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A theory of ordinal numbers with Ackermann's schema

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Introduction.

W. Ackermann introduced in [1] a system of axiomatic set theory. A typical character of the system is that the universe V of all sets is a part of the individual domain and besides it is an individual. The system has an interesting axiom schema which generates the sets. It is the following:

$$y_1 \cdots y_n \in V \rightarrow . \forall x[A(x) \rightarrow x \in V] \rightarrow \exists w \in V \forall x[x \in w \leftrightarrow A(x)],$$

where $A(x)$ contains neither the individual constant V nor free variables other than $y_1 \cdots y_n, x$.

In this paper, we modify the above schema to formalize a theory of ordinal numbers and study the strength of the theory. Some theories of ordinal number have been given in Takeuti [6]-[9]. The main purpose of [6]-[9] seems to construct theories of ordinal numbers in which a model of ZF can be constructed. Our interests here are the application itself of Ackermann's schema to the formalization of a theory of ordinal numbers and the degree of the strength of the theory.

The basic logic which we adopt is a second order calculus with the axiom of weak comprehension of the following form:

$$\exists P \forall x_1 \cdots x_n [P x_1 \cdots x_n \leftrightarrow O x_1 \wedge \cdots \wedge O x_n \wedge A],$$

where O is a predicate constant which means "...is an ordinal" and A is any formula. The reason why we weakened the axiom of comprehension in such a form consists of the following two:

1. The author could not estimate the strength of the corresponding theory formalized in the usual second order calculus.

2. Ackermann's set theory has the axiom schema (in the form of Levy and Vaught [4]) $\exists x \forall y [y \in x \leftrightarrow y \in V \wedge A]$ and our weakened axiom of comprehension could be regarded as its natural representation in a second order calculus.

In §1, we explain the source of the form of the mathematical axioms given in §2. In §2, we give the formal system of the theory denoted by OA. In §3, we develop the theory and show the existence of Ω which is defined by $\forall x[x<\Omega \leftrightarrow Ox]$ and the inaccessibility of Ω which corresponds to the replacement schema (in the form given in Hanazawa [2]) of Ackermann's set theory. The arguments proceed in parallel with the ones in Reinhardt [5]. In §4, we show that $OA \vdash A^{\circ}$ is equivalent to $ZFL-P \vdash A$ for every formula A of the language of ZF, where A° means the routine interpretation of A in ordinal number theory and $ZFL-P$ means ZF set theory without the axiom of power set and with the axiom of constructibility.

§1. The source of our axioms.

Let us consider the following procedure by which we construct an ordinal: For the set S of all ordinals which have been constructed (or defined) until now, we introduce an entity as the minimum ordinal which is larger than the ordinals in S. The following schema is an innocent formalization of this procedure:

$$\text{(A)} \qquad \forall x[A(x) \rightarrow \forall y<xA(y)] \wedge \forall x[A(x) \rightarrow Ox] \\ \rightarrow \exists! y \forall z[z<y \leftrightarrow A(z)] \wedge \exists y[Oy \wedge \forall z[z<y \leftrightarrow A(z)]],$$

where A is any formula and O denotes the class of the ordinals. But, if we assume the provability of $\forall x[Ox \rightarrow \forall y<xOy]$ and $\forall x \forall y[Ox \wedge Oy \wedge x<y \rightarrow x \neq y]$, the schema leads to the familiar contradiction. To avoid that contradiction, following the idea given in the system of Ackermann's set theory, we restrict the schema as follows: In the schema (A), A denotes a formula which contains neither the predicate symbol O nor parameters other than elements of O (we shall call such a formula a primitive formula). By (A′), we denote the schema (A) with this restriction.

On the other hand, we could assume that all the ordinals are generated only by that procedure. The following is a direct representation of this principle:

$$\text{(B1)} \qquad [(\forall \text{ primitive } A)\ B(P, A)] \rightarrow \forall x[Ox \rightarrow Px],$$

where $B(P, A)$ is a formula which results from the formula (A) by substituting the predicate variable P for the predicate constant O. We must replace "($\forall$ primitive A)..." by a formal expression. Note that (B1) is equivalent to the following schema:

$$\text{(B2)} \qquad \forall x[Px \rightarrow Ox] \wedge (\forall \text{ primitive } A)[B(P, A)] \rightarrow \forall x[Ox \rightarrow Px].$$

(It is evident that (B1) implies (B2). To see the converse, suppose the antecedent of (B1) and put Qx equal to $Px\wedge Ox$. Using (A′), we see that $B(P, A)$ implies $B(Q, A)$. So the desired conclusion follows by applying (B2) to Q.)
(B2) is written as follows:

$$\forall x[Px\rightarrow Ox]\wedge(\forall \text{ primitive } A)[\forall x[A(x)\rightarrow\forall y<xA(y)] \\ \wedge\forall x[A(x)\rightarrow Px]\rightarrow\cdots]\rightarrow\cdots$$

Note that $\forall x[A(x)\rightarrow Px]$ implies $\forall x[A(x)\rightarrow Ox]$ under the assumption $\forall x[Px\rightarrow Ox]$. On the other hand, if A is a primitive formula and satisfies $\forall x[A(x)\rightarrow\forall y<xA(y)]\wedge\forall x[A(x)\rightarrow Ox]$, then by (A′) there exists an ordinal u (i.e. an element of O) such that $\forall z[z<u\leftrightarrow A(z)]$. So the predicate $\lambda xA(x)$ in (B2) can be replaced by the predicate $\lambda x[x<u]$ for some ordinal u. Using this fact, we can finally rewrite (B2) equivalently to the following:

(B3) $$\forall u[Ou\wedge\forall x[x<u\rightarrow\forall y<x[y<u]]\wedge\forall x[x<u\rightarrow Px] \\ \rightarrow\exists!y\forall z[z<y\leftrightarrow z<u]\wedge Pu].\rightarrow.\forall x[Ox\rightarrow Px].$$

(The clause $\forall x[Px\rightarrow Ox]$ has been omitted from the antecedent by the same reason as above.)

Thus we have the mathematical axioms (A′) and (B3). In §2, we will separate each of (A′) and (B3) into two axioms because they are too long. (A′) is equivalent to the following (a) and (b):

(a) $$Ou_1\wedge\cdots\wedge Ou_n\rightarrow.\forall x[A(x)\rightarrow\forall y<xA(y)\wedge Ox] \\ \rightarrow\exists y[Oy\wedge\forall z[z<y\leftrightarrow A(z)]],$$

where A is a primitive formula which contains no free variables other than $u_1,\cdots,u_n, x$;

(b) $$Oa\wedge\forall x[x<a\leftrightarrow x<b]\rightarrow a=b.$$

(B3) is equivalent to the following (c) and (d) under (a) and (b):

(c) $$\forall u[Ou\wedge\forall x<uPx\rightarrow Pu]\rightarrow\forall x[Ox\rightarrow Px];$$

(d) $$\forall u[Ou\rightarrow\forall x<u\forall y<x[y<u]].$$

(The formula (d) can be proved from (B3) and hence the formula (c) can be proved from (B3) and (b).)

§ 2. The system of the theory OA.

This theory is developed in a second order language.

2.1. The language has the following symbols:

(a) Individual variables $v_1, v_2, \cdots$;

(b) Predicate variables $P_1, P_2, \cdots$, where P_i has j argument places, where $i=2^j(2k+1)$;

(c) Predicate constants $O*$, $*<*$, $*=*$;

(d) Logical symbols $\neg$, $\rightarrow$, $\exists$.

2.2. Formulas are defined as usual. The abbreviations $\wedge$, $\vee$, $\forall$, $\leqq$, $\forall x<a[\ldots]$, etc. are used in the standard way.

When a formula A contains neither predicate constant O nor free predicate variables, we shall call it a P-formula.

2.3. Axioms and inference rules.

2.3.1. Inference rules.

(a) $\dfrac{A \quad A\rightarrow B}{B}$.

(b) $\dfrac{A\rightarrow B}{\exists\alpha A\rightarrow B}$, where α is an individual variable or a predicate variable which is not free in B.

2.3.2. Logical axioms.

(a) $A\rightarrow[B\rightarrow A]$.

(b) $[A\rightarrow[B\rightarrow C]]\rightarrow[[A\rightarrow B]\rightarrow[A\rightarrow C]]$.

(c) $[\neg A\rightarrow\neg B]\rightarrow[B\rightarrow A]$.

(d) $A(a)\rightarrow\exists xA(x)$.

(e) $A(P)\rightarrow(\exists Q)A(Q)$.

(f) The axiom schema of comprehension:

$$\exists P\forall x_1\cdots x_n[Px_1\cdots x_n\leftrightarrow Ox_1\wedge\cdots\wedge Ox_n\wedge A],$$

where A is any formula in which P does not occur.

REMARK. Except for the axiom schema of comprehension, this is a standard system of second order predicate calculus.

2.3.3. Equality axioms.

2.3.4. Mathematical axioms.

A1: $\forall a, b[Oa \wedge \forall x[x<a \leftrightarrow x<b] \rightarrow a=b]$.

A2: $\forall a[Oa \wedge x<a \wedge y<x \rightarrow y<a]$.

A3: $\forall P[\forall x[Ox \rightarrow [(\forall y<x)Py \rightarrow Px]] \rightarrow \forall x[Ox \rightarrow Px]]$.

A4: $\forall a_1 \cdots a_n[Oa_1 \wedge \cdots \wedge Oa_n$
$\rightarrow [\forall x[A(x) \rightarrow (\forall y<x)A(y) \wedge Ox] \rightarrow \exists y[Oy \wedge \forall z[z<y \leftrightarrow A(z)]]]]$,

where $A(x)$ is a P-formula which contains no free variables other than $x, a_1, \cdots, a_n$.

§3. Development of the theory.

In this section, we raise some formal theorems of OA. To save the spaces, the formal proofs will be given through informal arguments (in OA).

CONVENTION 1. (a) We use well-known ι-symbol. An ι-term $\iota xA(x)$ is said to be a P-term if $A(x)$ is a P-formula. Note that every formula which results from a P-formula by substituting P-terms for the free variables is also a P-formula. (b) We abbreviate a formula of the form

$$\exists P\forall x_1 \cdots x_n[Px_1 \cdots x_n \leftrightarrow A(x_1 \cdots x_n)]$$

by "the predicate $\lambda x_1 \cdots x_n A(x_1 \cdots x_n)$ is admissible." Note that

(a) $OA \vdash$ "$\lambda x[Ox \wedge A]$ is admissible" and

(b) $OA \vdash$ "$\lambda x_1 \cdots x_n A$ is admissible" $\rightarrow [(\forall P)B(P) \rightarrow B(\lambda x_1 \cdots x_n A)]$.

3.1. THEOREM 1.

(a) $OA \vdash \forall x[Ox \wedge (\forall y<x)A(y) \rightarrow A(x)] \rightarrow \forall x[Ox \rightarrow A(x)]$,
for every formula $A(x)$.

(b) $OA \vdash \exists x[Ox \wedge A(x)] \rightarrow \exists x[Ox \wedge A(x) \wedge \forall y<x \neg A(y)]$,
for every formula $A(x)$.

PROOF. $\lambda x[Ox \wedge A(x)]$ and $\lambda x[Ox \wedge \neg A(x)]$ are admissible. So they follow immediately from A3.

3.2. THEOREM 2.

(a) $OA \vdash Oa \wedge b<a \rightarrow Ob$.

(b) $OA \vdash Oa \wedge Ob \wedge a<b \rightarrow a \neq b$.

(c) $OA \vdash Oa \wedge Ob \rightarrow a<b \vee a=b \vee b<a$.

PROOF.

(a) Substitute $(\forall y<x)Oy$ for Px in Theorem 1.

(b) Substitute $\forall z<x[z \neq x]$ for Px in Theorem 1.

(c) Let $A(x)$ be the formula $Ox \wedge \forall y[Oy \rightarrow x<y \vee y \leqq x]$. By Theorem 1, it suffices to show that Oa and $(\forall z<a)A(z)$ imply $A(a)$. Suppose Oa and $(\forall z<a)A(z)$. Let $B(y)$ be the formula $a<y \vee y \leqq a$. Again by Theorem 1, to prove $A(a)$, it suffices to show that Ob and $(\forall w<b)B(w)$ imply $B(b)$. Suppose Ob and $(\forall w<b)B(w)$. Now assume $\neg a<b \wedge \neg b<a$. Then we obtain the following (i) and (ii): (i) $\forall z<a[z<b]$ (Assume $z<a$. Then $A(z)$. So $z<b \vee b \leqq z$. On the other hand $\neg b \leqq z$ since $z<a \wedge \neg b<a \wedge Oa$ and A2. Hence $z<b$); (ii) $\forall w<b[w<a]$ (Assume $w<b$. Then $B(w)$, i.e. $a \leqq w \vee w<a$. On the other hand $\neg a \leqq w$ since $w<b \wedge \neg a<b \wedge Ob$. Hence $w<a$). From (i), (ii) and Oa, we obtain $a=b$ by A1.

3.3. THEOREM 3.

(a) $OA \vdash \exists ! x \forall y[\neg y<x]$. (*We write* 0 *for* $\iota x \forall y[\neg y<x]$.)

(b) $OA \vdash O0$.

PROOF. Its existence in O comes from A4 (Take the formula $x \neq x$ as $A(x)$). Its uniqueness comes from A1.

Note that 0 is a P-term.

3.4. The successor function. Let $S(x, y)$ be the formula $\forall z[z<y \leftrightarrow z \leqq x]$. By x', we denote the P-term $\iota y[\exists ! z S(x, z) \wedge S(x, y). \vee . \neg \exists ! z S(x, z) \wedge y=0]$.

THEOREM 4.

(a) $OA \vdash Oa \rightarrow Oa'$.

(b) $OA \vdash Oa \rightarrow a<a' \wedge \forall x \neg [a<x<a']$.

PROOF. Take the formula $x \leqq a$ as $A(x)$ in A4.

3.5. An infinite ordinal. Let $N(x)$ be the formula

$$\forall P[P0 \wedge \forall y[Py \rightarrow Py'] \rightarrow Px].$$

THEOREM 5.

(a) $OA \vdash \exists ! z \forall x [x<z \leftrightarrow N(x)]$. (*We write* ω *for* $\iota z \forall x [x<z \leftrightarrow N(x)]$.)

(b) $OA \vdash O\omega$.

(c) $OA \vdash 0<\omega \wedge \forall a [a<\omega \rightarrow a'<\omega]$.

(d) $OA \vdash A(0) \wedge \forall x [A(x) \rightarrow A(x')] \rightarrow \forall x<\omega A(x)$, *for every formula A.*

PROOF of (a). We see $\forall x [N(x) \rightarrow Ox]$, since $O0 \wedge \forall y [Oy \rightarrow Oy']$ by the previous two theorems. Besides $N(x) \rightarrow (\forall z<x) N(z)$. (To see this, suppose $N(x)$. The predicate $\lambda z (\forall w \leqq z) N(w)$ is admissible since $\forall x [N(x) \rightarrow Ox]$. So we can take this predicate as P in $\forall P [P0 \wedge \forall y [Py \rightarrow Py'] \rightarrow Px]$.) Since $N(x)$ is a P-formula, we obtain by A4 that $\exists z [Oz \wedge \forall x [x<z \leftrightarrow N(x)]]$. Its uniqueness comes from A1.

3.6. The individual Ω.

We shall show the existence of such an individual Ω that $\forall x [x<\Omega \leftrightarrow Ox]$ and exhibit its fundamental properties.

3.6.1. THEOREM 6. (*Undefinability of O*)

$$OA \vdash Oa_1 \wedge \cdots \wedge Oa_n \rightarrow \neg \forall x [A(x) \leftrightarrow Ox],$$

where A is any P-formula which contains no free variable other than $a_1, \cdots, a_n, x$.

PROOF. Suppose $Oa_1 \wedge \cdots \wedge Oa_n \wedge \forall x [A(x) \leftrightarrow Ox]$. Then by A4, there exists y such that $Oy \wedge \forall z [z<y \leftrightarrow A(z)]$. Then $y<y \leftrightarrow A(y)$. Since Oy implies $y \not< y$, we see that $\neg A(y)$ and Oy.

3.6.2. Introduction of Ω.

DEFINITION of the predicate O^*.

$L_1(b)$ is the formula $\forall x<b \forall y<x [y<b]$,

$L_2(b)$ is $\forall x [\forall y [y<x \leftrightarrow y<b] \rightarrow x=b]$,

$L_3(b)$ is $\forall x<b \forall y<b [x<y \vee y \leqq x]$,

$W_1(b)$ is $\exists P \forall x [Px \leftrightarrow x<b]$,

$W_2(b)$ is $\forall P [\exists x \leqq b Px \rightarrow \exists x \leqq b [Px \wedge \forall y<x \neg Py]]$,

$W_3(b)$ is $\forall P \exists Q \forall x [Qx \leftrightarrow x \leqq b \wedge \neg Px]$

and

$O^*(a)$ is $\forall b \leqq a [L_1(b) \wedge \cdots \wedge L_3(b) \wedge W_1(b) \wedge \cdots \wedge W_3(b)]$.

REMARK. All the members of O^* have the property $W_2(b)$. But this fact does not imply that O^* is well ordered by $<$. What we can assert is that if $A(x)$ is admissible, it holds that

$$\forall x[O^*(x) \wedge \forall y < x A(y) \to A(x)] \to \forall x[O^*(x) \to A(x)].$$

THEOREM 7. *The following formulas are provable in OA:*

(a) $O^*(a) \wedge x < a \to O^*(x)$.

(b) $Oa \to O^*(a)$.

(c) $O^*(a) \wedge O^*(b) \to a < b \vee b \leqq a$.

(d) $\exists x[O^*(x) \wedge \neg Ox]$.

(e) $\exists ! x[O^*(x) \wedge \neg Ox \wedge \forall y < x Oy]$.

PROOF. (a) Suppose $O^*(a)$ and $x < a$. Let $b \leqq x$. Then $b < a$ by $L_1(a)$ and $x < a$. Hence $L_1(b) \wedge \cdots \wedge W_3(b)$ by $O^*(a)$.

(b) $Ox \wedge y < x \to Oy$ and $Ox \to L_1(x) \wedge \cdots \wedge W_3(x)$.

(c) Suppose $O^*(a)$, $O^*(b)$, $\neg a < b$ and $\neg b < a$. Then the predicate $\lambda x[x < a]$ is admissible by $W_1(a)$. So, the predicate $\lambda x[x \leqq b \wedge \neg x < a]$ is admissible by $W_3(b)$. So, by $W_2(b)$ and $\neg b < a$, there exists a v such that $v \leqq b \wedge \neg v < a \wedge \forall x < v[x < a]$ because $x < v \wedge v \leqq b$ implies $x \leqq b$ by $L_1(b)$. Take such a v. Then $\neg a < v$ since $\neg a < b$. Since $v \leqq b$, we see $O^*(v)$ by Theorem 7 (a). So, in the same way as above, we obtain $\exists u \leqq a[\neg u < v \wedge \forall y < u[y < v]]$. Take such a u. Then we have $\forall y < v[y < u]$. (To see this, suppose $y < v$. Then $y < a$. So, $y < u \vee u \leqq y$ since $L_3(a)$ and $u \leqq a$. So, $y < u$ since $\neg u < v \wedge y < v \wedge L_1(v)$ implies $\neg u \leqq y$.) Thus we obtain $\forall y[y < u \leftrightarrow y < v]$. So $u = v$ by $L_2(v)$. Hence we obtain easily $a = v \leqq b$ which implies $a = b$ since $\neg a < b$.

(d) $O^*(x)$ is a P-formula. Hence $\exists x[O^*(x) \wedge \neg Ox]$ follows immediately from (b) and the undefinability of O.

(e) Take a b such that $O^*(b) \wedge \neg Ob$. Since $W_3(b)$ by $O^*(b)$, the predicate $\lambda x[x \leqq b \wedge \neg Ox]$ is admissible. Hence, by $W_2(b)$, there exists an x such that $x \leqq b \wedge \neg Ox \wedge \forall y < x \neg[y \leqq b \wedge \neg Oy]$ which is equivalent to $x \leqq b \wedge \neg Ox \wedge \forall y < x Oy$ since $L_1(b)$. Uniqueness of such an x follows from (a) and (c).

By Ω, we denote $\iota x[O^*(x) \wedge \neg Ox \wedge \forall y < x Oy]$.

THEOREM 8. (a) $OA \vdash \forall x[x < \Omega \leftrightarrow Ox]$.

(b) $OA \vdash \forall x[x < a \leftrightarrow Ox] \to a = \Omega$.

PROOF. (a) $a<\Omega \rightarrow Oa$ is trivial. Suppose Oa. Then $O^*(a)$ by Theorem 7 (b). Hence Ω and a are comparable by Theorem 7 (c). Hence $a<\Omega$ since $\Omega \leqq a$ leads to a contradiction.

3.6.3. THEOREM 9. *If $A(x)$ is a P-formula which contains no free variable other than $a_1, \cdots, a_n, x$, then the following hold:*

(a) (*Undefinability of Ω*) $OA \vdash Oa_1 \wedge \cdots \wedge Oa_n \rightarrow \neg \forall x[A(x) \leftrightarrow x=\Omega]$;

(b) (*Reflection principle for Ω*) $OA \vdash Oa_1 \wedge \cdots \wedge Oa_n \wedge A(\Omega) \rightarrow \exists x<\Omega A(x)$;

(c) (*Upward reflection principle for Ω*)

$$OA \vdash Oa_1 \wedge \cdots \wedge Oa_n \wedge A(\Omega) \rightarrow \exists x[\Omega<x \wedge A(x)].$$

PROOF. (a) Immediate from the undefinability of O. (b and c) Consider the P-formulas $A(x) \wedge O^*(x) \wedge \forall y<x \neg A(y)$ and $A(x) \wedge O^*(x) \wedge \forall y[O^*(y) \wedge x<y \rightarrow \neg A(y)]$ respectively. And apply them to the result (a).

3.7. We shall show two certain properties of Ω, which we shall call the weak inaccessibility of Ω and the inaccessibility of Ω respectively. The former is provided for the proof of the latter. These properties will play an important rôle in §4.

CONVENTION 2. (a) We shall use the abbreviation $\exists ! x<aA(x)$ for $\exists ! x[x<a \wedge A(x)]$ (not for $\exists x<a \forall z[z=x \leftrightarrow A(z)]$). (b) By $x_1 \cdots x_n<\Omega$, we denote $Ox_1 \wedge \cdots \wedge Ox_n$. (c) By $\mathrm{Min}[a; A(a)]$, we denote the formula $O^*(a) \wedge A(a) \wedge \forall w<a \neg A(w)$.

REMARK. $\exists x[O^*(x) \wedge A(x)]$ does not necessarily imply $\exists x\, \mathrm{Min}[x; A(x)]$.

3.7.1. THEOREM 10. (*Weak inaccessibility of Ω*)

$$OA \vdash a_1 \cdots a_n, b<\Omega \wedge \forall s<\Omega \exists ! t<\Omega A(s, t)$$
$$\rightarrow \exists u<\Omega \forall s<b \forall t<\Omega[A(s, t) \rightarrow t<u],$$

where $A(s, t)$ is a P-formula which contains no free variables other than $a_1 \cdots a_n$, b, s, t.

PROOF. Let $B(x)$ be the formula $\exists t[x \leqq t \wedge \exists s<b\ \mathrm{Min}[t; A(s, t)]]$. Now suppose $a_1 \cdots a_n, b<\Omega \wedge \forall s<\Omega \exists ! t<\Omega A(s, t)$. Then it holds that $\forall x[B(x) \rightarrow \forall y<xB(y)]$ and $\forall x[B(x) \rightarrow Ox]$. (To see this, assume $B(x)$. Then we have an s and a t such that $x \leqq t \wedge s<b \wedge O^*(t) \wedge A(s, t) \wedge \forall u<t \neg A(s, t)$. Let $y<x$. Then $y \leqq t$ follows from $O^*(t) \wedge x \leqq t$ by $L_1(t)$. Hence $B(y)$. On the other hand, there exists a $t_0<\Omega$ such that $A(s, t_0)$ by the assumption. Then t and t_0 are comparable because both are in O^*. So, by the minimality of t, we have $t \leqq t_0$. Hence t is an ordinal and hence so is x.) Hence, by A 4, there exists an ordinal u such that $\forall x[x<u \leftrightarrow B(x)]$. Then the ordinal u satisfies $\forall s, t[s<b \wedge t<\Omega \wedge A(s, t) \rightarrow t<u]$. (Suppose $s<b \wedge t<\Omega \wedge A(s, t)$. Then by the assumption, such a t is unique. So, we have $B(t)$. Hence $t<u$.)

3.7.2. DEFINITION. We define quasi-primitive formulas (abbreviated by Q-formulas) inductively as follows:

(a) Every P-formula is a Q-formula;

(b) If A and B are Q-formulas, then so are $\neg A$ and $A \rightarrow B$;

(c) If A is a Q-formula, then so is $\exists x[Ox \wedge A]$.

$\exists x[Ox \wedge A]$ is equivalent to $\exists x < \Omega A$. So, we write often $\exists x < \Omega A$ as an abbreviation of $\exists x[Ox \wedge A]$.

THEOREM 11. (*Inaccessibility of Ω*)

$$OA \vdash a_1 \cdots a_n,\ b < \Omega \wedge \forall s < \Omega \exists !\, t < \Omega A(s, t)$$
$$\rightarrow \exists u < \Omega \forall s < b \forall t < \Omega [A(s, t) \rightarrow t < u],$$

where $A(s, t)$ is a Q-formula which contains no free variable other than $a_1 \cdots a_n$, b, s, t.

This theorem differs from the previous theorem only in the respect that A is a Q-formula.

We shall provide two lemmas for its proof.

LEMMA 1. *If $A(y, x_1 \cdots x_n)$ is a P-formula, then there exists a P-formula $B(u_0, u_1 \cdots u_n, x_1 \cdots x_n)$ such that*

$$OA \vdash \exists u_0 \cdots u_n < \Omega \forall x_1 \cdots x_n < \Omega [\exists y < \Omega A(y, x_1 \cdots x_n)$$
$$\leftrightarrow B(u_0 \cdots u_n, x_1 \cdots x_n)],$$

where $u_0 \cdots u_n$ are variables not occurring in $A(y, x_1 \cdots x_n)$.

REMARK. We shall prove this only for the case $n=1$. To see it for the other cases, regard the letters x and u occurring in the proof below as finite sequences of variables.

PROOF. Let $A(y, x)$ be a P-formula. Then we shall use the following abbreviations:

$A^*(a, x)$ for $\mathrm{Min}[a\,;\, A(a, x)]$,

$C(b, a)$ for $\exists x < b A^*(a, x)$,

$D(c)$ for $\forall b \leqq c [\exists a [b \leqq a \leqq c \wedge C(b, a)] \rightarrow \exists d\, \mathrm{Min}[d\,;\, b \leqq d \leqq c \wedge C(b, d)]]$,

E for $\exists c [C(\Omega, c) \wedge D(c) \wedge \Omega \leqq c]$,

$F(d, u)$ for $\Omega \leqq d \wedge u < \Omega \wedge A^*(d, u) \wedge \forall v [\Omega \leqq v < d \rightarrow \neg C(\Omega, v)]$,

$B_1(u, x)$ for $\exists d [A^*(d, u) \wedge \exists y < d A^*(y, x)]$,

$B_2(x)$ for $\exists y [D(y) \wedge A^*(y, x)]$

and

$B(i, u, x)$ for $i=0 \wedge B_1(u, x). \vee . i=0' \wedge B_2(x)$.

Note that A^*, C, D, B_1, B_2 and B are P-formulas.

Then the following formulas (a)-(k) are theorems in OA:

(a) $A^*(a, x) \rightarrow O^*(a)$.

(b) $C(b, a) \rightarrow O^*(a)$.

(c) $\exists y<\Omega A(y, x) \leftrightarrow \exists y<\Omega A^*(y, x)$.

(d) $E \rightarrow \exists d \exists u F(d, u)$.

(e) $F(d, u) \rightarrow \forall x<\Omega [\exists y<\Omega A(y, x) \leftrightarrow B_1(u, x)]$.

(f) $F(d, u) \rightarrow u<\Omega$.

(g) $y<\Omega \rightarrow D(y)$.

(h) $\neg E \rightarrow \forall x<\Omega [\exists y<\Omega A(y, x) \leftrightarrow B_2(x)]$.

(i) $E \rightarrow \exists i, u<\Omega \forall x<\Omega [\exists y<\Omega A(y, x) \leftrightarrow B(i, u, x)]$.

(j) $\neg E \rightarrow \exists i, u<\Omega \forall x<\Omega [\exists y<\Omega A(y, x) \leftrightarrow B(i, u, x)]$.

(k) $\exists i, u<\Omega \forall x<\Omega [\exists y<\Omega A(y, x) \leftrightarrow B(i, u, x)]$.

The proof of the lemma is concluded by (k), since B is a P-formula.

PROOFS OF (a)-(k).

(a) Obvious. (See Convention 2 (b).)

(b) Obvious by (a).

(c) Obvious since O is well-ordered by $<$.

(d) Suppose $C(\Omega, c) \wedge D(c) \wedge \Omega \leqq c$. Then $\Omega \leqq c \wedge \exists a [\Omega \leqq a \leqq c \wedge C(\Omega, a)]$. So, by $D(c)$, we can obtain a d such that $\Omega \leqq d \leqq c \wedge C(\Omega, d) \wedge \forall v<d \neg [\Omega \leqq v \leqq c \wedge C(\Omega, v)]$. Then the d satisfies $\forall v [\Omega \leqq v<d \rightarrow \neg C(\Omega, v)]$. (Note that $O^*(c)$ follows from $C(\Omega, c)$ by (b). So, $v<d$ and $d \leqq c$ imply $v \leqq c$.) From $C(\Omega, d)$, it follows that there exists a u such that $u<\Omega \wedge A^*(d, u)$.

(e) Suppose $F(d, u) \wedge x<\Omega$.

1) Suppose $\exists y<\Omega A(y, x)$. Then, by (c), there exists a y such that $y<\Omega \wedge A^*(y, x)$. Take such a y. Then, by $F(d, u)$, we see $y<\Omega \wedge \Omega \leqq d \wedge O^*(d)$ and hence $y<d$. So $\exists y<d A^*(y, x)$. Since $A^*(d, u)$ by $F(d, u)$, we obtain $\exists d [A^*(d, u) \wedge \exists y<d A^*(y, x)]$.

2) Suppose $B_1(u, x)$. Then there exist a d and a y such that $A^*(d, u) \wedge y<d \wedge A^*(y, x)$. Since $x<\Omega$, we see $y<d \wedge C(\Omega, y)$, which implies $\neg \Omega \leqq y$ by $F(d, u)$. $\neg \Omega \leqq y$ implies $y<\Omega$ since $O^*(y)$ by $A^*(y, x)$ and (a). Hence $\exists y<\Omega A(y, x)$ by (c).

(f) Trivial.

(g) Because O is well-ordered by $<$.

(h) Suppose $\neg E$ and $x<\Omega$.

1) Suppose $\exists y<\Omega A(y, x)$. Then by (c), we have a y such that $y<\Omega$ and $A^*(y, x)$. Then by (g), we obtain $D(y) \wedge A^*(y, x)$.

2) Suppose $D(y) \wedge A^*(y, x)$. Then $D(y) \wedge C(\Omega, y)$ since $x<\Omega$. Hence, by $\neg E$, we obtain $\neg [\Omega \leqq y]$, which implies $y<\Omega$ since $O^*(y)$ by (a). Hence $\exists y<\Omega A(y, x)$ by (c).

(i) By (e) and (f), $F(d, u)$ implies $u<\Omega$ and $\forall x<\Omega [\exists y<\Omega A(y, x) \leftrightarrow B(0, u, x)]$. Hence by (d), E implies $\exists u<\Omega \forall x<\Omega [\exists y<\Omega A(y, x) \leftrightarrow B(0, u, x)]$.

(j) By (h), $\neg E$ implies $\exists u<\Omega \forall x<\Omega [\exists y<\Omega A(y, x) \leftrightarrow B(0', u, x)]$.

(k) Immediate from (i) and (j).

LEMMA 2. *If $A(x_1 \cdots x_n)$ is a Q-formula, then there exists a P-formula $B(u_1 \cdots u_m, x_1 \cdots x_n)$ such that*

$$OA \vdash \exists u_1 \cdots u_m<\Omega \forall x_1 \cdots x_n<\Omega [A(x_1 \cdots x_n) \leftrightarrow B(u_1 \cdots u_m, x_1 \cdots x_n)],$$

where $u_1 \cdots u_m$ are variables not occurring in $A(x_1 \cdots x_n)$.

PROOF. By induction corresponding to the inductive definition of the Q-formulas.

1) If $A(x_1 \cdots x_n)$ is a P-formula, then it is trivial.

2) If A is $\neg B$ or $B \rightarrow C$ where B and C are Q-formulas, then it follows immediately from the induction hypothesis.

3) Let $A(x_1 \cdots x_n)$ be $\exists y<\Omega B(y, x_1 \cdots x_n)$ and $B(y, x_1 \cdots x_n)$ be a P-formula. For simplicity, we write x and u for the sequences $x_1 \cdots x_n$ and $u_1 \cdots u_m$ respectively. Now, by the induction hypothesis, there exists a P-formula $C(u, y, x)$ such that

$$OA \vdash \exists u<\Omega \forall y, x<\Omega [B(y, x) \leftrightarrow C(u, y, x)].$$

By Lemma 1, there exists a P-formula $D(v, u, x)$ such that

$$OA \vdash \exists v<\Omega \forall u, x<\Omega [\exists y<\Omega C(u, y, x) \leftrightarrow D(v, u, x)],$$

where v is written for $v_0 \cdots v_{m+n}$. Then we have

$$OA \vdash \exists v, u<\Omega \forall x<\Omega [\exists y<\Omega B(y, x) \leftrightarrow D(v, u, x)].$$

PROOF OF THEOREM 11. Let $A(s, t)$ be a Q-formula which contains no free variable other than $a_1 \cdots a_n$, b, s, t. By Lemma 2, we have a P-formula $B(u_1 \cdots u_m, s, t)$ such that

$$OA \vdash \exists u_1 \cdots u_m < \Omega \forall s, t < \Omega [A(s, t) \leftrightarrow B(u_1 \cdots u_m, s, t)].$$

Now, suppose $a_1 \cdots a_n, b < \Omega \wedge \forall s < \Omega \exists ! t < \Omega A(s, t)$. There exist $u_1 \cdots u_m < \Omega$ such that $\forall s, t < \Omega [A(s, t) \leftrightarrow B(u_1 \cdots u_m, s, t)]$. Then $a_1 \cdots a_n, b, u_1 \cdots u_m < \Omega \wedge \forall s < \Omega \exists ! t < \Omega B(u_1 \cdots u_m, s, t)$. So, by Lemma 10, $\exists u < \Omega \forall s < b \forall t < \Omega [B(u_1 \cdots u_m, s, t) \rightarrow t < u]$. Hence

$$\exists u < \Omega \forall s < b \forall t < \Omega [A(s, t) \rightarrow t < u].$$

§ 4. Strength of the theory OA.

By $ZF-P$, we denote Zermelo-Fraenkel set theory minus the power set axiom, i.e. the theory which has the following axioms 1-8:

1. Extensionality.
2. Null set.
3. Pairs.
4. Unions.
5. Infinity.
6. Regularity.
7. Subsets: $\forall x \exists y \forall z [z \in y \leftrightarrow z \in x \wedge A(z, u_1 \cdots u_n)]$, where A is any formula in which y does not occur.
8. Collection: $\forall x [x \in u \rightarrow \exists z A(x, z, u, v_1 \cdots v_n)] \rightarrow \exists y \forall x [x \in u \rightarrow \exists z [z \in y \wedge A(x, z, u, v_1 \cdots v_n)]]$, where A is a formula in which y does not occur.

$ZFL-P$ means $ZF-P$ with the axiom of constructibility of the usual form: $\forall y \exists x [\mathrm{Ord}(x) \wedge y = F(x)]$. By $L_\in$ and L_{OA}, we denote the languages of $ZF-P$ and OA respectively.

In this section, we show that if we construct a model of set theory in the usual way (i.e. inductive definition of the constructible sets) in OA, then it is a mode lof $ZFL-P$ and conversely every formula which is valid in the model is provable in $ZFL-P$, i.e. $OA \vdash$ "A is valid in the model" is equivalent to $ZFL-P \vdash A$. Hence we see at least that OA and $ZF-P$ are equiconsistent.

4.1. Interpretation of $ZFL-P$ in OA.

DEFINITION of the predicates S, Se and J^* of L_{OA}.

$S(a, b, c, d)$ is $O^*(a) \wedge O^*(b) \wedge O^*(c) \wedge O^*(d) \wedge [\max\{a, b\} < \max\{c, d\}$
$\vee [\max\{a, b\} = \max\{c, d\} \wedge [a < c \vee [a = c \wedge b < d]]]]$;

$Se(a, b, c, d)$ is $S(a, b, c, d) \vee \{a = c \wedge b = d]$;

$J^*(a, b, c)$ is $\exists P[Pabc \wedge \forall x, y[Se(x, y, a, b) \rightarrow \forall z[Pxyz \leftrightarrow \mathrm{Min}[z ; \forall u, v, w[S(u, v, x, y) \wedge Puvw \rightarrow w<z]]]]]$.

Note that S, Se and J^* are P-formulas.

LEMMA 3. (*The schema of transfinite induction on* S)

$$OA \vdash \forall x, y<\Omega[\forall u, v[S(u, v, x, y) \rightarrow A(u, v)] \rightarrow A(x, y)] \rightarrow \forall x, y<\Omega A(x, y),$$

for every formula A.

DEFINITION of the functions J, K and L of L_{OA}.

$J(a, b)=\iota z[\exists ! w J^*(a, b, w) \wedge J^*(a, b, z). \vee . \neg \exists ! w J^*(a, b, w) \wedge z=0]$.

$K(c)=\iota x\, \mathrm{Min}[x ; \forall z[\mathrm{Min}[z ; \exists y \leqq c J^*(z, y, c)] \rightarrow z=x]]$.

$L(c)=\iota y\, \mathrm{Min}[y ; \forall z[\mathrm{Min}[z ; \exists x \leqq c J^*(x, z, c)] \rightarrow z=y]]$.

Note that $J(a, b)$, $K(c)$ and $L(c)$ are P-terms.

LEMMA 4. (a) $OA \vdash a, b<\Omega \rightarrow J(a, b)<\Omega$.

(b) $OA \vdash c<\Omega \rightarrow K(c), L(c)<\Omega$.

(c) $OA \vdash c<\Omega \rightarrow J(K(c), L(c))=c$.

(d) $OA \vdash c<\Omega \rightarrow \exists ! x<\Omega \exists ! y<\Omega[J(x, y)=c]$.

PROOF. We shall prove only (a), since the others are obtained as usual. To show it, it suffices to show $\forall x, y<\Omega \exists ! z<\Omega J^*(x, y, z)$. By induction on S. Suppose $x, y<\Omega$ and $\forall u, v[S(u, v, x, y) \rightarrow \exists ! z<\Omega J^*(u, v, z)]$. Let $t=\max\{x, y\}$. Let $A(u, v, w)$ be the formula $S(u, v, x, y) \wedge J^*(u, v, w). \vee . \neg S(u, v, x, y) \wedge w=0$. Then obviously $x, y, t<\Omega \wedge \forall u, v<\Omega \exists ! w<\Omega A(u, v, w)$. Note that $A(u, v, w)$ is a P-formula. So by inaccessibility of Ω, we obtain $\exists z<\Omega \forall u, v<t \forall w[A(u, v, w) \rightarrow w<t]$. This implies $\exists z<\Omega \forall u, v, w[S(u, v, x, y) \wedge J^*(u, v, w) \rightarrow w<z]$. The rest is routine.

Now we shall define a binary predicate $\lambda x, y[xEy]$ which means intuitively $\lambda x, y[F(x) \in F(y)]$, where F is Gödel's function for generating the constructible sets.

DEFINITION of the predicate $*E*$. It is not so important to write faithfully $*E*$ costing the wide spaces. So we shall give ${}_*E_*$ only in sketched form which will, however, indicate its character sufficiently. As the fundamental operations, we shall adopt the following (and fix through this paper):

$$F_1(x, y)=\{\langle a, b\rangle : \langle a, b\rangle \in x \text{ and } a \in b\}, \cdots, F_9(x, y)=x \cap \mathrm{dom}(y), F(x, y)=\{x, y\}.$$

($F_2, \cdots, F_8$ are also standard ones). aEb is the formula

$$\exists P[Pab \wedge \forall c, d[Se(c, d, a, b) \rightarrow [Pcd \leftrightarrow A(P, c, d)]]],$$

where $A(P, c, d)$ is the following formula:

$$\begin{aligned}\exists x<d[x\approx c\wedge[K(d)=0 \\ &\vee[K(d)=1\wedge\exists y, z<x[x\approx\langle y, z\rangle^\circ\wedge Pyz\wedge Pxd_1]] \\ &\vee\cdots \\ &\vee[K(d)=9\wedge Pxd_1\wedge\exists y, z<d_2[z\approx\langle x, y\rangle^\circ\wedge Pzd_2]] \\ &\vee[K(d)>9\wedge[x\approx d_1\vee x\approx d_2]]]]],\end{aligned}$$

where d_1 and d_2 stand for $K(L(d))$ and $L(L(d))$ respectively and

$a\approx b$ means $\quad \forall x<a[Pxa\rightarrow Pxb]\wedge\forall x<b[Pxb\rightarrow Pxa]$,

$a\approx\langle b, c\rangle^\circ$ means $\exists x, y<a[x\approx\{b, b\}^\circ\wedge y\approx\{b, c\}^\circ\wedge a\approx\{x, y\}^\circ]$,

$a\approx\{b, c\}^\circ$ means $Pba\wedge Pca\wedge\forall x<a[Pxa\rightarrow x\approx b\vee x\approx c]$.

REMARK. (a) Note that $c, d<\Omega\wedge\forall x, y[S(x, y, c, d)\rightarrow[Pxy\leftrightarrow Qxy]]$ implies $A(P, c, d)\leftrightarrow A(Q, c, d)$. So, the postulate $Pcd\leftrightarrow A(P, c, d)$ gives recursive definition (on S) of the predicate P. (b) The formula aEb is a P-formula since $K(x)$ and $L(x)$ are P-terms.

DEFINITION. $a\approx b$ is the formula

$$\forall x<a[xEa\rightarrow xEb]\wedge\forall x<b[xEb\rightarrow xEa].$$

Now we shall define an interpretation of $L_\in$ in OA. For this purpose, we provide a one-to-one correspondence between the set of all variables of $L_\in$ and a set of individual variables of L_{OA} and fix it. By x°, we denote the variable of L_{OA} corresponding to a variable x of $L_\in$.

DEFINITION. For each formula A of $L_\in$, the formula A° of L_{OA} is defined inductively as follows:

$[a=b]^\circ$ is $a^\circ\approx b^\circ$;

$[a\in b]^\circ$ is $a^\circ Eb^\circ$;

$[A\rightarrow B]^\circ$, $[\neg A]^\circ$ are $A^\circ\rightarrow B^\circ$, $\neg A^\circ$ respectively;

$[\exists xA]^\circ$ is $\exists x^\circ[Ox^\circ\wedge A^\circ]$.

REMARK. A° is a Q-formula for every formula A of $L_\in$, since $x\approx y$ and xEy are P-formulas.

We shall use the following abbreviations:

$a\approx\{b, c\}^{\circ}$ is the P-formula
$$\forall x\leqq\max\{a, b, c\}[xEa \leftrightarrow x\approx b\vee x\approx c];$$
$a\approx\langle b, c\rangle^{\circ}$ is the P-formula
$$\exists y, z<\max\{a, b, c\}[a\approx\{y, z\}^{\circ}\wedge y\approx\{b, b\}^{\circ}\wedge z\approx\{b, c\}^{\circ}];$$
$a\approx\langle x_1\cdots x_{n+1}\rangle^{\circ}$ and $\langle x_1\cdots x_n\rangle^{\circ}Ea$ means respectively
$$\exists y<a[y\approx\langle x_1\cdots x_n\rangle^{\circ}\wedge a\approx\langle y, x_{n+1}\rangle^{\circ}]$$
$$\text{and } \exists y<a[y\approx\langle x_1\cdots x_n\rangle^{\circ}\wedge yEa].$$

LEMMA 5. $OA\vdash\forall a<\Omega\exists b<\Omega\forall x, y<a[\langle x, y\rangle^{\circ}Eb]$.

PROOF. Consider the formula $\mathrm{Min}[c; c\approx\langle x, y\rangle^{\circ}]$. This is a P-formula and besides $\forall x, y<\Omega\exists!c<\Omega\,\mathrm{Min}[c; c\approx\langle x, y\rangle^{\circ}]$ holds. So by inaccessibility of Ω, $a<\Omega$ implies $\exists b<\Omega\forall x, y<a\forall c<\Omega[\mathrm{Min}[c: c\approx\langle x, y\rangle^{\circ}]\rightarrow c<b]$. Take such a b. Then $\forall x, y<a[\langle x, y\rangle^{\circ}EJ(0, b)]$.

LEMMA 6. For every formula A of $L_{\in}$,
$$OA\vdash\forall a<\Omega\exists b<\Omega\forall x_1\cdots x_n<\Omega[x_1Ea\wedge\cdots\wedge x_nEa$$
$$\rightarrow.\langle x_1\cdots x_n\rangle^{\circ}Eb\leftrightarrow A^{\circ}(x_1\cdots x_n)].$$

PROOF. By induction on the complexity of A. The case that A° is of the form $\exists y<\Omega B^{\circ}(y, x_1\cdots x_n)$. Let $C(y, x_1\cdots x_n)$ be the formula
$$B^{\circ}(y, x_1\cdots x_n)\wedge\forall z<y\neg B^{\circ}(z, x_1\cdots x_n).\vee.\neg\exists z<\Omega B^{\circ}(z, x_1\cdots x_n)\wedge y=0.$$
Since B° is a Q-formula, so is C. Besides it holds that $\forall x_1\cdots x_n<\Omega\exists!y<\Omega C(y, x_1\cdots x_n)$. So, by inaccessibility of Ω, we have $a<\Omega\rightarrow\exists b<\Omega\forall x_1\cdots x_n<a\forall y<\Omega[C(y, x_1\cdots x_n)\rightarrow y<b]$. The rest goes as usual.

THEOREM 12. *If* $ZFL-P\vdash A$, *then* $OA\vdash A^{\circ}$, *where* A *is any sentence of* $L_{\in}$.

PROOF. We shall show that $OA\vdash A^{\circ}$ for all axioms A of $ZFL-P$. For the axioms 1-7, it is evident by the previous lemma. The case of the axiom of collection follows from the inaccessibility of Ω and the fact that every A° is a Q-formula. We shall check the case of the axiom of constructibility: $\forall x\exists y[\mathrm{Ord}(y)\wedge F(x, y)]$, where $F(x, y)$ is a formula which represents $x=F(y)$. Let $Nm(a, b)$ be the following formula:
$$Oa\wedge Ob\wedge\exists P[Pab\wedge\forall x, y<\Omega[Pxy$$
$$\leftrightarrow[\forall v<y[\neg v\approx y]\wedge\forall z[zEy\leftrightarrow\exists w<x\exists v[Pwv\wedge v\approx z]]]]].$$
Then it holds that $\forall x<\Omega\exists!y<\Omega Nm(x, y)$. Let x^* stand for $\iota y[x<\Omega\wedge y<\Omega\wedge Nm(x, y).\vee.\neg x<\Omega\wedge y=0]$. Then we obtain the following: $[x<\Omega\rightarrow x^*<\Omega]$ and $\forall x<\Omega[\mathrm{Ord}^{\circ}(x^*)\wedge F^{\circ}(x, x^*)]$. These facts show $\forall x<\Omega\exists y<\Omega[\mathrm{Ord}^{\circ}(y)\wedge F^{\circ}(x, y)]$.

4.2. The auxiliary system ZFL_m-P.

In this section we define a system ZFL_m-P, which is a conservative extension of $ZFL-P$.

The language of ZFL_m-P (denoted by L_M) is $L_\in$ with a constant symbol M. The axioms of ZFL_m-P are $ZFL-P$ with the following:

10. $\exists x[\mathrm{Ord}(x)\wedge M=F``x]$.

11. $\forall x_1\cdots x_n\in M[\exists y\in M\ A(y, x_1\cdots x_n)\leftrightarrow\exists yA(y, x_1\cdots x_n)]$, where the formula A contains neither the constant symbol M nor free variable other than $y, x_1\cdots x_n$.

THEOREM 13. *ZFL_m-P is a conservative extension of $ZFL-P$.*

PROOF. It suffices to prove that

$$ZFL-P\vdash\exists m[\exists z[\mathrm{Ord}(z)\wedge 0<z\wedge m=F``z]$$
$$\wedge\bigwedge_{1\leq i\leq k}\forall x_1\cdots x_n\in m[\exists y\in mA_i(y, x_1\cdots x_n)\leftrightarrow\exists yA_i(y, x_1\cdots x_n)]],$$

for any finite set of formulas $\{A_i(y, x_1\cdots x_n)\}_{i=1\cdots k}$. Fix arbitrarily given formulas $A_i(y, x_1\cdots x_n)$, $i=1\cdots k$. Let $B(a, b)$ be the following formula:

$$\bigwedge_{1\leq i\leq k}\forall x_1\cdots x_n\in F``a[\exists y\in F``bA_i(y, x_1\cdots x_n)\vee\neg\exists yA_i(y, x_1\cdots x_n)].$$

Then by the axiom of collection and the axiom of constructibility, we have $\forall a\in\mathrm{Ord}\,\exists b\in\mathrm{Ord}\,B(a, b)$. Let $f(a)$ be the function $\iota b[\mathrm{Ord}(b)\wedge B(a, b)\wedge\forall x<b\neg B(a, x)]$ and $\{a_n\}_{n\in\omega}$ be the sequences of ordinals defined by $a_0=1$ and $a_{n+1}=f(a_n)$. By the axiom of collection, we see the existence of $a=\bigcup\{a_n : n\in\omega\}$ and $m=F``a$. They have the desired property.

THEOREM 14. *$ZFL_m-P\vdash A\leftrightarrow ZFL_m-P\vdash A^M$ for every sentence A of $L_\in$, where A^M means the relativization of A by M.*

PROOF. By induction on the complexity of A, using the axiom 11 of ZFL_m-P, we see that $ZFL_m-P\vdash\forall x_1\cdots x_n\in M[A\leftrightarrow A^M]$, where A is a formula of $L_\in$ which contains no free variable other than $x_1\cdots x_n$.

THEOREM 15. *$ZFL-P\vdash A\leftrightarrow ZFL_m-P\vdash A^M$ for every sentence A of $L_\in$.*

PROOF. By Theorems 13 and 14.

4.3. Interpretation of OA in ZFL_m-P.

Fix a one-to-one mapping from the union of the set of all predicate variables and the set of all individual variables of L_{OA} into a set of variables of $L_\in$. We shall denote the variables corresponding to the individual variables x or the predicate variables P by x^m or P^m respectively. But often for simplicity, we write x itself for x^m and a small letter p for P^m.

DEFINITION. For each formula A of L_{OA}, the formula A^m of L_M is defined inductively as follows:

1. $[Oa]^m$ is $\mathrm{Ord}(a)\wedge a\in M$,
2. $[a<b]^m$ is $a\in b$,
3. $[Pa_1\cdots a_n]^m$ is $\langle a_1\cdots a_n\rangle\in p$,
4. $[A\to B]^m$ and $[\neg A]^m$ are $A^m\to B^m$ and $\neg[A]^m$ respectively,
5. $[\exists xA]^m$ is $\exists xA^m$,
6. $[\exists PA]^m$ is $\exists pA^m$.

THEOREM 16. *$OA\vdash A\to ZFL_m-P\vdash A^m$, for every formula A of L_{OA}.*

PROOF. It suffices to show that A^m is provable in ZFL_m-P for every axiom A of OA.

(1) The case where A is an axiom of comprehension. A^m has the form of $\exists p\forall x_1\cdots x_k[\langle x_1\cdots x_k\rangle\in p\leftrightarrow\bigwedge_{i\leq k}[\mathrm{Ord}(x_i)\wedge x_i\in M]\wedge B^m]$. Hence A^m follows from the axiom of subset since $ZF-P\vdash\forall x\exists y[y=x\times\cdots\times x]$.

(2) $[\mathrm{A1}]^m$ follows from the extensionality.

(3) $[\mathrm{A2}]^m$ and $[\mathrm{A3}]^m$ is clear by the properties of Ord.

(4) The case of A4. Assume $\mathrm{Ord}(a_i)\wedge a_i\in M$, $i=1\cdots n$ and $\forall x[A^m(x)\to.\forall y\in xA^m(y)\wedge\mathrm{Ord}(x)\wedge x\in M]$, where $A(x)$ contains neither the predicate constant O nor free variable other than x, $a_1\cdots a_n$. $\forall x[A^m(x)\to x\in M]$ implies $\exists u\forall x[A^m(x)\to x\in u]$. Note that the formula $\exists u\forall x[A^m(x)\to x\in u]$ does not contain the constant symbol M since A does not contain the predicate symbol O. Hence, by Axiom 11, $\exists u\in M\forall x[A^m(x)\to x\in u]$ is provable from $a_1\cdots a_n\in M$. Put $y=\sup\{x: A^m(x)\}$. Then $y\in M\wedge\mathrm{Ord}(y)\wedge\forall z[z\in y\leftrightarrow A^m(z)]$.

(5) Other cases are evident.

4.4. Equivalence of OA and $ZFL-P$ with respect to the formulas of $L_\in$.

We shall show that $OA\vdash A^\circ\leftrightarrow ZFL-P\vdash A$ for every sentence A of $L_\in$. For this purpose we shall first prove the following theorem.

THEOREM 17. *$ZFL_m-P\vdash y_1\cdots y_n\in\mathrm{Ord}\cap M\wedge\bigwedge_{i\leq n}x_i=F(y_i)\to[A^{\circ m}\leftrightarrow A^M]$, where $x_1\cdots x_n$ are all the free variables in A (a formula of $L_\in$) and $y_1\cdots y_n$ stand for $(x_1{}^\circ)^m\cdots(x_n{}^\circ)^m$ respectively.*

PROOF. By induction on the complexity of A.

(1) $[x_i\in x_j]^{\circ m}$ is identical with $y_iE^my_j$. $y_iE^my_j$ does not contain the symbol M since $x_i{}^\circ Ex_j{}^\circ$ is a P-formula. So $y_iE^my_j$ is a formula of $L_\in$ and

it is easily seen that

$$ZFL-P \vdash \forall a, b[\mathrm{Ord}(a) \wedge \mathrm{Ord}(b) \rightarrow [aE^m b \leftrightarrow F(a) \in F(b)]].$$

Hence $\bigwedge_{i\leq n}[x_i = F(y_i) \wedge \mathrm{Ord}(y_i)]$ implies $y_i E^m y_j \leftrightarrow x_i \in x_j$. Since $[x_i \in x_j]^M$ is $x_i \in x_j$, we see that

$$\bigwedge_{i\leq n}[x_i = F(y_i) \wedge \mathrm{Ord}(y_i)] \quad \text{implies} \quad [x_i \in x_j]^{\circ m} \leftrightarrow [x_i \in x_j]^M.$$

(2) If A is $\exists xB$, then A^M is $\exists x \in M\ B^m$ and $A^{\circ m}$ is $\exists y[\mathrm{Ord}(y) \wedge y \in M \wedge B^{\circ m}]$, where y stands for $x^{\circ m}$. Suppose $y_1 \cdots y_n \in M \cap \mathrm{Ord} \wedge \bigwedge_{i\leq n} x_i = F(y_i)$. By the induction hypothesis, $\forall x, y[x = F(y) \wedge \mathrm{Ord}(y) \wedge y \in M \rightarrow [B^{\circ m} \leftrightarrow B^M]]$. Since M is a model of $ZFL-P$ (Theorem 15), $\mathrm{Ord}(y)$ and $y \in M$ imply $F(y) \in M$. By the same reason, $x \in M$ implies $\exists y \in M[\mathrm{Ord}(y) \wedge x = F(y)]$. Hence $\exists y[\mathrm{Ord}(y) \wedge y \in M \wedge B^{\circ m}] \leftrightarrow \exists x \in M\ B^M$.

(3) The other cases are evident.

THEOREM 18. *$OA \vdash A^\circ \leftrightarrow ZFL-P \vdash A$, for every sentence A of $L_\in$.*

PROOF. Let A be a sentence of $L_\in$ and suppose $OA \vdash A^\circ$. By Theorem 16, $ZFL_m - P \vdash A^{\circ m}$. By Theorem 17, $ZFL_m - P \vdash A^M$. By Theorem 15, $ZFL-P \vdash A$. The converse is just Theorem 12.

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