J. Math. Soc. Japan
Vol. 30, No. 3, 1978

Some remarks on simply invariant subspaces on compact abelian groups

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(Received Jan. 17, 1977)
(Revised Sept. 28, 1977)

§ 1. Introduction.

Many results have recently been obtained concerning simply invariant subspaces on compact abelian groups. The most fundamental result in this direction is due to Helson [4] and states the existence of unitary functions in any simply invariant subspace on compact abelian group with archimedean ordered dual. In this paper we shall give among other things a generalization of this result of Helson's to the case of function algebras: Let A be a logmodular algebra and m a representing measure for A. If g is a function in $L^2(m)$ whose zero-set is of measure zero, then the $L^2(m)$-closure of Ag contains unitary functions. Moreover we shall prove the following result concerning $\mathcal{A}$-continuous cocycles of Helson [5]: Let M be a simply invariant subspace corresponding to a non-trivial $\mathcal{A}$-continuous cocycle of some special form. Then M is generated by functions with absolutely convergent Fourier series.

§ 2. Preliminaries.

Let X be a compact Hausdorff space and A a logmodular algebra on X. As is well-known ([1], [8]), every non-zero complex homomorphism of A has unique representing measure. Let m be a representing measure for A. Note that, if m is not a point mass on X, then m is a continuous measure. For each positive number p, $H^p(m)$ denotes the closure of A in the normed space $L^p(X, m)$ and $H^\infty(m)$ denotes the w^*-closure of A in $L^\infty(X, m)$. An *outer function* g in $H^p(m)$ is a function in $H^p(m)$ such that the closure of Ag in $L^p(X, m)$ coincides with $H^p(m)$ and a *unitary fuction* q is a function in $L^\infty(X, m)$ with $|q|=1$ almost everywhere. By an *invariant subspace* we mean a closed subspace M of $L^2(X, m)$ such that $AM\subset M$. An invariant subspace M is *doubly invariant* if $\bar{A}M\subset M$. We shall call an invariant subspace M a *simply invariant* if M is not doubly invariant.

Next, let K be a compact abelian group, not a circle, dual to a subgroup

Γ of the discrete real line R_d. $\mathfrak{A}$ is the space of all continuous analytic functions on K, i.e., the set of all continuous functions on K whose Fourier coefficients a_λ vanish for all negative λ in Γ. Then $\mathfrak{A}$ is a Dirichlet algebra, so is logmodular, and the normalized Haar measure σ on K is a representing measure for $\mathfrak{A}$. Let T_t be the translation operator,

$$T_t f(x) = f(x+e_t),$$

where e_t is the element of K defined by $e_t(\lambda)=e^{it\lambda}$ for all λ in Γ. The mapping from t to e_t embeds the real line R continuously onto a dense subgroup K_0 of K. A family of unitary functions $A=\{A_t\}$ in $L^\infty(K, \sigma)$ with the following properties is called *cocycle:*

(i) $|A_t(x)|=1$ almost everywhere,

(ii) A_t moves continuously in $L^2(K, \sigma)$ as a function of t,

(iii) $A_{t+u}=A_t T_t A_u$ for each real t, u in R.

A cocycle is a *coboundary* if it is of the form $\varphi(x)\cdot\overline{\varphi(x+e_t)}$, where φ is a unitary function in $L^\infty(K, \sigma)$. A one to one correspondence was established in [**3**] between normalized simply invariant subspaces and cocycles on K.

In our discussion in the forthcoming sections, we frequently use the following lemma which is a corollary of Szgö's theorem.

LEMMA 2.1. *Let A be a logmodular algebra on X and let m be a representing measure for A. If f is a function in $L^2(X, m)$ such that* $\log|f|$ *is summable, then $f=ph$ with a unitary p in $L^\infty(X, m)$ and an outer h in $L^2(X, m)$. The factoring is unique, up to a constant factor of modulus one.*

§ 3. Existence theorem.

Helson [**4**] showed that every simply invariant subspace on K contains a function f in $L^2(K, \sigma)$ such that $\log|f|$ is summable. We shall extend this and a few other results to the case of logmodular algebras. In 3.1, 3.2, 3.5, and 3.6 we assume that A is a logmodular algebra on a compact Hausdorff space X and m is a representing measure for A. For any g in $L^2(X, m)$, M_g denotes the smallest invariant subspace containing g.

THEOREM 3.1. *If the zero-set of g in $L^2(X, m)$ is of m-measure zero, then the invariant subspace M_g generated by g contains a function h such that* $\log|h|$ *is summable.*

COROLLARY 3.2. *If the zero-set of g in $L^2(X, m)$ is of m-measure zero, then M_g contains a unitary function.*

In order to prove Theorem 3.1, we need two lemmas.

LEMMA 3.3 ([**5**; Chap. 2, 5, Lemma 1]). *Let μ be the normalized Haar measure on T^∞, the infinite dimensional torus, and $\{a_n\}$ be any square-summable sequence of numbers. Then*

$$\int_{T^\infty} \log \left| \sum_{n=1}^{\infty} a_n e^{i\theta_n} \right| d\mu(e^{i\theta_1}, e^{i\theta_2}, \cdots)$$
$$\geqq \max \{\log |a_n| : n=1, 2, 3, \cdots\}.$$

LEMMA 3.4. *Let ν be a bounded positive Borel regular measure on a compact Hausdorff space X, and let E be a Borel subset of X. If ν is continuous, then, for any α with $0 \leqq \alpha \leqq 1$, there exists Borel subset F_α of E such that $\nu(F_\alpha)=\alpha \cdot \nu(E)$.*

Lemma 3.4 is well-known, so we omit the proof.

PROOF OF THEOREM 3.1. We may assume that m is a continuous measure. Put $Z(g)=\{x \in X: g(x)=0\}$. By hypothesis, $m(Z(g))=0$. Let $p=\min(1, |g|^{-1})$, then $\log p$ is summable. Hence there is an outer function h such that $|h|=p$ by Lemma 2.1. Since $M_g=M_{hg}$ and hg is in $L^\infty(X, m)$, we may assume that g is in $L^\infty(X, m)$, and $\|g\|_\infty=1$. We set

$$H_n=\{x \in X : 1/n \leqq |g(x)| \leqq 1\}.$$

Since the complement of $\bigcup_{n=1}^{\infty} H_n$ is $Z(g)$, $m(\bigcup_{n=1}^{\infty} H_n)=1$. Therefore there exists k_1 such that $m(H_{k_1})>1/2$. We can choose a Borel subset G_1 of H_{k_1} such that $m(G_1)=1/2$ by Lemma 3.4. By induction, it is not hard to find sequences $\{k_n\}$ of indices and $\{G_n\}$ of Borel sets such that

$$H_{k_n} \setminus \bigcup_{i=1}^{n-1} G_i \supset G_n, \qquad m(G_n)=2^{-n}.$$

We define $p_n=\min(k_n^{-1}, |g|^{-1})$, so $\log p_n$ is summable. Hence there exists an outer function h_n in $H^\infty(m)$ such that $|h_n|=p_n$. Note that $h_n g$ is in M_g and $|h_n g|=1$ on G_n. Since $\|h_n g\|_2 \leqq 1$, the function

$$F_\theta(x)=\sum_{n=1}^{\infty} n^{-2} e^{i\theta_n}(h_n g)(x)$$

is in M_g for any point $\theta=(\theta_1, \theta_2, \cdots)$ in T^∞. By Fubini's theorem and Lemma 3.3, we have

$$\int_{T^\infty}\int_X \log |F_\theta(x)| dm(x) d\mu(\theta)$$
$$\geqq \int_X \sup_n \log |n^{-2}(h_n g)(x)| dm(x)$$
$$\geqq \sum_{n=1}^{\infty} \int_{G_n} \log (n^{-2}) dm(x)$$
$$= \sum_{n=1}^{\infty} \log (n^{-2}) 2^{-n} > -\infty .$$

Therefore $\log |F_\theta(x)|$ is summable for μ-almost all θ in T^∞. This completes the proof.

Next we shall give a generalization of one result in [4]. For any family $\mathcal{F}$ of measurable functions, we write:

$$|\mathcal{F}|=\{|f| : f \text{ is in } \mathcal{F}\}.$$

PROPOSITION 3.5. *Suppose that $H^\infty(m)$ is maximal among w^*-closed subalgebras of $L^\infty(X, m)$. If M is simply invariant subspace, then $|M|=|H^2(m)|$.*

PROOF. Let $\tilde{M}$ be the set of all h in $L^2(X, m)$ such that fh is in $H^1(m)$ for all f in M. Then $\tilde{M}$ is a simply invariant subspace. It follows from Szgö's theorem that the space of all bounded functions in M (resp. $\tilde{M}$) is dense in M (resp. $\tilde{M}$). Since M and $\tilde{M}$ are simply invariant, we see that there exist a bounded function f in M and a bounded function g in $\tilde{M}$ such that fg is not identically equal to zero. Since fg is in $H^\infty(m)$, it follows from [**9**; Theorem] that $Z(f)$ and $Z(g)$ are m-measure zero. Therefore, we see that both M and $\tilde{M}$ have unitary functions by Corollary 3.2. Thus we have $|M|=|H^2(m)|$.

PROPOSITION 3.6. *If g is a continuous function such that the zero set of g, $Z(g)$, is of m-measure zero, then M_g contains a continuous function h such that* $\log|h|$ *is in $L^1(X, m)$.*

PROOF. We may assume that m is a continuous measure and $\|g\|_\infty=1$. Since A is logmodular, for any positive real-valued continuous function p and any given $\varepsilon>0$, we can find f in A such that $\||f|-p\|_\infty<\varepsilon$. Let H_{k_n} and G_n be as in the proof of Theorem 3.1. Put $h_n=\min(n, |g|^{-1})$, so h_n is positive continuous function on X. Therefore there exists f_n in A such that $\||f_n|-h_n\|_\infty<2^{-1}$. Since

$$\|h_n|g|-|f_ng|\|_\infty<2^{-1} \quad\text{and}\quad |h_{n_k}g|=1 \quad\text{on}\quad G_k,$$

we have $|f_{n_k}g|>2^{-1}$ on G_k. On the other hand, $\|f_ng\|_\infty<3/2$, so for any $\theta=(\theta_1, \theta_2, \cdots)$ in T^∞,

$$F_\theta(x)=\sum_{n=1}^{\infty} n^{-2}e^{i\theta_n}(f_ng)(x)$$

is a continuous function in M_g. And we can see $\log|F_\theta(x)|$ is in $L^1(X, m)$ for μ-almost all θ by the same way as in the proof of Theorem 3.1. This completes the proof.

REMARK. We put $X=K$, a compact abelian group, not a circle, which has an archimedean ordered dual, then there exists a continuous function f such that

$$\rho(f)=\int_{-\infty}^{\infty}\log|f(x+e_t)|\frac{1}{1+t^2}dt>-\infty$$

and $\log|f|$ is not in $L^1(K, \sigma)$. So M_f is simply invariant, for it is known that this is the case if and only if $\rho(f)>-\infty$ (cf. [**5**; Theorem 22]). By Proposition 3.6, we see that M_f contains a continuous function h such that $\log|h|$ is in $L^1(K, \sigma)$.

We can extend Theorem 3.1 to the case of w^*-Dirichlet algebras which were introduced by Srinivasan and Wang [**10**]. Recall that by definition a

w^*-Dirichlet algebra is an algebra A of essentially bounded measurable function on a probability measure space $(X, \mathfrak{B}, m)$ such that A contains constant functions, $A+\bar{A}$ is w^*-dense in $L^\infty(X, m)$, and m is multiplicative on A (cf. [**10**]). We define $H^p(m)$, $0<p\leqq\infty$, and invariant subspaces in the same way as in section 2.

PROPOSITION 3.7. *Let A be a w^*-Dirichlet algebra on a probability measure space $(X, \mathfrak{B}, m)$. If the zero-set of g in $L^2(X, m)$ is of m-measure zero, then M_g contains a function h in $L^2(X, m)$ such that* $\log|h|$ *is summable.*

PROOF. We may regard $H^\infty(m)$ as a logmodular algebra on Ω which is the maximal ideal space of $L^\infty(X, m)$. On the other hand, the zero-set of Gelfand transform of g has $\hat{m}$-measure zero, where $\hat{m}$ is the Radonization of m (cf. [**10**; 2.4]). Therefore Proposition 3.7 follows from Theorem 3.1.

§ 4. $\mathcal{A}$-continuous cocycles.

Let $\mathcal{A}$ be the Banach algebra of all functions on K which have absolutely convergent Fourier series. A cocycle $A=\{A_t\}$ is a *$\mathcal{A}_H$-cocycle* if there exist a unitary function q in $\mathcal{A}$ and a function m in $\mathcal{A}$ with Fourier coefficient m_λ satisfying

$$\sum_{0<\lambda<1}|m_\lambda \log\lambda|<\infty$$

such that

$$A(t, x)=\exp\left\{i\int_0^t m(x+e_u)du\right\}\cdot q(x)\overline{q(x+e_t)}\,.$$

Note that $\mathcal{A}_H$-cocycle is an $\mathcal{A}$-continuous cocycle, i.e., $A_t\in\mathcal{A}$ for all t in R. Helson [**5**; Theorem 31] has shown that any simply invariant subspace corresponding to $\mathcal{A}_H$-cocycle has non-null elements of $\mathcal{A}$ (cf. [**11**; Theorem 2]). In this section we shall show that non-trivial invariant subspaces of this sort are generated by elements of $\mathcal{A}$, and give some remarks on closed ideals in function algebra $\mathfrak{A}$ which consists of all generalized analytic functions.

THEOREM 4.1. *Let M be a simply invariant subspace corresponding to a non-trivial $\mathcal{A}_H$-cocycle. Then M is generated by two unitary functions in $\mathcal{A}$.*

In order to prove Theorem 4.1, we need the following lemmas. The first one is a weaker version of [**5**; Theorem 32].

LEMMA 4.2. *If f is an element of $\mathcal{A}$ and non-vanishing on K (so* $\log|f|$ *is in $L^1(K, \sigma)$), then the unitary and outer factors of f are both in $\mathcal{A}$.*

LEMMA 4.3. *If $f_1, \cdots, f_n$ are continuous functions which have no common zeros on K, then there exist trigonometric polynomials $p_1, \cdots, p_n$ such that $p_1f_1+\cdots+p_nf_n$ is non-vanishing on K.*

PROOF. $C(K)$ denotes the space of all complex-valued continuous functions on K. Let J be the closed ideal of $C(K)$ generated by $f_1, \cdots, f_n$. Since $f_1, \cdots, f_n$

have no common zeros and the maximal ideal space of $C(K)$ is K, J coincides with $C(K)$. Since the set of all trigonometric polynomials is dense in $C(K)$, it follows that there exist trigonometric polynomials $p_1, \cdots, p_n$ such that $p_1 f_1 + \cdots + p_n f_n$ is non-vanishing on K.

PROOF OF THEOREM 4.1. We can find g in M such that g is an element of $\mathcal{A}$ and g is orthogonal to $\chi_\tau \cdot M$ for some positive τ in Γ (cf. [**5**; Theorem 31]). Since

$$x+K_0=\{x+e_t\,;\ t \ \text{ in } \ R\}$$

is dense in K, there exist $t_1, \cdots, t_n$ such that $g, T_{t_1}g, \cdots, T_{t_n}g$ have no common zeros. Since

$$A_t T_t g = -\int_0^\tau e^{it\lambda} dP_\lambda g$$

for the orthogonal projection P_λ from $L^2(K, \sigma)$ to $\chi_\lambda \cdot M$, it follows that $A_{t_1}T_{t_1}g, \cdots, A_{t_n}T_{t_n}g$ are continuous functions in M. From Lemma 4.3, we have trigonometric polynomials $p_0, \cdots, p_n$ such that

$$F' = p_0 g + p_1 A_{t_1} T_{t_1} g + \cdots + p_n A_{t_n} T_{t_n} g$$

is non-vanishing on K. Since $p_0, \cdots, p_n$ are trigonometric polynomials, there exists a positive λ in Γ such that $\chi_\lambda p_0, \cdots, \chi_\lambda p_n$ are analytic trigonometric polynomials. Hence $F = \chi_\lambda \cdot F'$ is an element of $\mathcal{A} \cap M$ which is orthogonal to $\chi_{\tau+\lambda} \cdot M$. We see that there exists a G in $\mathcal{A} \cap M$ such that G is orthogonal to $\chi_{\tau+\lambda} \cdot M$ and is not contained in $\chi_\nu \cdot M$ for any positive ν in Γ. In fact, if F is in $\chi_{\nu_1} \cdot M$ for some positive ν_1 in Γ, then there exists a positive μ_1 such that $F_1 = \bar{\chi}_{\mu_1} F$ is contained in M and not in $\chi_{\nu_1} \cdot M$. But F_1 may be contained in $\chi_{\nu_2} \cdot M$ where $0 < \nu_2 < (1/2)\nu_1$. Repeat the procedure to find a function F_2 in M and is not in $\chi_{\nu_2} \cdot M$. We continue in this way infinitely if necessary. $\|\cdot\|_{\mathcal{A}}$ denotes the norm of $\mathcal{A}$, and set

$$G = F + \sum_{n=1}^{\infty} a \cdot F_n 2^{-(n+1)} \|F_n\|_{\mathcal{A}}^{-1}$$

where $a = \min\{|F(x)|\,;\ x \text{ in } K\}$. Then it is not hard to see that G has the desired properties (cf. [**11**; Theorem 3]). Since $\log|G|$ is summable, $G = qh$ where q is unitary and h is outer. By Lemma 4.2, q and h are both elements in $\mathcal{A}$. So $B(t, x) = A(t, x)\overline{q(x)}q(x+e_t)$ is an $\mathcal{A}$-continuous cocycle. By the same way as in the proof of [**5**; Theorem 26], we see that $B(t, x)$ is a Blaschke cocycle such that the zeros of $B(z, x)$ do not accumulate on the real axis for almost all x. From the proof of [**5**; Theorem 33], we can choose u in R such that q and $A_u T_u q$ generate M. This completes the proof.

PROPOSITION 4.4. *There exist non-trivial analytic* (*Blaschke type*) *$\mathcal{A}_H$-cocycles.*

PROOF. We can construct non-trivial $\mathcal{A}_H$-cocycles by a method similar to the one used in [6]. From the proof of Theorem 4.1, we have the existence of such cocycles.

COROLLARY 4.5. *Let $\mathfrak{A}$ be the function algebra which consists of all continuous analytic functions on K. Then there exists a closed ideal I in $\mathfrak{A}$ such that the $L^2(K, \sigma)$-closure of I has a non-trivial cocycle.*

PROOF. Let $A=\{A_t\}$ be a non-trivial $\mathcal{A}_H$-cocycle which is analytic, and let M be the simply invariant subspace corresponding to $\bar{A}=\{\bar{A}_t\}$. Since A is analytic, M is contained in $H^2(\sigma)$ (cf. [5; Theorem 21]). On the other hand, M is generated by elements of $\mathcal{A}$ by Theorem 4.1. We set I is the set of all continuous functions in M. Then I is a closed ideal of $\mathfrak{A}$ which has desired properties.

REMARK. The closed ideals of the disc algebra are completely known (see [2]). But it must be difficult to describe the closed ideals of function algebra which consists of all generalized analytic functions by the similar way as in [2]. The corollary above shows that there exists an ideal whose $L^2(K, \sigma)$-closure is a peculiar invariant subspace.

ACKNOWLEDGEMENT. The author wishes to express his sincere gratitude to the referee whose suggestions improved the paper very much.

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Added in proof: After this paper was submitted, the author has found another proof of Theorem 3.1. For the proof, see our paper: *A note on Helson's existence theorem,* which will appear in Proc. Amer. Math. Soc.