P_{0,c_\alpha}$ ($1\leqq\alpha\leqq n$). If $\mathfrak{p}_\alpha$ is a prime ideal of A containing p_α, then it follows from Lemmas 1 and 2 that $P_{\mathfrak{p}_\alpha,x_\alpha}\in V_{\alpha-1}\cap V_\alpha$, which proves the first assertion.

Now suppose that λ satisfies the above condition. If two distinct ideals $P_{0,x}$ and $P_{0,y}$ are contained in the same connected component, then there exists a sequence $\{V_{j_\alpha}\}_{0\leq\alpha\leq n}$ such that $P_{0,x}\in V_{j_0}$, $P_{0,y}\in V_{j_n}$, and $V_{j_{\alpha-1}}\cap V_{j_\alpha}\neq\emptyset$ ($1\leqq\alpha\leqq n$). If we choose prime ideals $P_{\mathfrak{p}_\alpha,y_\alpha}\in V_{j_{\alpha-1}}\cap V_{j_\alpha}$ such that $\mathfrak{p}_\alpha\cap \boldsymbol{Z}=p_\alpha\boldsymbol{Z}$ for some prime numbers p_α ($1\leqq\alpha\leqq n$), then it follows from Lemma 5 that the p_α-regular factors of $c_{j_{\alpha-1}}$ and c_{j_α} are K-conjugate. Furthermore Proposition 2 implies that x and y are K-conjugate to c_{j_0} and c_{j_n}, respectively. The result is obtained by setting $x_\alpha = c_{j_\alpha}$ $(1 \leqq \alpha < n)$.

COROLLARY 1. *If no prime divisor of the order of* G *is a unit in* λ, *then* $\mathrm{Spec}(R_\lambda(G))$ *is connected. Furthermore, the converse is true when* λ *is the ring as in the proposition.*

PROOF. To prove the first part it suffices to show that for any $x \in G$, $P_{0,x}$ and $P_{0,e}$ are contained in the same connected component. Let $\{p_\alpha\}_{1 \leqq \alpha \leqq n}$ be the set of all prime divisors of the order of x, then each p_α is a non-unit in λ. Take x_α to be the p_α-regular factor of $x_{\alpha-1}$ $(1 \leqq \alpha \leqq n)$ where $x_0 = x$, and apply the proposition to $\{x_\alpha\}$ and $\{p_\alpha\}$.

We now assume that λ is the ring as asserted in the proposition and that $\mathrm{Spec}(R_\lambda(G))$ is connected. Let x be any element of G. It suffices to show that the order of x is a non-unit in λ. Since $P_{0,e}$ and $P_{0,x}$ are contained in the same connected component, there exist two sequences $\{x_\alpha\}$ and $\{p_\alpha\}$ which have the properties in the proposition. Then $x_0 = e$ means that the order of x_1 is a power of p_1. Similarly the prime divisors of the order of x_2 are at most p_1 and p_2, and so on. Thus the prime divisors of the order of x are at most $p_1, p_2, \cdots, p_n$. Hence the order of x is a non-unit in λ.

COROLLARY 2 (Serre [**10**, Proposition 31]). $\mathrm{Spec}(R(G))$ *is connected.*

By Proposition 6 we can determine the individual summand in the decomposition of $R_\lambda(G)$ into a direct sum of indecomposable ideals. In particular, we have the following:

PROPOSITION 7. *Suppose that* λ *is a discrete valuation ring in which* p *is a non-unit. Let* $\{C_i\}_{1 \leqq i \leqq s}$ *be the set of all* K*-conjugate classes in the* p*-regular elements of* G, *and let*

$$B_i = \bigcap_{y_p \notin C_i} P_{0,y} \quad (1 \leqq i \leqq s).$$

Then $R_\lambda(G)$ *is a direct sum of indecomposable ideals* B_i.

PROOF. Let $B'_i = \bigcap_{y_p \in C_i} P_{0,y}$, then $B_i \cap B'_i = 0$. If $R_\lambda(G) \neq B_i + B'_i$, then there exists a maximal ideal $P_{\mathfrak{p},x}$ such that $B_i + B'_i \subseteqq P_{\mathfrak{p},x}$. According to [**2**, Proposition 1.11] we have $y_p \notin C_i$ and $y'_p \in C_i$ such that $P_{0,y} + P_{0,y'} \subseteqq P_{\mathfrak{p},x}$. However, since $p \in \mathfrak{p}$, it follows from Lemma 5 that $y_p \underset{K}{\sim} y'_p$, which is a contradiction. Thus $R_\lambda(G) = B_i + B'_i$ for all i. Noting $\sum_{j \neq i} B_j \subseteqq B'_i$ and $\bigcap_i B'_i = 0$, we see that $R_\lambda(G)$ is a direct sum of the B_i.

Now suppose that B_i is not indecomposable, and let W be the set of all prime ideals of $R_\lambda(G)$ which contain B'_i. It is not hard to show that W is not a connected subset of $\mathrm{Spec}(R_\lambda(G))$. However, Proposition 6 asserts that W is connected, a contradiction.

REMARK. In the case where λ is a field K, every $P_{0,x}$ is a maximal ideal of $R_\lambda(G)$. Let $\{x_i\}_{1\leq i\leq t}$ be a full set of representatives of the K-conjugate classes of G, then $\bigcap_i P_{0,x_i}=0$. If we set $B_i=\bigcap_{j\neq i}P_{0,x_j}$, then $R_\lambda(G)=P_{0,x_i}+B_i$, hence $R_\lambda(G)$ is a direct sum of B_i. On the other hand, since P_{0,x_i} is the kernel of the map $R_\lambda(G)\to L$ which assigns to each f its value $f(x_i)$ at x_i, it follows that $R_\lambda(G)/P_{0,x_i}$ (and hence B_i) is isomorphic to the subfield of L generated by all $\chi(x_i)$ over K. Further results on this ring $R_\lambda(G)$ may be found in [**7**] and [**14**].

3.3. A characterization of p-groups. Finally we give a characterization of p-groups.

THEOREM 3. *Under the hypothesis for λ as in Proposition 7, the following conditions are equivalent.*

(1) *G is a p-group.*

(2) *$R_\lambda(G)$ is a Neotherian local ring.*

(3) *$R_\lambda(G)$ is Hausdorff with respect to the augmentation topology.*

(4) Spec$(R_\lambda(G))$ *is connected.*

PROOF. (1)⇒(2) follows from Proposition 1.

(2)⇒(3). Use the intersection theorem of Krull [**2**, Corollary 10.20].

(3)⇒(4). Since $I_\lambda(G)$ is a prime ideal, it follows that $R_\lambda(G)$ has no non-trivial idempotents. Therefore Spec$(R_\lambda(G))$ is connected.

(4)⇒(1) follows from Proposition 7.

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