J. Math. Soc. Japan
Vol. 30, No. 4, 1978

Product theorem of the fundamental group of a reducible curve

By Mutsuo OKA and Koichi SAKAMOTO

(Received April 23, 1976)

1. In this note, we study the fundamental groups of the complement of reducible curves and we prove the following generalization of the result of [2].

THEOREM. *Let C_1 and C_2 be plane algebraic curves in $\boldsymbol{C}^2$. Assume that the intersection $C_1 \cap C_2$ consists of distinct $d_1 d_2$ points where d_i $(i=1, 2)$ are respective degrees of C_1 and C_2. Then the fundamental group $\pi_1(\boldsymbol{C}^2 - C_1 \cup C_2)$ is isomorphic to the product of $\pi_1(\boldsymbol{C}^2 - C_1)$ and $\pi_1(\boldsymbol{C}^2 - C_2)$.*

2. PROOF.

Let (x, y) be a coordinate of $\boldsymbol{C}^2$, and let $f(x, y)$ and $g(x, y)$ be defining polynomials of C_1 and C_2 respectively. We can assume that the x-axis and y-axis are in general position with respect to C_1 and C_2. Consider the deformations $C_1(t)$ and $C_2(\tau)$ $(t, \tau \in \boldsymbol{C})$ of C_1 and C_2 defined by,

$$C_1(t):\ f(x, ty)=0\,,$$

$$C_2(\tau):\ g(\tau x, y)=0\,.$$

Obviously, each deformation is biholomorphic if $t \neq 0$ or $\tau \neq 0$, and $C_i(1)=C_i$ $(i=1, 2)$, so that $\boldsymbol{C}^2 - C_i(t)$ is homeomorphic to $\boldsymbol{C}^2 - C_i$ for all $t \neq 0$. The intersection $C_1(t) \cap C_2(\tau)$ consists of distinct $d_1 d_2$ points for $(t, \tau) \in U$ where U is a Zariski open set of $\boldsymbol{C}^2$. For any (t_0, τ_0) in U we can construct a one parameter family of curves $\{C_1(t(s)) \cup C_2(\tau(s))\,;\ 0 \leqq s \leqq 1\}$ such that $(t(s), \tau(s))$ is contained in U for each $0 \leqq s \leqq 1$, and $t(0)=\tau(0)=1$, $t(1)=t_0$, $\tau(1)=\tau_0$. Hence, $\boldsymbol{C}^2 - C_1 \cup C_2$ is homeomorphic to $\boldsymbol{C}^2 - C_1(t_0) \cup C_2(\tau_0)$. (See [2], for the precise proof.) So it is enough to show that $\pi_1(\boldsymbol{C}^2 - C_1(t_0) \cup C_2(\tau_0))$ is isomorphic to the product of $\pi_1(\boldsymbol{C}^2 - C_1(t_0))$ and $\pi_1(\boldsymbol{C}^2 - C_2(\tau_0))$, for a suitable $(t_0, \tau_0) \in U$.

The curve $C_1(0)$ consists of distinct d_1 lines which are parallel to the y-axis, and $C_2(0)$ consits of distinct d_2 lines which are parallel to the x-axis, because, by the assumption, the equations $f(x, 0)=0$ and $g(0, y)=0$ have distinct d_1 and d_2 roots respectively. We consider the following parallel lines: L_λ: $y=x+\lambda$ $(\lambda \in \boldsymbol{C})$. For a fixed general λ_0, we can take loops a_j $(j=1, \cdots, d_1)$ and b_k $(k=1, \cdots, d_2)$ generating $\pi_1(L_{\lambda_0} - L_{\lambda_0} \cap (C_1(0) \cup C_2(0)))$, so that $[a_j, b_k]=a_j b_k a_j^{-1} b_k^{-1}$ becomes the unit element in $\pi_1(\boldsymbol{C}^2 - C_1(0) \cup C_2(0))$. Here a_j (respectively b_k) is a small loop which goes around a point of $L_{\lambda_0}\cap C_1(0)$ (resp. $L_{\lambda_0}\cap C_2(0)$), and is joined to the base point. (To see this, one notes that $\boldsymbol{C}^2-C_1(0)\cup C_2(0)$ is homeomorphic to $(\boldsymbol{C}-d_1 \text{ points})\times(\boldsymbol{C}-d_2 \text{ points})$. See Figure 1 and Figure 2.)

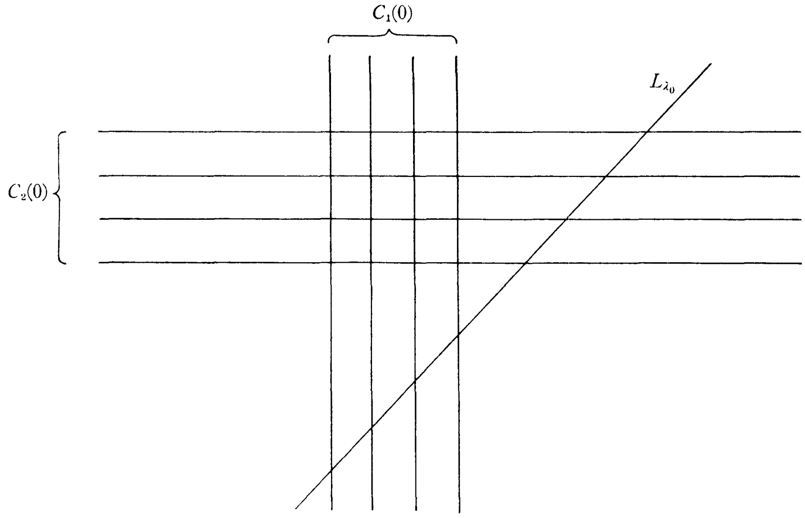

Figure 1

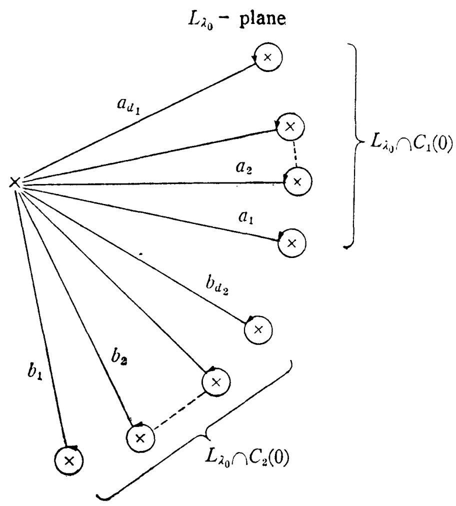

Figure 2

Let $D=\{(x, y)\in\boldsymbol{C}^2 \mid |x|^2+|y|^2<R\}$ be a sufficiently large disc which contains the intersections $C_1(0)\cap C_2(0)$ and $(C_1(0)\cup C_2(0))\cap L_{\lambda_0}$. We can see easily that $\boldsymbol{C}^2-C_1(0)\cup C_2(0)$ is homeomorphic to $D-C_1(0)\cup C_2(0)$. Now we can take (t_0, τ_0) near enough to the origin so that $D-C_1(t_0)\cup C_2(\tau_0)$ is homeomorphic to $D-C_1(0)\cup C_2(0)$ and the same loops a_j $(j=1, \cdots, d_1)$ and b_k $(k=1, \cdots, d_2)$ generate $\pi_1(L_{\lambda_0}-L_{\lambda_0}\cap(C_1(t_0)\cup C_2(\tau_0)))$. The generating relations are given by the monodromy relations around the singular fibers L_ξ. $(L_\xi\cap(C_1(t_0)\cup C_2(\tau_0))$ consists of

less than d_1d_2 points.) ([**1**]). At those lines L_ξ which pass through a point of the intersection $C_1(t_0)\cap C_2(\tau_0)$, we get:

$$a_jb_k=b_ka_j \qquad (j=1, \cdots, d_1,\ k=1, \cdots, d_2). \tag{1}$$

For other L_ξ's, the monodromy relations are following types:

$$\left.\begin{aligned} a_j&=A_{\xi,j}a_{\sigma_\xi(j)}A_{\xi,j}^{-1} && j=1, \cdots, d_1 \\ b_k&=B_{\xi,k}b_{\tau_\xi(k)}B_{\xi,k}^{-1} && k=1, \cdots, d_2 \end{aligned}\right\} \tag{2}$$

where $A_{\xi,j}$ and $B_{\xi,k}$ are words of a_l's and b_h's and σ_ξ and τ_ξ are permutations of the sets $\{1, \cdots, d_1\}$ and $\{1, \cdots, d_2\}$ respectively. Since $a_jb_k=b_ka_j$, we can express $A_{\xi,j}=A_{\xi,j}(a)\cdot A_{\xi,j}(b)$ and $B_{\xi,k}=B_{\xi,k}(a)B_{\xi,k}(b)$, where $A_{\xi,j}(a)$ and $B_{\xi,k}(a)$ (resp. $A_{\xi,j}(b)$ and $B_{\xi,k}(b)$) are words of a_l's (resp. b_h's). Hence the relations (2) become

$$a_j=A_{\xi,j}a_{\sigma_\xi(j)}A_{\xi,j}^{-1}=A_{\xi,j}(a)a_{\sigma_\xi(j)}A_{\xi,j}(a)^{-1}$$

$$b_k=B_{\xi,k}b_{\tau_\xi(k)}B_{\xi,k}^{-1}=B_{\xi,k}(b)b_{\tau_\xi(k)}B_{\xi,k}(b)^{-1}.$$

Therefore, we may assume that the words $A_{\xi,j}$ are generated by a_l $(l=1, \cdots, d_1)$ and $B_{\xi,k}$ are generated by b_h $(h=1, \cdots, d_2)$ for each ξ, j and k. On the other hand, the group $\pi_1(\boldsymbol{C}^2-C_1(t_0))$ is generated by a_j $(j=1, 2, \cdots, d_1)$ and the generating relations are given by

$$a_j=A_{\xi,j}a_{\sigma_\xi(j)}A_{\xi,j}^{-1} \qquad (j=1, \cdots, d_1) \tag{3}$$

and $\pi_1(\boldsymbol{C}^2-C_2(\tau_0))$ is generated by b_k $(k=1, 2, \cdots, d_2)$ and the generating relations are given by

$$b_k=B_{\xi,k}b_{\tau_\xi(k)}B_{\xi,k}^{-1} \qquad (k=1, \cdots, d_2). \tag{4}$$

Thus we obtain

$$\begin{aligned} \pi_1(\boldsymbol{C}^2-C_1(t_0)\cup C_2(\tau_0)) &\cong\langle a_j, b_k\,;\ (1), (3), (4)\rangle \\ &\cong\langle a_j\,;\ (3)\rangle\times\langle b_k\,;\ (4)\rangle \\ &\cong\pi_1(\boldsymbol{C}^2-C_1(t_0))\times\pi_1(\boldsymbol{C}^2-C_2(\tau_0)). \end{aligned}$$

This completes the proof.

3. REMARK.

Let C_1 and C_2 be projective algebraic curves in $\boldsymbol{C}P^2$. If the line $z=0$ is in general position to C_1 and C_2, then $\pi_1(\boldsymbol{C}P^2-C_1\cup C_2)$ is decided by the following central extension

$$1\longrightarrow \boldsymbol{Z}\longrightarrow \pi_1(\boldsymbol{C}^2-C_1\cup C_2)\longrightarrow \pi_1(\boldsymbol{C}P^2-C_1\cup C_2)\longrightarrow 1$$

where $\boldsymbol{C}^2=\boldsymbol{C}P^2-\{z=0\}$. The generator of infinite cyclic group $\boldsymbol{Z}$ corresponds to a large circle in a general affine line L which contains $L\cap(C_1\cup C_2)$ ([**2**]).

References

[1] E.R. Van Kampen, On the fundamental group of an algebraic curve, Amer. J. Math., **55** (1933), 255-260.
[2] M. Oka, On the fundamental group of the complement of a reducible curve in $\boldsymbol{P}^2$, J. London Math. Soc. (2), **12** (1976), 239-252.

Mutsuo OKA
Department of Mathematics
Faculty of Science
University of Tokyo
Tokyo, Japan

Koichi SAKAMOTO
Department of Mathematics
Tsuda College
Kodaira Tokyo
Japan