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Volume estimate of submanifolds in compact Riemannian manifolds

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0. Introduction.

In a study of a given Riemannian manifold $\bar{M}$, it is important and also interesting by itself to know a precise value of the injectivity radius of $\bar{M}$. Here the injectivity radius $i(\bar{M})$ of $\bar{M}$ is, by definition, the supremum of a number t such that every geodesic in $\bar{M}$ with length$<t$ is the shortest connection between its end points. And in general $i(\bar{M})$ can be estimated from below by using a number which relates to half of the infimum of length of all closed geodesics in $\bar{M}$ and hence it is needed to know the infimum of length of all closed geodesics in $\bar{M}$ for the estimate of $i(\bar{M})$. As a remarkable result in this field, J. Cheeger in [2] gave a lower bound of length of all closed geodesics in $\bar{M}$ depending on the volume, the sectional curvature and the diameter of $\bar{M}$.

Now if we consider a closed geodesic in $\bar{M}$ as a 1-dimensional compact totally geodesic submanifold of $\bar{M}$, then the problem to estimate length of closed geodesics in $\bar{M}$ can be generalized as follows. "Is it possible to estimate the volume of compact totally geodesic submanifolds of $\bar{M}$ by using the geometrical terms of $\bar{M}$?". Then from this point of view, a result obtained by N. Grossman in [5], which gives an estimate of the volume of totally geodesic hypersurface in a certain pinched manifold, may be regard as a partial answer to this problem. And with respect to the Grossman's result, we can give a slight generalization, see Theorem A. In Section 3, we will give an answer to the problem mensioned above in a more generalized form. Namely the volume of a compact submanifold M of $\bar{M}$ is estimated from below by using the principal curvatures of the second fundamental forms on M. Furthermore when the codimension of M is 1, a lower bound for the volume of M is given in a further generalized form in the sense that the mean curvature, the mean value of the principal curvatures, is used instead of the principal curvatures. This is shown in Section 2. We will give an upper bound for the volume of M with codimension one or two in Sections 2 and 4.

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1. Totally geodesic submanifolds.

All Riemannian manifolds considered in this paper are connected, complete and without boundary.

Let M be an m-dimensional Riemannian manifold immersed isometrically and totally geodesically in an $\bar{m}$-dimensional Riemannian manifold $\bar{M}$. Let i_M be the injectivity radius of M i.e. $i_M := \inf\{d(q, C(q))\,;\, q\in M\}$ where $C(q)$ is the cut locus of q in M and d is the distance function of M. And $i_{\bar{M}}$ is defined similarly. Then we have the following

LEMMA 1. $i_M \geqq i_{\bar{M}}$.

PROOF. Since the immersion $x: M\to \bar{M}$ is totally geodesic, for any geodesic $\gamma\subset M$, $x\circ\gamma$ is also a geodesic in $\bar{M}$. From this fact, the result can easily be obtained. q.e.d.

For a point $p\in M$ and a number $r>0$, set $B(r, p) := \{q\in M\,;\, d(p, q)<r\}$. And for a real number k, $B(r\,;\,k)$ denotes an open metric ball with radius r in the m-dimensional simply connected space form with constant sectional curvature k.

Now furthermore, we assume that $K_{\bar{M}}\leqq b^2$, where $K_{\bar{M}}$ denotes the sectional curvature of $\bar{M}$ and b is a nonnegative real number or pure imaginary. $d(\bar{M})$ denotes the diameter of $\bar{M}$. Then, from Lemma 1 and from the comparison theorem for volume by R.L. Bishop [**1**; Theorem 15, p. 253], for any point $p\in M$,

$$Volume\,(M)\geqq Volume\, B(i_M, p)\geqq Volume\, B(i_{\bar{M}}, p)\geqq Volume\, B(i_{\bar{M}}\,;\, b^2)\,.$$

So we have the following

THEOREM A. *Let M be an m-dimensional Riemannian manifold immersed isometrically and totally geodesically in a Riemannian manifold $\bar{M}$. If either*

(i) *$\bar{M}$ is simply connected and $b^2/4<K_{\bar{M}}\leqq b^2$, $b>0$ on $\bar{M}$*

or

(ii) *$\bar{M}$ is orientable, even-dimensional and $0<K_{\bar{M}}\leqq b^2$ on $\bar{M}$ then*

$$Volume\,(M)\geqq w_m/b^m\,,$$

where w_m is the volume of the standard unit m-sphere.

PROOF. In either case, from W. Klingenberg's theorem, we have $i_{\bar{M}}\geqq \pi/b$. q.e.d.

2. Volume estimates of hypersurfaces.

Let M be an m-dimensional Riemannian manifold with Riemannian metric g immersed isometrically in a Riemannian manifold $\bar{M}$. $\bar{g}$ (or $\langle , \rangle$) and $\|\cdot\|$ denote the Riemannian metric of $\bar{M}$ and its associated norm respectively. ∇ denotes the Riemannian connection on $\bar{M}$ with respect to $\bar{g}$. Let $T\bar{M}$, TM be the tangent bundle of $\bar{M}$, M respectively, NM the normal bundle of M and π the natural projection. For a point $p \in M$, T_pM is the tangent space of M' at p and N_pM the normal space of M at p. Let $\exp : T\bar{M} \rightarrow \bar{M}$ be the exponential mapping of $\bar{M}$ and $\exp^{\perp} : NM \rightarrow \bar{M}$ be the normal exponential mapping given by the restriction $\exp | NM$. $N_1M := \{v \in NM ; \|v\| = 1\}$ be the unit normal bundle of M. For a unit $v \in N_pM$, we have the second fundamental form $s_v : T_pM \rightarrow T_pM$ with respect to v defined by

$$s_v(X) := (\nabla_X \tilde{v})^T, \qquad \text{for} \quad X \in T_pM$$

where $T : T_p\bar{M} \rightarrow T_pM$ is the orthogonal projection and $\tilde{v}$ is a normal vector field on M such that $\tilde{v}(p) = v$. The norm $\|s_v\|$ of s_v is, by definition,

$$\|s_v\|^2 := \sum_{i,j=1}^{m} \langle s_v(e_i), e_j \rangle^2$$

where $e_1, \cdots, e_m$ is an orthonormal basis of T_pM. M is called a minimal submanifold of M when for all point $p \in M$ and for an orthonormal basis $v_1, \cdots, v_q$ of N_pM

$$\mathit{trace}\, S_{v_\alpha} = \sum_{i=1}^{m} \langle s_{v_\alpha}(e_i), e_i \rangle = 0, \qquad \alpha = 1, \cdots, q.$$

A vector $v \in NM$ is called a focal point of M if the differential of $\exp^{\perp} =: \exp^{\perp}_{*} : TNM \rightarrow T\bar{M}$ is singular at v, where TNM is the tangent bundle of NM. And $\exp(v)$ is called a focal point of M along the geodesic $\gamma(t) := \exp tv/\|v\|$ perpendicular to M. For a positive number $t > 0$, set $U(t) := \{v \in NM ; \|v\| < t\}$. Then we can easily see that the number $t_0 := \sup\{t ; \exp^{\perp} | U(t) \text{ is injective}\}$ is positive when M is compact.

Now, we assume that M is compact and dimension $\bar{M} = m+1$. Let U be a coordinate neighborhood of M with coordinate functions $\{u_1, \cdots, u_m\}$ and v a unit normal vector field on U. Let $\varphi : U \times (-\infty, \infty) \rightarrow \bar{M}$ be the map defined by

$$\varphi(q, t) := \exp^{\perp} tv(q) \qquad \text{for} \quad (q, t) \in U \times (-\infty, \infty).$$

Let $dv_{\bar{M}}$, dv_M and dr be the canonical volume elements of $\bar{M}$, M and $(-\infty, \infty) = E^1$ respectively. We define a function $j : U \times (-\infty, \infty) \rightarrow [0, \infty)$ as follows:

Let $\{v_1, \cdots, v_{m+1}\}$ be local coordinates of $\bar{M}$ defined on $V \subset \bar{M}$ such that for a point $(p, r)\in U\times(-\infty, \infty)$, $\varphi(p, r)\in V$. Then $dv_{\bar{M}}=\sqrt{\det \bar{g}_{kl}}dv_1\cdots dv_{m+1}$ and $dv_M=\sqrt{\det g_{ij}}\,du_1\cdots du_m$ where we put $\bar{g}_{kl}=\bar{g}(\partial/\partial v_k, \partial/\partial v_l)$ and $g_{ij}=g(\partial/\partial u_i, \partial/\partial u_j)$, $1\leqq k$, $l\leqq m+1$, $1\leqq i$, $j\leqq m$. Let $\tilde{j}:(U\times(-\infty, \infty))\cap\varphi^{-1}(V)\to(-\infty, \infty)$ be a function given by

$$\varphi^*(\sqrt{\det g_{kl}}\,dv_1\wedge\cdots\wedge dv_{m+1})=\tilde{j}\sqrt{\det g_{ij}}\,du_1\wedge\cdots\wedge du_m\wedge dr\,.$$

And define $j:=|\tilde{j}|$ on $(U\times(-\infty, \infty))\cap\varphi^{-1}(V)$.

LEMMA 2. *Under the above assumption, we have*

(1) *if* $a^2\leqq \mathrm{Ricci}_{\bar{M}}$ *and* $|\mathrm{trace}\, s_{v(p)}|\leqq\lambda$ *for some* $\lambda\geqq 0$, *then*

$$j(p, r)\leqq\exp.((-ma^2/2)r^2+\lambda r)$$

as long as there exist no focal point of M on $\exp tv(p)|[0, r]$, *where* $\mathrm{Ricci}_{\bar{M}}$ *is the Ricci curvature of* $\bar{M}$,

(2) *if* $a^2\leqq K_{\bar{M}}\leqq b^2$ *and* $\|s_{v(p)}\|\leqq\lambda$ *for some* $\lambda\geqq 0$, *then*

$$j(p, r)\geqq\exp.\left\{-mb^2\int_0^r G(s, b^2)^{-2}\left\{\int_0^s F(t, a^2)^2dt+\lambda\right\}ds\right\},$$

$$\textit{as long as } r<t_2:=\begin{cases}(1/b)\cdot\mathrm{arccot}\,\lambda/b\,, & b^2>0\\ 1/\lambda\,, & b^2=0\\ (1/|b|)\cdot\mathrm{arccoth}\,\lambda/|b|\,, & b^2<0\end{cases}$$

$$\textit{where}\quad F(t, a^2):=\begin{cases}\cos at+(\lambda/a)\sin at\,, & a^2>0\\ \lambda t+1\,, & a^2=0\\ \cosh|a|t+(\lambda/|a|)\sinh|a|t\,, & a^2<0\end{cases}$$

$$\textit{and}\quad G(t, b^2):=\begin{cases}\cos bt-(\lambda/b)\sin bt\,, & b^2>0\\ -\lambda t+1\,, & b^2=0\\ \cosh|b|t-(\lambda/b)\sinh|b|t\,, & b^2<0\,.\end{cases}$$

PROOF. (1) Let $\gamma:[0, \infty)\to M$ be the geodesic such that $\gamma(t)=\exp tv(p)$ and $e_1, \cdots, e_m, \dot{\gamma}(r)$ an orthonormal basis of $T_{\gamma(r)}M$. Then from Gauss Lemma and the fact that $\gamma(r)$ is not a focal point of M, there exists a basis $v_1, \cdots, v_m\in T_pM$ such that $\varphi_*(v_i\times 0_r)=e_i$, $i=1, \cdots, m$ where $0_r\in T_rE^1$ is the zero vector. Let Z_i be the Jacobi field along γ satisfying

$$Z_i(0)=v_i \quad\text{and}\quad Z_i(r)=e_i\,,\qquad i=1, \cdots, m\,.$$

Then, by definition, it is easily seen that

$$j(p, t)=\|Z_1(t)\wedge\cdots\wedge Z_m(t)\|/\|v_1\wedge\cdots\wedge v_m\| \qquad\text{for}\quad 0\leqq t\leqq r\,,$$

and $j(p, 0)=1$. And also it can be checked that

$$j'(p, r)/j(p, r)=\sum_{i=1}^{m}\langle Z_1(r)\wedge\cdots\wedge Z_i'(r)\wedge\cdots\wedge Z_m(r), Z_1(r)\wedge\cdots\wedge Z_m(r)\rangle$$

$$=\sum_{i=1}^{m}\langle Z_i(r), Z_i'(r)\rangle$$

where Z_i' denotes the covariant differentiation of Z_i along γ. Let E_i, $1\leqq i\leqq m$ be the parallel vector field along γ such that $E_i(r)=e_i$. Since there exist no focal point of M on $\gamma|[0, r]$, from [**9**; Corollary 4.3, p. 343], we have

$$0\leqq\int_0^r\{\|(E_i-Z_i)'\|^2-K_{(\dot\gamma(t),(E_i-Z_i)(t))}\|E_i-Z_i\|^2\}dt$$

$$+\langle s_{\dot\gamma(0)}(E_i-Z_i)(0), (E_i-Z_i)(0)\rangle$$

$$=-\int_0^r K_{(\dot\gamma(t),E_i(t))}dt+\langle Z_i, Z_i'\rangle|_0^r-2\langle E_i, Z_i'\rangle|_0^r$$

$$+\langle s_{\dot\gamma(0)}E_i(0), E_i(0)\rangle+\langle s_{\dot\gamma(0)}Z_i(0), Z_i(0)\rangle-2\langle s_{\dot\gamma(0)}E_i(0), Z_i(0)\rangle$$

$$=:(*).$$

Here $K_{(v,w)}$ denotes the sectional curvature of the plane spanned by the vectors v and w, and we have used the facts that Z_i is a Jacobi field and $\langle s_{\dot\gamma(0)}E_i(0), Z_i(0)\rangle=\langle s_{\dot\gamma(0)}Z_i(0), E_i(0)\rangle$. Noticing the fact that $s_{\dot\gamma(0)}Z_i(0)=Z_i'(0)$, because M is a hypersurface, we have

$$(*)=-\int_0^r K_{(\dot\gamma(t),E_i(t))}dt-\langle Z_i(r), Z_i'(r)\rangle+\langle s_{\dot\gamma(0)}E_i(0), E_i(0)\rangle\,.$$

Hence

$$\sum_{i=1}^{m}\langle Z_i(r), Z_i'(r)\rangle\leqq-m\int_0^r \mathrm{Ricci}\,(\dot\gamma(t))dt+\sum_{i=1}^{m}\langle s_{\dot\gamma(0)}E_i(0), E_i(0)\rangle$$

$$\leqq-ma^2r+trace\ s_{\dot\gamma(0)}$$

$$\leqq-ma^2r+\lambda\,.$$

Here Ricci $(\dot\gamma(t))$ is the Ricci curvature in the direction $\dot\gamma(t)$. So

$$j'(p, r)/j(p, r)\leqq-ma^2r+\lambda\,.$$

Thus we get

$$j(p, r)\leqq\exp.\{(-ma^2/2)r^2+\lambda r\}\,.$$

(2). Since Z_i is a Jacobi field,

$$\sum_{i=1}^{m}\langle Z_i(r), Z_i'(r)\rangle=\sum_{i=1}^{m}\int_0^r\{\|Z_i'(t)\|^2-K_{(\dot\gamma(t),Z_i(t))}\|Z_i(t)\|^2\}dt$$

$$+\sum_{i=1}^{m}\langle Z_i(0),\ Z_i'(0)\rangle$$

$$\geqq -b^2\sum_{i=1}^{m}\int_0^r \|Z_i(t)\|^2 dt - \lambda\sum_{i=1}^{m}\|Z_i(0)\|^2 .$$

We now want to estimate the norm of $Z_i(t)$. In order to estimate it below (above), we set up an m-dimensional hypersurface in the $(m+1)$-dimensional space form of constant sectional curvature $b^2(a^2)$, all of whose eigenvalues of the second fundamental form with respect to a unit normal are equal to $-\lambda(\lambda)$. And apply Warner's comparison theorem [**9**; Theorem 4.4, p. 352] to $\tilde{Z}_i(t) := Z_i(t)/\|Z_i(0)\|$, to obtain

$$G(t,\ b^2) \leqq \|\tilde{Z}_i(t)\| \leqq F(t,\ a^2) \qquad \text{for} \quad t \leqq t_2 .$$

(Hereafter, we will call this theorem Warner's comparison theorem). Thus

$$G(r,\ b^2) \leqq 1/\|Z_i(0)\| = \|\tilde{Z}_i(r)\| \leqq F(r,\ a^2) .$$

Hence

$$F(r,\ a^2)^{-1} \leqq \|Z_i(0)\| \leqq G(r,\ b^2)^{-1}, \qquad i=1, \cdots, m .$$

So

$$\|Z_i(t)\| \leqq F(t,\ a^2)\cdot\|Z_i(0)\| \leqq F(t,\ a^2)G(r,\ b^2)^{-1}$$

and

$$\|Z_i(t)\| \geqq G(t,\ b^2)\cdot\|Z_i(0)\| \geqq G(t,\ b^2)F(r,\ a^2)^{-1} .$$

Hence

$$j'(p,\ r)/j(p,\ r) = \sum_{i=1}^{m}\langle Z_i(r),\ Z_i'(r)\rangle$$

$$\geqq -b^2\sum_{i=1}^{m}\int_0^r \|Z_i\|^2 dt - \lambda\sum_{i=1}^{m}\|Z_i(0)\|^2$$

$$\geqq -mb^2 G(r,\ b^2)^{-2}\left\{\int_0^r F(t,\ a^2)^2 dt + \lambda\right\} .$$

Thus we get

$$j(p,\ r) \geqq \exp.\left\{-mb^2\int_0^r G(s,\ b^2)^{-2}\left\{\int_0^s F(t,\ a^2)^2 dt + \lambda\right\} ds\right\} .$$

q. e. d.

From this Lemma 2, we have

THEOREM B. *Let $\bar{M}$ be an $(m+1)$-dimensional compact Riemannian manifold and M an m-dimensional compact Riemannian manifold immersed isometrically in $\bar{M}$.*

(1) *If $a^2 \leqq \mathrm{Ricci}_{\bar{M}}$ and $|\mathrm{trace}\, s_v| \leqq \lambda$ for all $v \in N_1M$, then*

$$Volume\,(M) \geqq Volume\,(\bar{M})\left\{2\int_0^{d(\bar{M})} \exp.\{-(ma^2/2)r^2+\lambda r\}\,dr\right\}^{-1}.$$

(2) *If* $a^2 \leqq K_{\bar{M}} \leqq b^2$ *and* $\|s_v\| \leqq \lambda$ *for all* $v \in N_1M$ *and* M *is an imbedded submanifold of* $\bar{M}$, *then*

$$Volume\,(M) \leqq Volume\,(\bar{M})\cdot$$

$$\left\{2\int_0^{t_0'} \exp.\left\{-mb^2\int_0^r G(s,\,b^2)^{-2}\left\{\int_0^s F(t,\,a^2)^2 dt+\lambda\right\}ds\right\}dr\right\}^{-1},$$

where $t_0' := \min\{t_0, t_2\}$, $t_0 := \sup\{t\,;\,\exp^{\perp}|U(t)$ *is injective*$\}$ *and* G, F *are the functions defined in Lemma 2.*

REMARK. In (1), if we assume $a^2 \leqq K_{\bar{M}}$, then we can use the number $\min\{d(\bar{M}),\,t_1\}$ instead of $d(\bar{M})$ where

$$t_1 := \begin{cases} (1/a)\,\mathrm{arccot}\,(-\lambda/a)\,, & a^2>0 \\ -1/\lambda\,, & a^2=0 \\ (1/|a|)\,\mathrm{arccoth}\,(-\lambda/|a|)\,, & a^2<0\,. \end{cases}$$

PROOF. (1) Since $\bar{M} \subset \exp^{\perp}(\overline{U(d(\bar{M}))})$, if $a^2 \leqq \mathrm{Ricci}_{\bar{M}}$,

$$Volume\,(\bar{M}) = Volume\,(\exp^{\perp}(\overline{U(d(\bar{M}))}) \leqq 2\int_M\int_0^{d(\bar{M})} \exp.\{(-ma^2/2)r^2+\lambda r\}\,dv_M dr$$

$$= Volume\,(M)\cdot 2\int_0^{d(\bar{M})} \exp.\{(-ma^2/2)r^2+\lambda r\}\,dr\,.$$

When $a^2 \leqq K_{\bar{M}}$, all focal points of M are in $\overline{U(t_1)}$ by Warner's comparison theorem and hence $\exp^{\perp}(\overline{U(t_1)}) \supset \bar{M}$.

(2) By Lemma 2, we have

$$Volume\,(\bar{M}) \geqq Volume\,(\exp^{\perp}(\overline{U(t_0)})) \geqq 2\int_M\int_0^{t_0'} j\,dv_M dr$$

$$\geqq Volume\,(M)\cdot 2\int_0^{t_0'} \exp.\left\{-mb^2\int_0^r G(r,\,s^2)^{-2}\left\{\int_0^s F(t,\,b^2)^2 dt+\lambda\right\}ds\right\}dr\,.$$

q. e. d.

Now, in the following we will calculate the number t_0 stated in Theorem B for a certain pair $\bar{M}$ and M.

LEMMA 3. *Let* $\bar{M}$ *be an* $(m+1)$*-dimensional Riemannian manifold with* $K_{\bar{M}} \leqq b^2$, $b>0$ *and* $\mathrm{Ricci}_{\bar{M}}>0$ *and* M *a compact minimally imbedded hypersurface of* $\bar{M}$. *Let* $\lambda \geqq 0$ *be a number satisfying* $\|s_v\| \leqq \lambda$ *for all* $v \in N_1M$. *Then*

$$t_0 := \sup\{t\,;\,\exp^{\perp}|U(t) \text{ is injective}\} \geqq (1/b)\cdot\mathrm{arccot}\,(\lambda/b)\,.$$

PROOF. Suppose $t_0<(1/b)\cdot\mathrm{arccot}\,(\lambda/b)$. Then by Warner's comparison theorem, there exist no focal point of M in $U(t_0+\varepsilon)$ for some $\varepsilon>0$, i.e. $\exp^{\perp}|U(t_0+\varepsilon)$ is a local diffeomorphism. Then from the definition of t_0 and compactness of M, we see that $\exp^{\perp}|\overline{U(t_0)}$ is not injective. Thus there exist two disjoint geodesics $c_1, c_2:[0, t_0]\to\bar{M}$ such that $\dot{c}_1(0), \dot{c}_2(0)\in N_1M$ and $c_1(t_0)=c_2(t_0)$. And sufficiently small neighborhoods of $t_0\dot{c}_1(0)$ and $t_0\dot{c}_2(0)$ in the closed hypersurface $\partial U(t_0)\subset NM$ are mapped by $\exp^{\perp}$ into sets V and W which are hypersurfaces in $\bar{M}$ intersecting at $c_1(t_0)$ and orthogonal to c_1 and c_2 respectively, because of Gauss Lemma. If $\sphericalangle(\dot{c}_1(t_0), \dot{c}_2(t_0))$, the angle between $\dot{c}_1(t_0)$ and $\dot{c}_2(t_0)$, $\neq\pi$, V and W meet transversally at $c_1(t_0)$ and we can find $v, w\in U(t_0)$, $v\neq w$ such that $\exp^{\perp}v=\exp^{\perp}w$. This contradicts the definition of t_0. Hence $\sphericalangle(\dot{c}_1(t_0), \dot{c}_2(t_0))=\pi$. So $c_1(t_0+t)=c_2(t_0-t)$ for $t\in[0, t_0]$. Let $e_1, \cdots, e_m$ be an orthonormal basis of $T_{c_1(0)}M$ and $E_1, \cdots, E_m$ the vector fields along the extension $c_1:[0, 2t_0]\to M$, obtained by the parallel translation of $e_1, \cdots, e_m$. Then $E_1(2t_0), \cdots, E_m(2t_0)$ is an orthonormal basis of $T_{c_1(2t_0)}M$, because M is a hypersurface. For each E_i, we consider a variation $V_i:[0, 2t_0]\times(-\rho, \rho)\to\bar{M}$, $\rho>0$, of c_1 such that $V_i(\{0\}\times(-\rho, \rho)\cup\{2t_0\}\times(-\rho, \rho))\subset M$ and $\partial(V_i(t, s))/\partial s|_{s=0}=E_i(t)$, $i=1, \cdots, m$, see [**3**; Generalized Hadamard Theorem, p. 69]. Let $L(V_{i,s})$ denote the length of the curve $V_{i,s}(t):=V_i(t, s)$. Then from the second variational formula,

$$\frac{d^2}{ds^2}L(V_{i,s})_{|s=0}=\langle s_{\dot{c}_1(0)}E_i(0), E_i(0)\rangle-\langle s_{\dot{c}_1(2t_0)}E_i(2t_0), E_i(2t_0)\rangle$$

$$-\int_0^{2t_0}K_{(\dot{c}_1(t),E_i(t))}dt\,.$$

Since M is minimal

$$\sum_{i=1}^{m}\langle s_{\dot{c}_1(0)}E_i(0), E_i(0)\rangle=\sum_{i=1}^{m}\langle s_{\dot{c}_1(2t_0)}E_i(2t_0), E_i(2t_0)\rangle=0\,.$$

So we have

$$\sum_{i=1}^{m}\frac{d^2}{ds^2}L(V_{i,s})_{|s=0}=-\int_0^{2t_0}m\cdot\mathrm{Ricci}\,(\dot{c}_1(t))dt<0\,.$$

Hence there exists some number i such that

$$\frac{d^2}{ds^2}L(V_{i,s})_{|s=0}<0\,.$$

Then we can derive a contradiction by the same argument used in [**6**; Theorem 2, p. 330]. q.e.d.

Thus from Lemma 2, 3, in a special case, Theorem B can be restated as follows;

THEOREM C. *Let $\bar{M}$ be a compact $(m+1)$-dimensional Riemannian manifold.*

(1) *If M is a compact m-dimensional Riemannian manifold immersed minimally in $\bar{M}$ and $a^2 \leqq \mathrm{Ricci}_{\bar{M}}$, then*

$$Volume(M) \geqq Volume(\bar{M}) \left\{ 2\int_0^{d(\bar{M})} \exp.((-ma^2/2)r^2\, dr \right\}^{-1}.$$

(2) *Let M be an m-dimensional compact Riemannian manifold imbedded minimally in $\bar{M}$. Let $0 < a^2 \leqq K_{\bar{M}} \leqq b^2$ and $\lambda \geqq 0$ be a number satisfying $\|s_v\| \leqq \lambda$ for all $v \in N_1 M$. Then, putting $t_0' := (1/b) \cdot \operatorname{arccot}(\lambda/b)$,*

$$Volume(M) \leqq Volume(\bar{M})$$

$$\cdot \left\{ 2\int_0^{t_0'} \exp.\left\{ -mb^2 \int_0^r G(r, b^2)^{-2} \left\{ \int_0^s F(t, a^2)^2 dt + \lambda \right\} ds \right\} dr \right\}^{-1}.$$

3. Submanifolds with codimension $\geqq 1$.

In this section, we will give an estimate of volume of a submanifold with codimension $\geqq 1$.

Let M be an m-dimensional compact Riemannian manifold with the Riemannian metric g, immersed isometrically in an $(m+q)$-dimensional Riemannian manifold $\bar{M}$, $q \geqq 1$, with the Riemannian metric $\bar{g}$. On the normal bundle NM of M, we introduce a Riemannian metric $\tilde{g}$ as follows; for any $X, Y \in TNM$,

$$\tilde{g}(X, Y) := g(\pi_* X, \pi_* Y) + \bar{g}(K^N(X), K^N(Y)).$$

Here $\pi : NM \to M$ is the natural projection and $K^N : TNM \to NM$ is the connection map defined by, using the connection map $K : TT\bar{M} \to T\bar{M}$,

$$K^N(X) := \text{orthogonal projection of } K(X) \text{ to } NM.$$

For the definition of K, see [**4**; 2.4, p. 43]. It is checked that $\tilde{g}$ is a Riemannian metric on NM. Let dv_{NM} and $dv_{\bar{M}}$ be the volume elements of NM and $\bar{M}$ induced from the Riemannian metrics $\tilde{g}$ and $\bar{g}$ respectively. Let $j : NM \to [0, \infty)$ be the function defined as follows; Fix a point $v \in NM$. Let $\{v_1, \cdots, v_{m+q}\}$ be a local coordinate system of NM defined on $W \subset NM$ such that $v \in W$. Let $\{u_1, \cdots, u_{m+q}\}$ be a local coordinate system defined on $U \subset \bar{M}$ such that $\exp^\perp(v) \in U$. Put $\bar{g}_{rs} := \bar{g}(\partial/\partial u_r, \partial/\partial u_s)$ and $\tilde{g}_{rs} := \tilde{g}(\partial/\partial v_r, \partial/\partial v_s)$ for $1 \leqq r, s \leqq m+q$. Then

$$dv_{\bar{M}} = \sqrt{\det \bar{g}_{rs}}\, du_1 \cdots du_{m+q} \text{ and } dv_{NM} = \sqrt{\det \tilde{g}_{rs}}\, dv_1 \cdots dv_{m+q}.$$

Then on $W \cap (\exp^\perp)^{-1}(U)$, we have

$$(\exp^\perp)^* \sqrt{\det \bar{g}_{rs}}\, du_1 \wedge \cdots \wedge du_{m+q} = \hat{j} \sqrt{\det \tilde{g}_{rs}}\, dv_1 \wedge \cdots \wedge dv_{m+q}$$

for a function $\hat{j} : W \cap (\exp^\perp)^{-1}(U) \to (-\infty, \infty)$. And define $j := |\hat{j}|$ on

$W\cap(\exp^{\perp})^{-1}(U)$. And in the following, we will estimate this function j.

Fix a point $p_0\in M$ and let $V(p_0)$ be a normal neighborhood of p_0 in M and $p_i: V(p_0)\to R$, $i=1, \cdots, m$ the coordinate functions. Let $n_1, \cdots, n_q$ be an orthonormal vector fields on $V(p_0)$ normal to M given by the parallel translation (with respect to the connection of NM) of an orthonormal basis $n_1(p_0), \cdots, n_q(p_0)$ of $N_{p_0}M$ along each geodesic in M starting from p_0. Using these n_α, we introduce local coordinate $\phi: NM|V(p_0)\to R^{m+q}$ as

$$NM|V(p_0)\ni\tilde{y}=\sum_{\alpha=1}^{q}y_\alpha n_\alpha(p)\longrightarrow(p_1, \cdots, p_m(p); y_1, \cdots, y_q).$$

By the canonical isomorphism, we can use the notation $\partial/\partial p_i\in TTM$. Then it can be checked that

$$\pi_*(\partial/\partial p_i)=\partial/\partial p_i, \qquad \pi_*(\partial/\partial y_\alpha)=0,$$

$$K^N(\partial/\partial p_i)=\sum_{\alpha,\beta=1}^{q}y_\alpha\Gamma_{\alpha i}^{\beta}n_\beta \text{ and } K^N(\partial/\partial y_\alpha)=n_\alpha, \qquad \text{for } i=1, \cdots, m$$

and $\alpha=1, \cdots, q$. Here $\Gamma_{\alpha i}^{\beta}:=\tilde{g}(\overset{\perp}{\nabla}_{\partial/\partial p_i}n_\alpha, n_\beta)$ and $\overset{\perp}{\nabla}$ is the connection on NM. Then it follows

$$\tilde{g}_{ij}:=\tilde{g}(\partial/\partial p_i, \partial/\partial p_j)=g_{ij}+\sum_{\alpha,\beta,\gamma}y_\alpha y_\gamma\Gamma_{\alpha i}^{\beta}\Gamma_{\gamma j}^{\beta},$$

$$\tilde{g}_{i\alpha}:=\tilde{g}(\partial/\partial p_i, \partial/\partial y_\alpha)=\sum_{\beta}y_\beta\Gamma_{\beta i}^{\alpha},$$

and

$$\tilde{g}_{\alpha\beta}:=\tilde{g}(\partial/\partial y_\alpha, \partial/\partial y_\beta)=\delta_{\alpha\beta} \qquad \text{for } 1\leqq i,\ j\leqq m,\ 1\leqq\alpha,\ \beta,\ \gamma\leqq q.$$

And from the choice of n_α,

$$\Gamma_{\alpha i}^{\beta}(p_0)=0, \qquad 1\leqq i\leqq m, \qquad 1\leqq\alpha,\ \beta\leqq q.$$

So from these facts, we have

$$(*) \qquad \det\tilde{g}_{rs}|N_{p_0}M=\det g_{ij}(p_0) \qquad \text{i.e., } dv_{NM}|N_{p_0}M=dv_M\cdot dy,$$

$1\leqq r,\ s\leqq m+q$, where dy is the canonical volume element on the q-dimensional Euclidean space E^q.

Now, fix $\tilde{x}\in N_{p_0}M$, $\tilde{x}\neq 0$ such that $t\tilde{x}$, $0<t\leqq 1$, is not a focal point of M. We will estimate $j(\tilde{x})$. For this purpose, we may suppose that $n_q(p_0)=\tilde{x}/\|\tilde{x}\|$. Let $e_1, \cdots, e_m$ be an orthonormal basis of $T_{p_0}M$. Let Z_i, $i=1, \cdots, m$ and W_α, $\alpha=1, \cdots, q-1$, be the Jacobi fields along the geodesic $\gamma(t):=\exp tn_q(p_0)$, satisfying the following boundary conditions,

$$Z_i(0)=e_i, \qquad Z_i'(0)=s_{n_q(p_0)}(e_i), \qquad i=1, \cdots, m$$

and

$$W_\alpha(0)=0\,, \qquad W'_\alpha(0)=n_\alpha\,, \qquad \alpha=1, \cdots, q-1\,.$$

Then from the above consideration, we can verify the following;

$$j(\tilde{x})=\|Z_1(\|\tilde{x}\|)\wedge \cdots \wedge Z_m(\|\tilde{x}\|)\wedge W_1(\|\tilde{x}\|)\wedge \cdots \wedge W_{q-1}(\|\tilde{x}\|)\|/\|\tilde{x}\|^{q-1}\,.$$

LEMMA 5. *Let $a^2\leqq K_{\bar{M}}$, where $a\geqq 0$ or a pure imaginary and $\lambda\geqq 0$ be a number satisfying $\|s_v\|\leqq\lambda$ for all $v\in N_1M$. If all M-Jacobi fields along γ split in the sense of F. Warner, then*

$$j(\tilde{x})\leqq J(a^2, \lambda, \|\tilde{x}\|, m, q)$$

where $J(a^2, \lambda, r, m, q)$

$$:=\begin{cases} (\sin ar/ar)^{q-1}(\cos ar+(\lambda/a)\sin ar)^m\,, & a^2>0 \\ (\lambda r+1)^m\,, & a^2=0 \\ (\sinh|a|r/|a|r)^{q-1}(\cosh|a|r+(\lambda/|a|)\sinh|a|r)^m\,, & a^2<0 \end{cases}$$

as long as there exist no focal point of M on $\gamma|[0, \|\tilde{x}\|]$.

PROOF.

$$\begin{aligned} j(\tilde{x})&=\|Z_1(\|\tilde{x}\|)\wedge \cdots \wedge Z_m(\|\tilde{x}\|)\wedge W_1(\|\tilde{x}\|)\wedge \cdots \wedge W_{q-1}(\|\tilde{x}\|)\|/\|\tilde{x}\|^{q-1} \\ &\leqq \|Z_1(\|\tilde{x}\|)\|\cdot \cdots \cdot\|Z_m(\|\tilde{x}\|)\|\cdot\|W_1(\|\tilde{x}\|)\|\cdot \cdots \cdot\|W_{q-1}(\|\tilde{x}\|)\|/\|\tilde{x}\|^{q-1}\,. \end{aligned}$$

We only prove the case $a^2>0$. The other cases are proved similarly. For $\alpha=1, \cdots, q-1$, by Rauch's comparison theorem

$$\|W_\alpha(t)\|\leqq\|W(t)\|$$

as long as there exist no conjugate point of p_0 on $\gamma|[0, t]$, where W is the Jacobi field along a geodesic c in $S^{m+q}(1/a)$ satisfying $W\perp\dot{c}$, $W(0)=0$ and $\|W'(0)\|=\|W'_\alpha(0)\|=1$, where $S^{m+q}(1/a)$ is the $(m+q)$-dimensional standard sphere with radius $1/a$. Let $\tilde{U}$ be the parallel field along c such that $\tilde{U}(0)=W'(0)$. Then $W(t)=(1/a)\sin at\tilde{U}(t)$. So

$$\|W_\alpha(\|\tilde{x}\|)\|\leqq\|W(\|\tilde{x}\|)\|=(1/a)\sin a\|\tilde{x}\|\,, \qquad 1\leqq\alpha\leqq q-1\,.$$

For Z_i, $i=1, \cdots, m$, since Z_i is a strong M-Jacobi field (see [9]), from Warner's comparison theorem, we have

$$\|Z_i(t)\|\leqq\|Z(t)\|$$

as long as there exist no focal point of M on $\gamma|[0, t]$, where Z is a Jacobi field along a geodesic c in $S^{m+q}(1/a)$ satisfying $Z\perp\dot{c}$, $\|Z(0)\|=\|Z_i(0)\|=1$ and $Z'(0)=\lambda Z(0)$. Let V be a parallel field along c such that $V(0)=Z(0)$. Then $Z(t)=(\cos at+(\lambda/a)\sin at)V(t)$. Hence, for $i=1, \cdots, m$

$$\|Z_i(\|\tilde{x}\|)\| \leqq \|Z(\|\tilde{x}\|)\| = \cos a\|\tilde{x}\| + (\lambda/a)\sin a\|\tilde{x}\| .$$

q. e. d.

From this Lemma 5, we have

THEOREM D. *Let $\bar{M}$ be an $(m+q)$-dimensional compact Riemannian manifold with the sectional curvature satisfying $a^2 \leqq K_{\bar{M}}$, where $a \geqq 0$ or a pure imaginary, and M an m-dimensional compact Riemannian manifold immersed isometrically in $\bar{M}$. Let $\lambda \geqq 0$ be a number satisfying $\|s_v\| \leqq \lambda$ for all $v \in N_1 M$. If all M-Jacobi equation split, then*

$$Volume\,(M) \geqq Volume\,(\bar{M}) \cdot \left\{ w_{q-1} \int_0^l J(a^2, \lambda, r, m, q) r^{q-1} dr \right\}^{-1}$$

where

$$l := \min\{d(\bar{M}),\ t_1\}, \quad t_1 := \begin{cases} (1/a)\operatorname{arccot}(-\lambda/a), & a^2 > 0 \\ -(1/\lambda), & a^2 = 0 \\ (1/|a|)\operatorname{arccoth}(-\lambda/|a|), & a^2 < 0 \end{cases}$$

and J the function given in Lemma 5 (we consider $w_0 = 2$).

PROOF. From (*), we can choose for any $\varepsilon > 0$ and a point $p_0 \in M$ a small normal neighborhood $V(p_0)$ of p_0 in M used in the above argument such that on the set $U(l) | V(p_0) = \{\tilde{y} \in NM | V(p_0) ; \|\tilde{y}\| < l\} \ni \tilde{y}$

$$(**) \qquad \sqrt{\det g_{ij}(\pi(\tilde{y}))} \cdot (1-\varepsilon) \leqq \sqrt{\det \tilde{g}_{rs}(\tilde{y})} \leqq \sqrt{\det g_{ij}(\pi(\tilde{y}))} \cdot (1+\varepsilon) .$$

This is possible, because of continuity of the function $\det \tilde{g}_{rs}$ and compactness of the set $\overline{U(l) | V(p_0)}$. Then putting $\tilde{U}(l) := \{v \in U(l) ; tv,\ 0 < t \leqq 1$ is not a focal point of $M\}$, we have

$$\int_{\tilde{U}(l)|V(p_0)} j\, dv_{NM} = \int_{\tilde{U}(l)|V(p_0)} j \sqrt{\det \tilde{g}_{rs}}\, dp_1 \cdots dp_m dy_1 \cdots dy_q$$

$$\leqq \int_{U(l)|V(p_0)} J \sqrt{\det g_{ij}}\, (1+\varepsilon) dp_1 \cdots dp_m dy_1 \cdots dy_q$$

$$= (1+\varepsilon) \int_{V(p_0)} dv_M \cdot \int_{D_l := \{y \in E^q ; |y| < l\}} J\, dy$$

$$= (1+\varepsilon) \cdot Volume\,(V(p_0)) \cdot \int_{D_l} J\, dy .$$

Then, by using compactness of M, we can choose points $p_i \in M$ and normal neighborhood $V(p_i)$, $i = 1, \cdots, k$ such that $V(p_i)$ are mutually disjoint and $\bigcup_{i=1}^{k} \overline{V(p_i)} = M$ and moreover the property (**) is satisfied on each $V(p_i)$, $i = 1, \cdots, k$. From these properties of $V(p_i)$, it follows that

$$\int_{\tilde{U}(l)} j dv_{NM} \leqq (1+\varepsilon) \cdot Volume\,(M) \cdot \int_{D_l} J\,dy\,.$$

Since $\varepsilon>0$ is arbitrary, letting $\varepsilon\to 0$, we get

$$Volume\,(\bar{M}) \leqq \int_{\tilde{U}(l)} j\,dv_{NM} \leqq Volume\,(M) \cdot \int_{D_l} J\,dy\,.$$

Here we have used the fact that all focal points of M are in $\overline{U(t_1)}$ by Warner's comparison theorem. q. e. d.

This Theorem D may be regarded as a sort of generalization of a theorem obtained by J. Cheeger in [2; Theorem 5.8, p. 96], which is

THEOREM (J. CHEEGER). *Let $\bar{M}$ be an m-dimensional compact Riemannian manifold satisfying*

$$\begin{aligned} &\textit{the sectional curvature of } \bar{M} &&\geqq H \\ &\textit{the diameter of } \bar{M} &&\leqq d \\ \textit{and}\quad &\textit{the volume of } \bar{M} &&\geqq V>0 \end{aligned}$$

for some constants H, d and V, then there exists a constant $c_m(H, d, V)>0$ depending on m, H, d and V such that for any closed geodesic γ in $\bar{M}$,

$$\textit{the length of } \gamma \geqq c_m(H, d, V)\,.$$

Since any closed geodesic in $\bar{M}$ is a 1-dimensional compact totally geodesic submanifold of $\bar{M}$, from Theorem D and [9], we have

COROLLARY OF THEOREM D. *Let $\bar{M}$ be a manifold stated in Cheeger's theorem. If $d \leqq \dfrac{\pi}{2\sqrt{c}}$ when $c:=\max K_{\bar{M}}>0$, then there exists a constant $c'_m(H, d, V)>0$ depending on m, H, d and V such that for any closed geodesic γ in $\bar{M}$*

$$\textit{the length of } \gamma \geqq c'_m(H, d, V)\,.$$

$c'_m(H, d, V)$ is given by

$$c'_m(H, d, V) := V\left\{w_{m-2}\int_0^{l'} J(H, 0, r, 1, m-1) r^{m-2} dr\right\}^{-1},$$

where $l' := \min\{d, t_1\}$.

REMARK. It is not obvious which one of the constants will give a better lower bound for the length of any closed geodesics in $\bar{M}$. But it seems that calculation of $c'_m(H, d, V)$ is considerably easier than Cheeger's one, see [7].

4. Submanifolds with codimension 2.

In this section, we will give an estimate of volume of a submanifold with codimension 2 from above.

Let M be a compact m-dimensional Riemannian manifold imbedded isometrically in a compact $(m+2)$-dimensional Riemannian manifold $\bar{M}$. By putting $q=2$, we use the same notation used in Section 3. Let $p_0 \in M$, $V(p_0)$, and n_1, n_2 be those defined in Section 3. Let $\tilde{x} \in N_{p_0}M$, $\tilde{x} \neq 0$ be a vector such that $t\tilde{x}$, $0<t\leqq 1$ is not a focal point of M. And also set $n_2(p_0)=\tilde{x}/\|\tilde{x}\|$ for the calculation of $j(\tilde{x})$. Let γ be the geodesic normal to M given by $\gamma(t):=\exp tn_2(p_0)$. Let $v_1, \cdots, v_m$ be the basis of $T_{p_0}M$ such that $(\exp^{\perp}\circ\psi^{-1})_*(v_i\times 0_x)=:e_i$, $i=1, \cdots, m$ are orthonormal, where 0_x is the zero vector of T_xE^2, $\psi(\tilde{x})=(p_1(p_0), \cdots, p_m(p_0), x)$, $x\in E^2$, and $\psi: NM|V(P_0)\to R^{m+2}$ is a coordinate function defined in Section 3. This is possible, because $\tilde{x}$ is not a focal point of M. Note that because of Gauss Lemma, e_i, $i=1, \cdots, m$ are orthogonal to $\dot{\gamma}(\|\tilde{x}\|)$. Let Z_i, $i=1, \cdots, m$ and W the Jacobi fields along γ satisfying the following boundary conditions:

$$Z_i(0)=v_i\,, \qquad Z_i(\|\tilde{x}\|)=e_i\,, \qquad i=1, \cdots, m$$

and

$$W(0)=0\,, \qquad W'(0)=n_1(p_0)\,.$$

Then from the choice of n_α, it is obvious that $Z_i'(0)=s_{n_2(p_0)}Z_i(0)$. And it can be checked that

$$\begin{aligned} j(\tilde{x}) &= \|Z_1(\|\tilde{x}\|)\wedge \cdots \wedge Z_m(\|\tilde{x}\|)\wedge W(\|\tilde{x}\|)\|/(\|\tilde{x}\wedge v_1\wedge \cdots \wedge v_m\|) \\ &\geqq \|e_1\wedge \cdots \wedge e_m\wedge W(\|\tilde{x}\|)\|/(\|\tilde{x}\|\cdot\|v_1\|\cdot \cdots \cdot\|v_m\|)\,. \end{aligned}$$

Put $\|\tilde{x}\|=:r$ for convenience. And now assume the sectional curvature of $\bar{M}$ satisfies $a^2\leqq K_{\bar{M}}\leqq b^2$, $a\geqq 0$, $b\geqq 0$ or pure imaginaries, and $\|\tilde{x}\|=r<t_2$ where the number t_2 is the one defined in Lemma 2 by replacing λ with $-\lambda$. Then in the proof of Lemma 2 (ii), it has been proved that

$$\|v_i\|=\|Z_i(0)\|\leqq G(r, b^2)^{-1}\,, \qquad i=1, \cdots, m\,.$$

Thus

$$\begin{aligned} j(\tilde{x})^2 &\geqq G(r, b^2)^{-2m}\|e_1\wedge \cdots \wedge e_m\wedge (W(r)/r)\|^2 \\ &= G(r, b^2)^{-2m}\{\langle W(r)/r, W(r)/r\rangle - \sum_{i=1}^m \langle W(r)/r, e_i\rangle^2\}\,. \end{aligned}$$

Since the term $\langle W(r)/r, W(r)/r\rangle$ is estimable from Rauch's comparison theorem, it remains the estimate of $\sum_{i=1}^m \langle W(r)/r, e_i\rangle^2$. We fix any Z_i and write it as Z for simplicity of our argument. From the proof of Lemma 2 and Rauch's comparison theorem, we have

$$G(t, b^2)F(r, a^2)^{-1}\leqq \|Z(t)\|\leqq F(t, a^2)G(r, b^2)^{-1}$$

and

$$H(t, b^2)\leqq \|W(t)\|\leqq H(t, a^2) \qquad \text{for} \quad t\leqq r\,,$$

where

$$H(t, k^2) := \begin{cases} (1/k)\cdot\sin kt, & k^2>0 \\ t, & k^2=0 \\ (1/|k|)\cdot\sinh |k|t, & k^2<0. \end{cases}$$

Let E be the parallel field along γ such that $E(0)=Z(0)/\|Z(0)\|$. And consider the function $f(t):=\langle Z(t), E(t)\rangle$. Then $f(0)=\|Z(0)\|$ and

$$f'(t)=\langle Z', E\rangle(t)=\langle Z', E\rangle|_0^t+\langle Z'(0), E(0)\rangle$$

$$=-\int_0^t \langle R(Z,\dot{\gamma})\dot{\gamma}, E\rangle dt+\langle Z'(0), E(0)\rangle.$$

From a property of curvature tensor, we have

$$\langle R(Z,\dot{\gamma})\dot{\gamma}, E\rangle(t)=\langle R(E,\dot{\gamma})\gamma, Z\rangle(t)\leqq\|Z\|\cdot\|R(E,\dot{\gamma})\dot{\gamma}\|(t).$$

And $\|R(E,\dot{\gamma})\dot{\gamma}\|(t)$ does not exceed the maximum absolute eigenvalue of curvature transformation $R(\cdot, \dot{\gamma}(t))\dot{\gamma}(t): T^{\perp}_{\gamma(t)}\bar{M}\to T^{\perp}_{\gamma(t)}\bar{M}$, where $T^{\perp}_{\gamma(t)}\bar{M}:=\{v\in T_{\gamma(t)}\bar{M}\,;\,\langle\dot{\gamma}(t), v\rangle=0\}$, because $\|E\|=1$. Thus

$$-\langle R(Z,\dot{\gamma})\dot{\gamma}, E\rangle(t)\geqq-\|Z\|\cdot\|R(E,\dot{\gamma})\dot{\gamma}\|(t)\geqq-F(t, a^2)G(r, b^2)^{-1}\max\{|a^2|, |b^2|\}.$$

Put $A(t):=-F(t, a^2)G(r, b^2)^{-1}\max\{|a^2|, |b^2|\}$. For $\langle Z'(0), E(0)\rangle$, since

$$\langle Z'(0), E(0)\rangle=\langle Z'(0), Z(0)\rangle/\|Z(0)\|=\langle s_{\dot{\gamma}(0)}Z(0), Z(0)\rangle/\|Z(0)\|,$$

we have

$$\langle Z'(0), E(0)\rangle\geqq-\lambda\|Z(0)\|\geqq-\lambda G(r, b^2)^{-1}.$$

So

$$f'(t)\geqq\int_0^t A(u)\,du-\lambda G(r, b^2)^{-1}.$$

Thus

$$f(s)=\int_0^s f'(t)dt+f(0)$$

$$\geqq\int_0^s\left(\int_0^t A(u)\,du-\lambda G(r, b^2)^{-1}\right)dt+F(r, a^2)^{-1}.$$

So

$$f(r)\geqq\int_0^r\left\{\int_0^t A(u)\,du-\lambda G(r, b^2)^{-1}\right\}dt+F(r, a^2)^{-1}.$$

Hence putting

$$l_1(r):=\max\left\{-1, \int_0^r\left\{\int_0^t A\,du-\lambda G(r, b^2)^{-1}\right\}dt+F(r, a^2)^{-1}\right\},$$

we have

$$\sphericalangle(Z(r), E(r)) \leqq \arccos l_1(r) \leqq \pi\,, \tag{1}$$

because $f(r)=\langle Z(r), E(r)\rangle=\cos \sphericalangle(Z(r), E(r))$. Note that $\arccos l_1(r) \to 0$ as $r \to 0$.

Now, let $\tilde{E}$ be the parallel vector field along γ such that $\tilde{E}(0)=W'(0)=n_1$. And put $g(s):=\langle W(s), \tilde{E}(s)\rangle$. Then $g(0)=0$ and

$$g'(s)=\langle W', \tilde{E}\rangle(s)=\langle W', \tilde{E}\rangle|_0^s+\langle W', \tilde{E}\rangle(0)$$

$$=-\int_0^s \langle R(W, \dot{\gamma})\dot{\gamma}, E\rangle dt+1\,,$$

where we have used the fact that W is a Jacobi field. Then from the same reason for the estimate of $f'(t)$, we have

$$g'(s) \geqq \int_0^s B(t)\, dt+1$$

where $B(t):=-H(t, a^2)\max\{|a^2|, |b^2|\}$. Thus

$$g(r)=\int_0^r g'(s)\, ds+g(0) \geqq \int_0^r\int_0^s B(t)\, dtds+r\,.$$

So putting $l_2(r):=\max\left\{-1, \left(\int_0^r\int_0^s B(t)\, dtds+r\right)\Big/ H(r, a^2)\right\}$, we have

$$l_2(r) \leqq g(r)/\|W(r)\|=\cos \sphericalangle(W(r), \tilde{E}(r))\,, \qquad \text{i.e.,}$$

$$\sphericalangle(W(r), \tilde{E}(r)) \leqq \arccos l_2(r) \leqq \pi\,. \tag{2}$$

And it is easily verified that $l_2(r) \to 1$ as $r \to 0$. From (1) and (2), we have

$$\sphericalangle(W(r), Z(r)) \geqq \sphericalangle(E(r), \tilde{E}(r))-\sphericalangle(W(r), \tilde{E}(r))-\sphericalangle(Z(r), E(r))$$

$$\geqq \pi/2-\arccos l_1(r)-\arccos l_2(r)\,,$$

and

$$\sphericalangle(W(r), Z(r)) \leqq \sphericalangle(E(r), \tilde{E}(r))+\sphericalangle(W(r), \tilde{E}(r))+\sphericalangle(Z(r), E(r))$$

$$\leqq \pi/2+\arccos l_1(r)+\arccos l_2(r)\,,$$

as long as $\arccos l_1(r)+\arccos l_2(r) \leqq \pi/2$. Thus

$$-\sin(\arccos l_1(r)+\arccos l_2(r)) \leqq \cos \sphericalangle(W(r), Z(r))$$

$$\leqq \sin(\arccos l_1(r)+\arccos l_2(r))\,,$$

and hence $\cos^2 \sphericalangle(W(r), Z(r)) \leqq \sin^2(\arccos l_1(r)+\arccos l_2(r))$ as long as $\arccos l_1(r)+\arccos l_2(r) \leqq \pi/2$. Put

$$l(r):=\sin(\arccos l_1(r)+\arccos l_2(r))=l_2(r)\sqrt{1-l_1(r)^2}+l_1(r)\sqrt{1-l_2(r)^2}.$$

Then from the remarks stated above, $l(r)\to 0$ as $r\to 0$. So we have

$$\langle W/r, W/r\rangle-\sum_i\langle W/r, e_i\rangle^2=\|W\|^2(1-\sum_i \cos^2\measuredangle(W(r), Z_i(r))/r^2)$$
$$\geqq(H(r, b^2)^2/r^2)(1-ml^2(r))$$

and $l(r)\to 0$ as $r\to 0$.

Summarizing the above, we have

LEMMA 6.

$$j^2(\tilde{x})\geqq G(\|\tilde{x}\|, b^2)^{-2m}(H(\|\tilde{x}\|, b^2)/\|\tilde{x}\|)^2(1-ml^2(\|\tilde{x}\|)).$$

Let $r_0\leqq t_2$ be the smallest number such that $1-ml^2(r_0)=0$. If such r_0 does not exist, we consider $r_0=\infty$. Let $r_1:=\min\{r>0; \arccos l_1(r)+\arccos l_2(r)=\pi/2\}$ and $t_3:=\min\{r_0, r_1, t_0\}$ where $t_0:=\sup\{t; \exp^{\perp}|U(t)$ is injective$\}$. As in the proof of Theorem C, we have

THEOREM E. *Let $\bar{M}$ be an $(m+2)$-dimensional Riemannian manifold with the sectional curvature $a^2\leqq K_{\bar{M}}\leqq b^2$, where a, $b\geqq 0$ or pure imaginaries, and M a compact m-dimensional Riemannian manifold imbedded isometrically in $\bar{M}$. Let $\lambda\geqq 0$ be a number satisfying $\|s_v\|\leqq\lambda$ for all $v\in N_1M$. Then*

$$\text{Volume }(M)\leqq\text{Volume }(\bar{M})\left\{2\pi\int_0^{t_3}G(r, b^2)^{-m}(H(r, b^2)/r)\sqrt{1-ml^2(r)}\,dr\right\}^{-1},$$

where the function G, H, l and number t_3 are those given above.

Corresponding to Lemma 3, we will show that there exists a pair of manifolds $\bar{M}$ and M for which the number t_0 can be estimated.

LEMMA 7, *Let $\bar{M}$ be a Kähler manifold with complex dimension $m+1$ and with the sectional curvature $0<K_{\bar{M}}\leqq b^2$, $b>0$ and M a compact Kähler submanifold (imbedded) of $\bar{M}$ with complex dimension m. Let $\lambda\geqq 0$ be a number satisfying $\|s_v\|\leqq\lambda$ for all $v\in N_1M$. Then*

$$t_0:=\sup\{t; \exp^{\perp}|U(t)\text{ is injective}\}\geqq(1/b)\operatorname{arccot}(\lambda/b).$$

PROOF. Suppose $t_0<(1/b)\operatorname{arccot}(\lambda/b)$. Then by the same reason in Lemma 3, there exists a geodesic $\sigma:[0, 2t_0]\to M$ such that $\dot{\sigma}(0)\perp T_{\sigma(0)}M$ and $\dot{\sigma}(2t_0)\perp T_{\sigma(2t_0)}M$. Since M is a complex submanifold of $\bar{M}$, we can choose an orthonormal basis $e_1, \cdots, e_m, Je_1, \cdots, Je_m$ of $T_{\sigma(0)}M$ where J is the complex structure of $\bar{M}$. And let $E_1(t), \cdots, E_m(t)$ be the parallel vector fields along σ such that $E_1(0)=e_1, \cdots, E_m(0)=e_m$. Since J is parallel so JE_i and $J\dot{\sigma}$ are parallel, we see that $E_1(2t_0), \cdots, E_m(2t_0), JE_1(2t_0), \cdots, JE_m(2t_0)$ is an orthonormal basis of $T_{(2t_0)}M$. Again, since J is parallel and $s_{\dot{\sigma}(0)}$ is symmetric, we have $\langle s_{\dot{\sigma}(0)}Je_i, Je_i\rangle=-\langle s_{\dot{\sigma}(0)}e_i, e_i\rangle$ for $i=1, \cdots, m$. So trace $s_{\dot{\sigma}(0)}=\sum_{i=1}^m\{\langle s_{\dot{\sigma}(0)}e_i, e_i\rangle+\langle s_{\dot{\sigma}(0)}Je_i, Je_i\rangle\}=0$. And by the same reason, we have

$$\mathit{trace}\ s_{\dot{\sigma}(2t_0)}=\sum_{i=1}^{m}\{\langle s_{\dot{\sigma}(2t_0)}E_i(2t_0),E_i(2t_0)\rangle+\langle s_{\dot{\sigma}(2t_0)}JE_i(2t_0),JE_i(2t_0)\rangle\}=0\,.$$

Thus

$$\begin{aligned}\sum_{i=1}^{m}\Big[\int_0^{2t_0}&\{\|E_i'\|^2-\langle R(E_i,\dot{\sigma})\dot{\sigma},E_i\rangle\}dt+\langle s_{\dot{\sigma}(0)}e_i,e_i\rangle\\&-\langle s_{\dot{\sigma}(2t_0)}E_i(2t_0),E_i(2t_0)\rangle\\&+\int_0^{2t_0}\{\|JE_i'\|^2-\langle R(JE_i,\dot{\sigma})\dot{\sigma},JE_i\rangle\}dt+\langle s_{\dot{\sigma}(0)}Je_i,Je_i\rangle\\&-\langle s_{\dot{\sigma}(2t_0)}JE_i(2t_0),JE_i(2t_0)\rangle\Big]\\&=-\sum_{i=1}^{m}\int_0^{2t_0}\{\langle R(E_i,\dot{\sigma})\dot{\sigma},E_i\rangle+\langle R(JE_i,\dot{\sigma})\dot{\sigma},JE_i\rangle\}dt<0\,.\end{aligned}$$

And hence there exists a vector field X along σ normal to σ such that

$$\int_0^{2t_0}\{\|X'\|^2-\langle R(X,\dot{\sigma})\dot{\sigma},X\rangle\}dt+\langle s_{\dot{\sigma}(0)}X(0),X(0)\rangle-\langle s_{\dot{\sigma}(2t_0)}X(2t_0),X(2t_0)\rangle<0\,.$$

Then by the same reason stated in Lemma 3, we can derive a contradiction.

q. e. d.

Thus from Lemma 7, we have

THEOREM F. *Let $\bar{M}$ be a Kähler manifold with complex dimension $m+1$ and with the sectional curvature $0<K_{\bar{M}}\leqq b^2$, $b>0$ and M a compact Kähler submanifold of M with complex dimension m. Let λ be a number satisfying $\|s_v\|\leqq\lambda$ for all $v\in N_1M$. Then*

$$\mathit{Volume}\ (M)\leqq \mathit{Volume}\ (\bar{M})\Big\{2\pi\int_0^{t_3}G(r,b^2)^{-2m}(\sin br/br)\sqrt{(1-2ml^2(r))}dr\Big\}^{-1}$$

where $l(r)$ and t_3 are those used in Theorem E by putting $a^2=0$ and $t_0:=(1/b)\cdot$ $\mathrm{arccot}\,(\lambda/b)$ respectively.

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