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Tôki covering surfaces and their applications

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An infinite and unbounded covering surface $R^{\sim}$ of an open Riemann surface R is referred to as a *Tôki covering surface* if any bounded harmonic function on $R^{\sim}$ is constant on $\pi^{-1}(q)$ for each q in R where π is the projection. The primary purpose of this paper is to show the existence of a Tôki covering surface $R^{\sim}$ of *any* given open Riemann surface R (Main theorem in no. 1.2). We can construct $R^{\sim}$ so that the projections of branch points in $R^{\sim}$ is discrete in R. Remove a parametric disk V from R. We will show that any bounded harmonic function on $R^{\sim}-\pi^{-1}(\bar{V})$ vanishing on its boundary relative to $R^{\sim}$ is constant on $\pi^{-1}(q)$ for each q in $R-\bar{V}$, and actually we will prove this assertion for a more general subset than V (Theorem in no. 2.5). As an application of this we will see that $\pi^{-1}(V)$ always clusters to the Royden harmonic boundary of $R^{\sim}$ which consists of a single point (Theorem in no. 2.3). Based on these results we will show that there exists a single point of positive harmonic measure but no isolated point in the Royden harmonic boundary of $R^{\sim}-\pi^{-1}(\bar{V})$ (Theorem in no. 3.1). The most effective application of Tôki covering surfaces is the following: For any compact Stonean space Δ which is a Wiener harmonic boundary of a hyperbolic Riemann surface, there exists an open Riemann surface whose Royden harmonic boundary consists of a single point and whose Wiener harmonic boundary is Δ (Theorem in no. 4.3). We denote by $b(W)$ (the B-harmonic dimension) the number of isolated points in the Wiener harmonic boundary of an open Riemann surface W and by $d(W)$ (the D-harmonic dimension) and $d^{\sim}(W)$ (the $D^{\sim}$-harmonic dimension) the numbers of isolated points and points with positive harmonic measures, respectively, in the Royden harmonic boundary of W. Based on the above results we will determine the triples $(b, d, d^{\sim})$ of countable cardinal numbers such that $(b, d, d^{\sim})=(b(W), d(W), d^{\sim}(W))$ for a certain open Riemann surface W (Theorem in no. 5.3).

Tôki covering surfaces.

1.1. We start by fixing terminologies. Let $R^{\sim}$ and R be Riemann surfaces. The triple $(R^{\sim}, R, \pi)$ is said to be a *covering surface* if $\pi: R^{\sim}\to R$ is a nonconstant analytic mapping. The surface R is referred to as the *base surface* and π the *projection* of the covering surface. The surface $R^{\sim}$ itself is often called the covering surface. A curve γ in R is a continuous mapping of the interval [0, 1] into R. We say that the covering surface $(R^{\sim}, R, \pi)$ is *unbounded* if the following condition is satisfied: For any curve γ in R and any point $a^{\sim}$ in $R^{\sim}$ with $\pi(a^{\sim})=\gamma(0)$ there always exists a curve $\gamma^{\sim}$ in $R^{\sim}$ such that $\gamma^{\sim}(0)=a^{\sim}$ and $\gamma(t)\equiv\pi\circ\gamma^{\sim}(t)$ on [0, 1]. Let $a^{\sim}\in R^{\sim}$ and $a\in R$ with $\pi(a^{\sim})=a$ and $z=\pi(z^{\sim})=a+(z^{\sim}-a^{\sim})^m$ $(m\geqq 1)$ be the local representation of π. If $m\geqq 2$, then $a^{\sim}$ is said to be a *branch point* of order m of the covering surface. Let $a\in R$ and $\pi^{-1}(a)=\{a_n^{\sim}\}(1\leqq n<N\leqq\infty)$. For convenience we say that $a\in R$ is an *even* base point if we can find a parametric disk V at a with the following property: There exist an $m\geqq 1$ and $N-1$ connected components $V_n^{\sim}$ of $\pi^{-1}(V)$ $(1\leqq n<N\leqq\infty)$ such that $V_n^{\sim}$ is a parametric disk at $a_n^{\sim}$ and $z=\pi(z^{\sim})=a+(z^{\sim}-a_n^{\sim})^m$ is a mapping of $V_n^{\sim}$ onto V $(1\leqq n<N)$. A covering surface $(R^{\sim}, R, \pi)$ is referred to as an *even* covering surface if every point $a\in R$ is an even base point. In this case there exist no branch points in $\pi^{-1}(a)$ for every $a\in R$ except for an isolated subset of R. Even covering surfaces are unbounded. For unbounded covering surfaces $(R^{\sim}, R, \pi)$ the number of points in $\pi^{-1}(a)$ is a constant $\leqq\infty$ for every $a\in R$ where branch points are counted repeatedly according their orders. This number is referred to as the sheet number. If it is finite (infinite, resp.), then $(R^{\sim}, R, \pi)$ is said to be *finite* (*infinite*, resp.).

1.2. For any covering surface $(R^{\sim}, R, \pi)$ we can consider the *lift up* π^* which is an injective map from the space of functions on R to that on $R^{\sim}$ defined by $\pi^*f=f\circ\pi$ for functions f on R. The lift up π^* preserves constants, ring operations, positiveness, boundedness, analyticity, super and subharmonicity, and so forth. In particular the mapping

$$\pi^*: HB(R)\longrightarrow HB(R^{\sim}) \tag{1}$$

is well defined and injective, where $H(R)$ is the space of harmonic functions on R and $HB(R)$ is the subspace of $H(R)$ consisting of bounded functions. We say that $(R^{\sim}, R, \pi)$ or simply $R^{\sim}$ is a *Tôki covering surface* of R if $(R^{\sim}, R, \pi)$ is infinite and unbounded and the mapping (1) is surjective, i. e.

$$\pi^*(HB(R))=HB(R)\circ\pi=HB(R^{\sim}). \tag{2}$$

The primary purpose of this paper is to prove the following

MAIN THEOREM. *For any open Riemann surface R there always exists a Tôki covering surface $R^{\sim}$ of R.*

The above result was originally proved by Tôki [7] when the base surface R is the open unit disk $|z|<1$. We adopted the terminology Tôki covering surface in honor of this very important work in the classification theory of Riemann surfaces. The covering surface $R^{\sim}$ can be constructed so as to satisfy the following two more properties: $R^{\sim}$ is even; every point in an arbitrarily given compact subset K of R is not the projection of any branch point of $R^{\sim}$. The proof will be given in nos. 1.3-1.8.

1.3. We denote by $\boldsymbol{N}$ the set of positive integers. Consider the mapping

$$(m, n) \longrightarrow \mu=\mu(m, n)=2^{m-1}(2n-1) \tag{3}$$

of $\boldsymbol{N}\times\boldsymbol{N}$ to $\boldsymbol{N}$. Observe that the mapping (3) is bijective. Moreover $\mu(m, n) \leqq \mu(m', n')$ if $m\leqq m'$ and $n\leqq n'$. It is also clear that $\mu(m, n)\to\infty$ if $m\to\infty$ or $n\to\infty$ or m and $n\to\infty$.

1.4. Since R is open, we can find an exhaustion $\{R^{\alpha}\}_{\alpha\in N}$ of R such that $R^{2\mu}-\bar{R}^{2\mu-1}$ consists of a finite number $l(\mu)$ of annuli $A_{\mu\lambda}$ ($\lambda=1, \cdots, l(\mu)$) for each $\mu\in\boldsymbol{N}$. We denote by $\operatorname{mod} A_{\mu\lambda}$ the logarithmic modulus of $A_{\mu\lambda}$, i.e. $\operatorname{mod} A_{\mu\lambda}=t$ if the conformal representation of $A_{\mu\lambda}$ is $1<|z|<e^t$. We choose an arbitrary but fixed sequence $\{k(\mu)\}_{\mu\in N}$ in $\boldsymbol{N}$ such that

$$4/k(\mu)<\min_{1\leq\lambda\leq l(\mu)} \operatorname{mod} A_{\mu\lambda}$$

for every $\mu\in\boldsymbol{N}$. Since $\operatorname{mod} A<\operatorname{mod} A'$ for $\bar{A}\subset A'$, we can find an annulus $B_{\mu\lambda}$ with $\bar{B}_{\mu\lambda}\subset A_{\mu\lambda}$ for each (μ, λ) such that $B_{\mu\lambda}$ separates one component of $\partial A_{\mu\lambda}$ from the other and

$$\operatorname{mod} B_{\mu\lambda}=4/k(\mu)$$

for $\lambda=1, \cdots, l(\mu)$. Therefore we can view $B_{\mu\lambda}$ as a spherical ring, i.e.

$$B_{\mu\lambda}=\{re^{i\theta}\,;\ 0<\log r<4/k(\mu)\}. \tag{4}$$

We then consider the slits $S^{\nu}_{mn\lambda}$ in each $B_{\mu\lambda}$ with $\mu=\mu(m, n)$ given by

$$S^{\nu}_{mn\lambda}=\{re^{i\theta}\,;\ 1/k(\mu)<\log r<3/k(\mu),\ \theta=2\pi\nu/k(\mu)\}$$

for $\nu=1, \cdots, k(\mu)$.

1.5. We denote by R_0 the surface R less all the slits $S^{\nu}_{mn\lambda}$ ($(m, n)\in\boldsymbol{N}\times\boldsymbol{N}$, $\lambda=1, \cdots, l(\mu(m, n))$, $\nu=1, \cdots, k(\mu(m, n))$), i.e.

$$R_0=R-\bigcup_{(m,n)\in N\times N}\ \bigcup_{1\leq\lambda\leq l(\mu(m,n))}\ \bigcup_{1\leq\nu\leq k(\mu(m,n))} S^{\nu}_{mn\lambda}.$$

Consider two sequences $\{R(h)\}_{h\in N}$ and $\{\hat{R}(h)\}_{h\in N}$ of duplicates $R(h)$ and $\hat{R}(h)$ of R_0.

1.6. We join $R(h)$ ($h=1, 2, \cdots$) with $\hat{R}(h')$ ($h'=1, 2, \cdots$) suitably crosswise along every slit $S^{\nu}_{mn\lambda}$ described as follows. For convenience we introduce the following notation: $\hat{m}=0$ for $m=1$ and $\hat{m}=2^{m-2}$ for $m>1$. First, for $m=1$, join

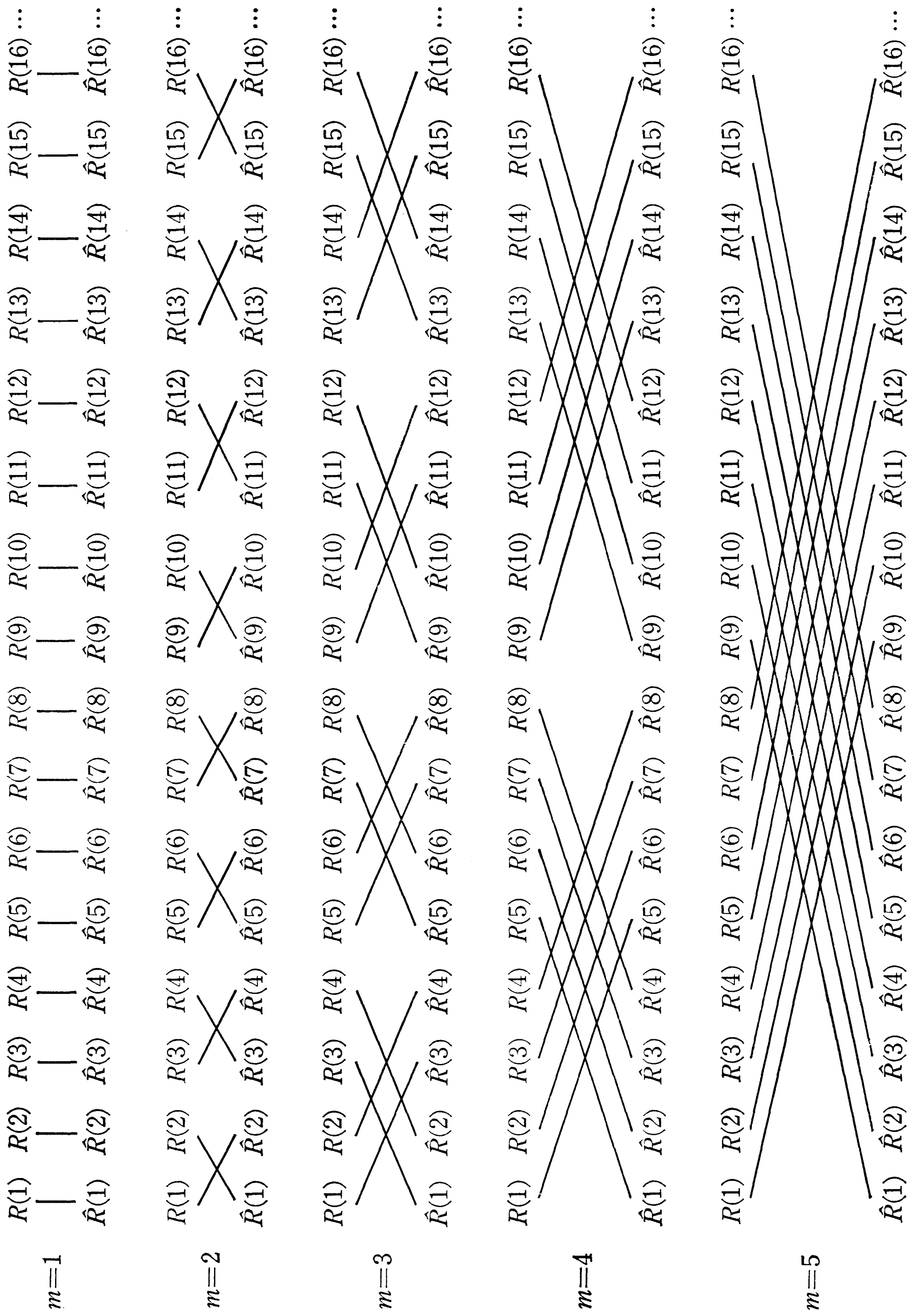
m=1
R(1) R(2) R(3) R(4) R(5) R(6) R(7) R(8) R(9) R(10) R(11) R(12) R(13) R(14) R(15) R(16) ...
R̂(1) R̂(2) R̂(3) R̂(4) R̂(5) R̂(6) R̂(7) R̂(8) R̂(9) R̂(10) R̂(11) R̂(12) R̂(13) R̂(14) R̂(15) R̂(16) ...
m=2
R(1) R(2) R(3) R(4) R(5) R(6) R(7) R(8) R(9) R(10) R(11) R(12) R(13) R(14) R(15) R(16) ...
R̂(1) R̂(2) R̂(3) R̂(4) R̂(5) R̂(6) R̂(7) R̂(8) R̂(9) R̂(10) R̂(11) R̂(12) R̂(13) R̂(14) R̂(15) R̂(16) ...
m=3
R(1) R(2) R(3) R(4) R(5) R(6) R(7) R(8) R(9) R(10) R(11) R(12) R(13) R(14) R(15) R(16) ...
R̂(1) R̂(2) R̂(3) R̂(4) R̂(5) R̂(6) R̂(7) R̂(8) R̂(9) R̂(10) R̂(11) R̂(12) R̂(13) R̂(14) R̂(15) R̂(16) ...
m=4
R(1) R(2) R(3) R(4) R(5) R(6) R(7) R(8) R(9) R(10) R(11) R(12) R(13) R(14) R(15) R(16) ...
R̂(1) R̂(2) R̂(3) R̂(4) R̂(5) R̂(6) R̂(7) R̂(8) R̂(9) R̂(10) R̂(11) R̂(12) R̂(13) R̂(14) R̂(15) R̂(16) ...
m=5
R(1) R(2) R(3) R(4) R(5) R(6) R(7) R(8) R(9) R(10) R(11) R(12) R(13) R(14) R(15) R(16) ...
R̂(1) R̂(2) R̂(3) R̂(4) R̂(5) R̂(6) R̂(7) R̂(8) R̂(9) R̂(10) R̂(11) R̂(12) R̂(13) R̂(14) R̂(15) R̂(16) ...

Fig. 1.

$R(h)$ with $\hat{R}(h)$ $(h=1, 2, \cdots)$ crosswise along every slit $S_{1n\lambda}^{\nu}$ with $n\in\boldsymbol{N}$, $\lambda=1, \cdots, l(\mu(1, n))$, and $\nu=1, \cdots, k(\mu(1, n))$. Next for each fixed $m\in\boldsymbol{N}$ with $m>1$ and subsequently fixed $j=0, 1, \cdots$ and $i=1, \cdots, \hat{m}$, join $R(i+\hat{m}j)$ with $\hat{R}(i+\hat{m}(j+1))$ for even j and $R(i+\hat{m}j)$ with $\hat{R}(i+\hat{m}(j-1))$ for odd j, crosswise along every slit $S_{mn\lambda}^{\nu}$ with $n\in\boldsymbol{N}$, $\lambda=1, \cdots, l(\mu(m, n))$, and $\nu=1, \cdots, k(\mu(m, n))$. This rather intricate procedure can be intuitively clarified by the scheme in Fig. 1.

The covering surface $R^{\sim}$ over R thus constructed with π the natural projection $R^{\sim}\rightarrow R$ is easily seen to be unbounded and infinite. It is also clear that $R^{\sim}$ is even. For any compact subset K of R, we could take R^{1} large enough so that $R^{1}\supset K$. Then there is no branch point of $R^{\sim}$ over any point of K. We will prove that $R^{\sim}$ is a Tôki covering surface of R. For this purpose we only have to show that (2) is valid for the above constructed $R^{\sim}$.

1.7. Set $R_{mn\lambda}=\pi^{-1}(B_{\mu(m,n)\lambda})$ and $L_{mn\lambda}=\pi^{-1}(l_{\mu(m,n)\lambda})$ where

$$l_{\mu(m,n)\lambda}=\{re^{i\theta}\,;\,\log r=2/k(\mu)\}$$

in $B_{\mu(m,n)\lambda}$ as represented by (4) with $\mu=\mu(m, n)$. We also set

$$R_{mn}=\bigcup_{1\leq\lambda\leq l(\mu(m,n))}R_{mn\lambda}\,,\quad L_{mn}=\bigcup_{1\leq\lambda\leq l(\mu(m,n))}L_{mn\lambda}\,.$$

Observe that R_{mn} contains all the copies of $S_{mn\lambda}^{\nu}$ $(\lambda=1, \cdots, l(\mu(m, n)), \nu=1, \cdots, k(\mu(m, n)))$ and L_{mn} passes through every copy of $S_{mn\lambda}^{\nu}$ above. We maintain the existence of a constant $\sigma\in(0, 1)$ such that

$$\sup_{L_{mn}}|v|\leqq\sigma\sup_{R_{mn}}|v| \tag{5}$$

for every $v\in HB(R_{mn})$ vanishing at branch points in R_{mn}, i.e. end points of all the copies of $S_{mn\lambda}^{\nu}$ in R_{mn} $(\lambda=1, \cdots, l(\mu(m, n)), \nu=1, \cdots, k(\mu(m, n)))$. We only have to show (5) for $L_{mn\lambda}$ and $R_{mn\lambda}$ instead of L_{mn} and R_{mn}. For this purpose let $R_{mn\lambda;s}$ be any connected component of $R_{mn\lambda}$ and set $L_{mn\lambda;s}=L_{mn\lambda}\cap R_{mn\lambda;s}$. Observe that $R_{mn\lambda;s}$ is a two sheeted covering surface over $B_{\mu(m,n)\lambda}$. We can make further reduction to prove (5). Namely we only have to prove (5) for $L_{mn\lambda;s}$ and $R_{mn\lambda;s}$ instead of $L_{mn\lambda}$ and $R_{mn\lambda}$. Again let $R_{mn\lambda;s}^{\nu}$ be the part of $R_{mn\lambda;s}$ lying over

$$2\pi(\nu-1)/k(\mu)<\theta<2\pi(\nu+1)/k(\mu)$$

and $L_{mn\lambda;s}^{\nu}$ be the part of $L_{mn\lambda;s}$ over

$$2\pi(\nu-1/2)/k(\mu)\leqq\theta\leqq 2\pi(\nu+1/2)/k(\mu)$$

for $\nu=1, \cdots, k(\mu)$ with $\mu=\mu(m, n)$. The crucial point in our reasoning is the following: Configurations $(R_{mn\lambda;s}^{\nu}, L_{mn\lambda;s}^{\nu})$ are conformally equivalent to each other for any $m\in\boldsymbol{N}$, $n\in\boldsymbol{N}$, $\lambda=1, \cdots, l(\mu(m, n))$, any s, and $\nu=1, \cdots, k(\mu)$. Therefore, as our final reduction, we only have to show the existence of a constant $\sigma\in(0,1)$ such that

$$\sup_{L^1_{mn\lambda;s}}|v|\leqq\sigma \tag{6}$$

for every $v\in H(R^1_{mn\lambda;s})$ such that $|v|\leqq 1$ on $R^1_{mn\lambda;s}$ and v vanishes at the end points of $S^1_{mn\lambda;s}$, in order to establish (5). If (6) were not the case, then there would exist a sequence $\{v_q\}$ in $H(R^1_{mn\lambda;s})$ with $|v_q|<1$ on $R^1_{mn\lambda;s}$ such that each v_q vanishes at the end points of $S^1_{mn\lambda;s}$ and that

$$\lim_{q\to\infty}(\sup_{L^1_{mn\lambda;s}}|v_q|)=1.$$

We may assume, by choosing a subsequence if necessary, that $\{v_q\}$ converges to a $v_0\in H(R^1_{mn\lambda;s})$. Obviously the $|v_0|\leqq 1$ on $R^1_{mn\lambda;s}$ and vanishes at the end points of $S^1_{mn\lambda;s}$. Clearly the supremum of $|v_0|$ on $L^1_{mn\lambda;s}$ is 1 and a fortiori the maximum principle yields that $|v_0|\equiv 1$ on $R^1_{mn\lambda;s}$ which contradicts that v_0 vanishes at the end points of $S^1_{mn\lambda;s}$.

1.8. Let T_1 be the cover transformation of $R^\sim$ such that two points in $R(h)$ and $\hat{R}(h)$ $(h=1, 2, \cdots)$ with the same projections are interchanged. For $m>1$, let T_m be the cover transformation of $R^\sim$ such that two points in $R(i+\hat{m}j)$ and $\hat{R}(i+\hat{m}(j+1))$ with the same projections are interchanged for even j and two points in $R(i+\hat{m}j)$ and $\hat{R}(i+\hat{m}(j-1))$ with the same projections are interchanged for odd j (cf. no. 1.6). Again the scheme in Fig. 1 will be helpful to see the mapping property of T_m $(m=1, 2, \cdots)$ intuitively and to be convinced that it is well defined. Take an arbitrary $u\in HB(R^\sim)$. We only have to show that u is constant on $\pi^{-1}(z)$ for any $z\in R$ in order to conclude the validity of (2). For this aim consider

$$u_m=(u-u\circ T_m)/2$$

for each fixed $m\in\boldsymbol{N}$. It is clear that $u_m\in HB(R^\sim)$ and $|u_m|\leqq M$ on $R^\sim$ where $M=\sup_{R^\sim}|u|$. Observe that u_m is qualified to be a v in (5) and therefore

$$\sup_{L_{mn}}|u_m|\leqq\sigma M.$$

This then implies that $|u_m|\leqq\sigma M$ on $R_{m,n-1}$, and again by (5) we deduce that

$$\sup_{L_{m,n-1}}|u_m|\leqq\sigma^2 M.$$

Repeating this process $n-1$ times we arrive at the conclusion

$$\sup_{L_{m,1}}|u_m|\leqq\sigma^n M.$$

Since $n\in\boldsymbol{N}$ is arbitrary, we deduce that $u_m=0$ on $L_{m,1}$, and a fortiori $u_m=0$ on $R^\sim$. Therefore $u\equiv u\circ T_m$ on $R^\sim$ for every $m\in\boldsymbol{N}$. This means that u is constant on $\pi^{-1}(z)$ for any $z\in R$.

The proof of the main theorem is herewith complete.

Minimal functions and compactifications.

2.1. We denote by $HX(R)$ the space of harmonic functions on R with a boundedness property X. In addition to $X=B$ (the finiteness of the supremum norm) we consider $X=D$ (the finiteness of the Dirichlet seminorm $D_R(u)^{1/2}=(\int_R du\wedge {}^*du)^{1/2}$) and $X=BD$ (both B and D). We also consider the class $HD^{\sim}(R)$ of nonnegative harmonic functions u on R such that there exists a decreasing sequence $\{u_n\}\subset HD(R)$ with $u_n\rightarrow u$ on R. A function u is said to be *HX-minimal* on R provided that R is hyperbolic, u is a strictly positive function in $HX(R)$, and there exists a positive constant c_v for any $v\in HX(R)$ with $u\geqq v>0$ on R such that $v=c_v u$ ($X=B, D, D^{\sim}, BD$ and $BD^{\sim}$). It is known that HX-minimal functions ($X=D, D^{\sim}$) are automatically bounded (cf. e.g. [**6**]). Therefore the notion should only be considered for $X=B, D$ and $D^{\sim}$. We will denote by $x(R)$ the cardinal number of HX-minimal functions on R when two HX-minimal functions u_1 and u_2 are identified if u_1/u_2 is a constant ($x=b, d$ and $d^{\sim}$ according as $X=B, D$ and $D^{\sim}$). Let u be an HX-minimal function on a subsurface S of a Riemann surface R such that each point in the relative boundary ∂S of S is regular with respect to the Dirichlet problem for S. Then it is well known that u has the vanishing boundary values on ∂S (cf. e.g. [**6**]).

2.2. We denote by $\Gamma_{\mathfrak{R}}(R)$ ($\Gamma_{\mathcal{W}}(R)$, resp.) the *Royden* (*Wiener*, resp.) boundary of a Riemann surface R and by $\Delta_{\mathfrak{R}}(R)$ ($\Delta_{\mathcal{W}}(R)$, resp.) the Royden (Wiener, resp.) *harmonic* boundary of R. The space $R\cup\Gamma_{\mathfrak{R}}(R)$ ($R\cup\Gamma_{\mathcal{W}}(R)$, resp.) is a compact Hausdorff space containing R as its dense subspace and is referred to as the Royden (Wiener, resp.) compactification of R. The space $HBD(R)$ ($HB(R)$, resp.) can be considered to be a subspace of $C(R\cup\Gamma_{\mathfrak{R}}(R))$ ($C(R\cup\Gamma_{\mathcal{W}}(R))$, resp.). We denote by $\mu_{\mathfrak{R}}$ ($\mu_{\mathcal{W}}$, resp.) the harmonic measure on $\Gamma_{\mathfrak{R}}(R)$ ($\Gamma_{\mathcal{W}}(R)$, resp.) with respect to a fixed center $z_0\in R$. Then $\mu_{\mathfrak{X}}(\Gamma_{\mathfrak{X}}(R)-\Delta_{\mathfrak{X}}(R))=0$ and $\Delta_{\mathfrak{X}}(R)$ is a compact subset of $\Gamma_{\mathfrak{X}}(R)$ ($\mathfrak{X}=\mathfrak{R}, \mathcal{W}$). Based on the fact that $HBD(R)|\Delta_{\mathfrak{R}}$ is dense in $C(\Delta_{\mathfrak{R}})$ and $HB(R)|\Delta_{\mathcal{W}}=C(\Delta_{\mathcal{W}})$, we see that $b(R)$ and $d(R)$ are the numbers of isolated points in $\Delta_{\mathfrak{R}}$ and $\Delta_{\mathcal{W}}$, respectively, and $d^{\sim}(R)$ is the number of points in $\Delta_{\mathfrak{R}}$ with positive $\mu_{\mathfrak{R}}$-mass. Thus in particular $x(R)$ is the countable cardinal number ($x=b, d, d^{\sim}$). For these we refer to e.g. monographs of Constantinescu-Cornea [**1**] or [**6**]. We are interested in the mapping $R\rightarrow(b(R), d(R), d^{\sim}(R))$ of hyperbolic Riemann surfaces into triples of countable cardinal numbers. In these studies the Tôki covering surfaces are very useful.

2.3. Consider a *hyperbolic* Riemann surface R and a Tôki covering surface $(R^{\sim}, R, \pi)$ of R. Then $R^{\sim}$ is also hyperbolic along with R, i.e. $R^{\sim} \notin O_G$ (the class of parabolic Riemann surfaces). In view of (2), $HBD(R^{\sim})=\boldsymbol{R}$ (the real number field), and since $HBD(R^{\sim})$ is dense in $HD(R^{\sim})$ with respect to the Dirichlet seminorm and the supremum norm on each compact subset of $R^{\sim}$, $HD(R^{\sim})=\boldsymbol{R}$. Therefore $R^{\sim} \in O_{HD}=O_{HBD}$ where O_{HX} is the class of Riemann surfaces F such that $HX(F)=\{\text{constants}\}$. Hence $\Delta_{\mathfrak{R}}(R^{\sim})$ consists of a single point. Take a sequence $\{B_n\}$, $n \in \boldsymbol{N}$, of closed parametric disks B_n such that $B_n \cap B_m=\phi$ $(n \neq m)$ and $\{B_n\}$ is locally finite in $R^{\sim}$. Here and hereafter parametric disks are assumed to be relatively compact. It is known (cf. [**6**]) that

$$\overline{\left(\bigcup_{n\in N} B_n\right)} \cap (\Gamma_{\mathfrak{R}}(R^{\sim})-\Delta_{\mathfrak{R}}(R^{\sim})) \neq \phi$$

where the closure is taken in $R^{\sim} \cup \Gamma_{\mathfrak{R}}(R^{\sim})$. We are interested in the question when the relation

$$\overline{\left(\bigcup_{n\in N} B_n\right)} \cap \Delta_{\mathfrak{R}}(R^{\sim}) \neq \phi \tag{7}$$

is valid. The following result intuitively clarifies the location of $\Delta_{\mathfrak{R}}(R^{\sim})$:

THEOREM *If there exists a closed parametric disk B in R such that $\pi^{-1}(B) = \bigcup_{n\in N} B_n$, then the relation* (7) *is valid.*

We will derive this result as a consequence of a more general assertion discussed in nos. 2.4–2.5 below.

2.4. Take a nonempty open subset S of an open Riemann surface R such that each point in ∂S is regular with respect to the Dirichlet problem for S. We denote by $HB(S; \partial S)$ the relative class consisting of $u \in HB(S) \cap C(R)$ such that $u|(R-S)=0$. We denote by $\lambda=\lambda_S$ the *inextremization* λ: $HB(R) \to HB(S; \partial S)$ and by $\mu=\mu_S$ the *extremization* μ: $HB(S; \partial S) \to HB(R)$ (cf. e.g. Noshiro [**5**, p. 103]; see Fig. 2). The composition $\lambda \circ \mu$ is always an identity map of $HB(S; \partial S)$

$$\begin{tikzcd} HB(S;\partial S) \arrow[r, "\mu"] \arrow[dr, "\lambda\circ\mu=id."'] & HB(R) \arrow[d, "\lambda"] \\ & HB(S;\partial S) \end{tikzcd} \qquad \begin{tikzcd} HB(R) \arrow[r, "\lambda"] \arrow[dr, "\mu\circ\lambda"'] & HB(S;\partial S) \arrow[d, "\mu"] \\ & HB(R) \end{tikzcd}$$

Fig. 2.

onto itself but $\mu \circ \lambda$ is not necessarily so. A subset $E \subset R$ is said to be B-*negligible* (cf. [**2**]) if there exists an S such that $R-S \supset E$ and $\mu_S \circ \lambda_S$ is an identity map of $HB(R)$ onto itself. Roughly speaking E is B-negligible if the 'closure' of E has a 'small' intersection with the ideal boundary of R, and trivial examples of B-negligible sets are compact subsets of R.

2.5. Let S be as in no. 2.4 and $S^{\sim}=\pi^{-1}(S)$. Then each point in $\partial S^{\sim}$ is also regular with respect to the Dirichlèt problem. Clearly $\pi^{*}: HB(S; \partial S) \rightarrow HB(S^{\sim}; \partial S^{\sim})$ is injective and we ask when it is surjective, viz.

$$\pi^{*}(HB(S; \partial S))=HB(S; \partial S)\circ\pi=HB(S^{\sim}; \partial S^{\sim}), \tag{8}$$

a *localization* of (2). As an answer we maintain the following

THEOREM. *If $R-S$ is B-negligible (and in particular compact), then the relation* (8) *is valid.*

We only have to show that there exists a $\hat{u}\in HB(S; \partial S)$ for any given nonnegative $u\in HB(S^{\sim}; \partial S^{\sim})$ such that $u=\hat{u}\circ\pi$. Let $v=\mu_{S^{\sim}}u$. By (2) there exists a $\hat{v}\in HB(R)$ with $v=\hat{v}\circ\pi\geqq 0$. Since μ_S is surjective (by the B-negligibility of $R-S$), there exists a $\hat{u}\in HB(S; \partial S)$ such that $\hat{v}=\mu_S\hat{u}$. Observe that $v-u\geqq 0$ and $\hat{v}-\hat{u}\geqq 0$. On setting $h=u-\hat{u}\circ\pi$, we see that $|h|\leqq(v-u)+(\hat{v}-\tilde{u})\circ\pi$. By the definition of μ, $v-u$ is a *potential* on $R^{\sim}$. Let k be a harmonic minorant of $(\hat{v}-\hat{u})\circ\pi$ on $R^{\sim}$. In view of (2) there exists a $\hat{k}\in HB(R)$ with $k=\hat{k}\circ\pi$ and a fortiori $\hat{v}-\hat{u}\geqq\hat{k}$ on R. Since $\hat{v}-\hat{u}$ is a potential on R, $\hat{k}$ and therefore k is nonpositive. Namely, any harmonic minorant of $(\hat{v}-\hat{u})\circ\pi$ is nonpositive, and hence $(\hat{v}-\hat{u})\circ\pi$ is a potential. We have seen that $|h|$ is dominated by a potential and therefore $h\equiv 0$, i.e. $u=\hat{u}\circ\pi$ with $\hat{u}\in HB(S; \partial S)$.

2.6. We prove Theorem in no. 2.3 as an application of the foregoing theorem. Suppose (7) is invalid. Then there exists a nonconstant $u\in HBD(S^{\sim}; \partial S^{\sim})$, $S^{\sim}=R^{\sim}-\bigcup_{n\in N}B_n$, such that $u|\Delta_{\mathfrak{R}}(R^{\sim})=1$ and $u|(R^{\sim}-S^{\sim})=0$. Since B is B-negligible, $S^{\sim}=\pi^{-1}(S)$ and $S=R-B$, we have (8), viz. there exists a $\hat{u}\in HB(S; \partial S)$ such that $u=\hat{u}\circ\pi$. Therefore $D_{R^{\sim}}(u)=D_R(\hat{u})\cdot\infty=\infty$, a contradiction.

Subsurfaces of Tôki covering surfaces.

3.1. We denote by $\mathcal{P}(R)$ the set of projections of the branch points of $R^{\sim}$ in R. In this section we consider only those Tôki covering surfaces $R^{\sim}$ of *hyperbolic* R such that $\mathcal{P}(R)$ is *isolated* in R. The $R^{\sim}$ constructed in Section 1 belongs to this category since even $R^{\sim}$ clearly has this property. For convenience we say that a subsurface $S^{\sim}$ of $R^{\sim}$ is *admissible* if it has a form

$$S^{\sim}=\pi^{-1}(S), \qquad S=R-K$$

where K is a compact subset contained in a region W such that each component of $\pi^{-1}(W)$ is a copy of W and each point in ∂S is regular with respect to the Dirichlet problem. The simplest example of $S^{\sim}$ is when $S=R-\bar{V}$ where V is a parametric disk with $\bar{V}\subset R-\mathscr{P}(R)$. As an extention of our former result [3] we maintain the following

THEOREM. *There exists a unique (up to multiplicative constants) $HD^{\sim}$-minimal function but no HD-minimal function on any admissible subsurface $S^{\sim}$ of a Tôki covering surface $R^{\sim}$ with an isolated set of projections of branch points in a hyperbolic Riemann surface R.*

Suppose that there exists an HD-minimal function u on $S^{\sim}$. Then $u\in HBD(S^{\sim}; \partial S^{\sim})$ and, by Theorem in no. 2.5, there exists a $\hat{u}\in HBD(S; \partial S)$ with $u=\hat{u}\circ\pi$. Since $D_{R^{\sim}}(u)=D_R(\hat{u})\cdot\infty<\infty$, u must be a constant zero, a contradiction. Therefore we only have to show the existence of a unique $HD^{\sim}$-minimal function on $S^{\sim}$, which will be carried over in nos. 3.2–3.5.

3.2. We denote by $\hat{w}$ the harmonic measure of the ideal boundary of R and hence of $S=R-K$ with respect to S. On letting $\hat{w}\equiv 0$ on K we see that $\hat{w}\in HBD(S; \partial S)$ and $\mu_S\hat{w}\equiv 1$. We set $K_\rho=\{\hat{w}\leqq\rho\}$ $(\rho\in(0, 1))$ and $K_0=K$. There exists an $\eta\in(0, 1)$ such that $K_\rho\cap\mathscr{P}(R)=\phi$, K_ρ is compact, and ∂K_ρ consists of a finite number of piecewise analytic Jordan curves for every $\rho\in(0, \eta]$. Observe that $\pi^{-1}(K_\rho)=\sum_{n\in N}(K_\rho)_n$ (disjoint union) where $(K_\rho)_n$ $(n\in \boldsymbol{N})$ are copies of K_ρ. Take any *positive* $u\in HBD(S^{\sim})$ dominating an $\hat{h}\circ\pi$ $(\hat{h}\in HB(S; \partial S))$ on $S^{\sim}$. Then, for any $\rho\in(0, \eta]$,

$$\liminf_{n\to\infty}\,(\min_{\partial(K_\rho)_n} u)\geqq\sup_S \hat{h}\,. \tag{9}$$

To prove this, fix an arbitrary positive number ε and then an $a\in S-\mathscr{P}(R)$ such that $\hat{h}(a)\geqq\sup_S \hat{h}-\varepsilon$. We can find a regular subregion $W\subset S-\mathscr{P}(R)$ such that $W\supset K_\rho\cup\{a\}$ $(\rho\in[0, \eta])$ and $\pi^{-1}(W)=\sum_{n\in N} W_n$ (disjoint union) where W_n $(n\in\boldsymbol{N})$ are copies of W with $W_n\supset(K_\rho)_n$ $(n\in\boldsymbol{N})$. Let $u_n=u|(W_n-(K_0)_n)$. Since $W_n-(K_0)_n=W_n-K_n$ may be identified with $W-K$, $\{u_n\}$ can also be viewed as a sequence of functions on $W-K$. The key observation to the proof of (9) is the following simple relation:

$$\sum_{n\in N} D_{W-K}(u_n-u(a))=\sum_{n\in N} D_{W-K}(u_n)=\sum_{n\in N} D_{W_n-K_n}(u)\leqq D_{S^{\sim}}(u)<\infty\,.$$

As a consequence of this we have

$$\lim_{n\to\infty} D_{W-K}(u_n-u_n(a))=0\,.$$

Therefore $\{u_n-u_n(a)\}$ converges to zero uniformly on each compact subset of $W-K$ and in particular on $\partial(K_\rho)_n$ $(\rho\in(0, \eta])$. Since $u_n\geqq\hat{h}$ on $W-K$, $u_n(a)\geqq\hat{h}(a)$ and a fortiori $u_n\geqq\hat{h}(a)+(u_n-u_n(a))$. Hence

$$\liminf_{n\to\infty}(\min_{\partial K_\rho} u_n)\geqq\hat{h}(a)\geqq\sup_S \hat{h}-\varepsilon\,.$$

On letting $\varepsilon \to 0$ we conclude the validity of (9).

3.3. We set $w=\hat{w}\circ\pi$ which is in $HB(S; \partial S)$. We denote by p the single point in $\Delta_{\mathfrak{R}}(R^{\sim})$. Since $\bigcup_{j=1}^{n} K_j$ is compact in $R^{\sim}$, $\Gamma_{\mathfrak{R}}(R^{\sim})$ and $\bigcup_{j=1}^{n} K_j$ are disjoint in $R^{\sim}\cup\Gamma_{\mathfrak{R}}(R^{\sim})$ and therefore there exists a unique $w_n\in HBD(R^{\sim}-\bigcup_{j=1}^{n} K_j)\cap C(R^{\sim}\cup\Gamma_{\mathfrak{R}}(R^{\sim}))$ such that $w_n(p)=1$ and $w_n|(\bigcup_{j=1}^{n} K_j)=0$ for each $n\in\boldsymbol{N}$. We maintain that

$$w=\lim_{n\to\infty} w_n \in HD^{\sim}(S^{\sim})\cap HB(S^{\sim}; \partial S^{\sim}). \tag{10}$$

Since $\{w_n\}$ $(n\in\boldsymbol{N})$ is decreasing on $R^{\sim}$, we see that $w^{\sim}=\lim_{n\to\infty} w_n$ belongs to $HD^{\sim}(S^{\sim})\cap HB(S^{\sim}; \partial S^{\sim})$. Since $\liminf_{z\to z^*}(w_n(z)-w(z))\geqq 0$ for every $z^*\in(\partial S^{\sim})\cup\{p\}$, the maximum principle (cf. e.g. [6]) yields $w_n\geqq w$ $(n\in\boldsymbol{N})$ and a fortiori $w^{\sim}\geqq w$. On the other hand, by (8), $w^{\sim}=\hat{w}^{\sim}\circ\pi$ with a $\hat{w}^{\sim}\in HB(S; \partial S)$. Here in view of $0\leqq w^{\sim}\leqq 1$ on $R^{\sim}$, we also have $0\leqq\hat{w}^{\sim}\leqq 1$ on R and a fortiori $\hat{w}^{\sim}\leqq\hat{w}$ on R. Hence $w^{\sim}=\hat{w}^{\sim}\circ\pi\leqq\hat{w}\circ\pi=w$. We thus conclude that $w^{\sim}=w$, i.e. (10) is valid.

3.4. We come to an essential part of our proof. We maintain that w is $HD^{\sim}$-minimal on $S^{\sim}$. Suppose that $0<u\leqq w$ on $S^{\sim}$ with $u\in HD^{\sim}(S^{\sim})$. Since $0<w<1$ on $S^{\sim}$, $\alpha=\sup_{S^{\sim}}u\in(0, 1]$. We will prove that $u\equiv\alpha w$ on $S^{\sim}$. Observe that $\sup_S\hat{u}=\sup_{S^{\sim}}u=\alpha$, where $\hat{u}\in HB(S; \partial S)$ with $u=\hat{u}\circ\pi$ whose existence is a consequence of $u\in HB(S^{\sim}; \partial S^{\sim})$ and (8). Hence $\hat{u}\leqq\alpha\hat{w}$ on S and a fortiori $u\leqq\alpha w$. Thus we only have to show that $u\geqq\alpha w$ on $S^{\sim}$. Let $\{u^i\}(i\in\boldsymbol{N})$ be a decreasing sequence in $HD(S^{\sim})$ converging to u on $S^{\sim}$. Replacing u^i by $u^i\wedge\alpha$ (the greatest harmonic minorant of u^i and α), if necessary, we may assume that $\alpha\geqq u^i\geqq u=\hat{u}\circ\pi$ on $S^{\sim}$. Fixing an arbitrary $\rho\in(0, \eta]$, (9) yields

$$\alpha=\sup_S \hat{u}\leqq\liminf_{n\to\infty}\left(\min_{\partial(K_\rho)_n} u^i\right)\leqq\limsup_{n\to\infty}\left(\max_{\partial(K_\rho)_n} u^i\right)\leqq\alpha.$$

This implies that

$$\lim_{n\to\infty}\left(\max_{\partial(K_\rho)_n}|u^i-\alpha|\right)=0.$$

Fix an arbitrary positive number ε and then an $m\in\boldsymbol{N}$ such that $u^i+\varepsilon>\alpha$ on $\partial(K_\rho)_n$ for every $n\geqq m$. Let $\bar{u}^i=u^i$ on $S^{\sim}-\bigcup_{n=1}^{m}(K_\rho)_n$ and $\bar{u}^i$ be in $H((K_\rho)_n-\partial(K_\rho)_n)\cap C((K_\rho)_n)$, with $\bar{u}^i=u^i$ on $\partial(K_\rho)_n$, on $(K_\rho)_n$ for $1\leqq n\leqq m$. Then $\bar{u}^i$ is a piecewise smooth continuous function on $R^{\sim}-\bigcup_{n>m}K_n=S^{\sim}\cup(\bigcup_{n=1}^{m}K_n)$ and has the finite Dirichlet integral over there. Set $v^i=\min(\bar{u}^i+\varepsilon, \alpha)$ on $R^{\sim}-\bigcup_{n>m}(K_\rho)_n$ and $v^i=\alpha$ on $\bigcup_{n>m}(K_\rho)_n$. Then v^i is piecewise smooth and has the finite Dirichlet integral over $R^{\sim}$. Therefore $v^i \in C(R^{\sim} \cup \Gamma_{\mathfrak{R}}(R^{\sim}))$ (cf. e.g. [6]). In view of (7), the closure of $\bigcup_{n\in N}(K_\rho)_n$ in $R^{\sim} \cup \Gamma_{\mathfrak{R}}(R)$ contains p, and since $\bigcup_{n=1}^{m}(K_\rho)_n$ is compact in $R^{\sim}$, the closure of $\bigcup_{n>m}(K_\rho)_n$ in $R^{\sim} \cup \Gamma_{\mathfrak{R}}(R^{\sim})$ contains p. Therefore $v^i = \alpha$ on $\bigcup_{n>m}(K_\rho)_n$ implies that $v^i(p)=\alpha$. Observe that

$$\liminf_{z\to z^*} \{(v^i(z)+\rho)-\alpha w(z)\} \geqq 0$$

for every $z^* \in (\partial(\pi^{-1}(K_\rho))) \cup \{p\}$. Hence the maximum principle yields

$$(u^i+\varepsilon)+\rho \geqq \alpha w$$

on $R^{\sim}-\pi^{-1}(K_\rho)$. On letting $\varepsilon \to 0$ we deduce that $u^i+\rho \geqq \alpha w$ on $R^{\sim}-\pi^{-1}(K_\rho)$. Then by making $\rho \to 0$ we have $u^i \geqq \alpha w$ on $R^{\sim}-\pi^{-1}(K)=S^{\sim}$ for every $i \in \boldsymbol{N}$. Again by $i \to \infty$, we conclude that $u \geqq \alpha w$ on $S^{\sim}$.

3.5. The uniqueness of the $HD^{\sim}$-minimal function is easy to see. Let u be an $HD^{\sim}$-minimal function on $S^{\sim}$. We may assume that $0<u<1$ on $S^{\sim}$. By the minimality of u, $u|\partial S^{\sim}=0$, and thus $u \in HB(S^{\sim}; \partial S^{\sim})$ by setting $u\equiv 0$ on $R^{\sim}-S^{\sim}$. By (8), $u=\hat{u}\circ\pi$ with a $\hat{u} \in HB(S; \partial S)$. Since $0<\hat{u}<1$ on S with $\hat{u}|\partial S=0$, we have $\hat{u} \leqq \hat{w}$ on S. Therefore $u=\hat{u}\circ\pi \leqq \hat{w}\circ\pi = w$ on $S^{\sim}$. By the minimality of w, there exists a constant c such that $u=cw$, viz. there exists a unique $HD^{\sim}$-minimal function w on $S^{\sim}$ up to multiplicative constants.

Classification of fibers.

4.1. We denote by $\tau=\tau_R$ the *natural mapping* of $R \cup \Gamma_{\mathfrak{W}}(R)$ onto $R \cup \Gamma_{\mathfrak{R}}(R)$, viz. τ is a continuous mapping of $R \cup \Gamma_{\mathfrak{W}}(R)$ onto $R \cup \Gamma_{\mathfrak{R}}(R)$ such that $\tau|R$ is an identity mapping. Take a $q \in \Gamma_{\mathfrak{R}}(R)$. The set $\pi^{-1}(q)$ is compact and is referred to as a *fiber* over q. In view of the relation (cf. e.g. [6])

$$\mu_{\mathfrak{W}}(\tau^{-1}(q)) = \mu_{\mathfrak{R}}(q), \tag{11}$$

it is interesting to study the fiber $\tau^{-1}(q)$ over a $q \in \Delta_{\mathfrak{R}}(R)$ with $\mu_{\mathfrak{R}}(q)>0$. We classify such fibers into three types. We say that $\tau^{-1}(q)$ is of *type* I or more precisely type I_n if there exists a sequence $\{p_j\}(1 \leqq j < n+1)$ of distinct points p_j in $\tau^{-1}(q)$ with $\mu_{\mathfrak{W}}(p_j)>0$ such that $\mu_{\mathfrak{W}}(\tau^{-1}(q)-\{p_j\})=0$. Here $n \in \bar{\boldsymbol{N}} = \boldsymbol{N} \cup \{\infty\}$, the set of countable cardinal numbers except zero, and $\infty+1=\infty$. The fiber $\tau^{-1}(q)$ is said to be of *type* II if $\mu_{\mathfrak{W}}(p)=0$ for every $p \in \tau^{-1}(q)$. If there exist a sequence $\{p_j\}(1 \leqq j < n+1)$ of distinct points p_j in $\tau^{-1}(q)$ with $\mu_{\mathfrak{W}}(p_j)>0$ and a subset E of $\tau^{-1}(q)$ with the property that $\mu_{\mathfrak{W}}(E)>0$ and $\mu_{\mathfrak{W}}(p)=0$ for any $p \in E$ such that $\tau^{-1}(q)=\{p_j\} \cup E$, then we say that the fiber $\tau^{-1}(q)$ is of *type* III or more precisely type III_n $(n \in \bar{\boldsymbol{N}})$.

4.2. Let $q\in\Delta_{\mathfrak{R}}(R)$ with $\mu_{\mathfrak{R}}(q)>0$. We maintain that the fiber $\tau^{-1}(q)$ is either of type I_n $(n\in\bar{\boldsymbol{N}})$, type II, or type III_n $(n\in\bar{\boldsymbol{N}})$. In fact, let $F=\{p_j\in\tau^{-1}(q)\,;\,\mu_{\mathfrak{W}}(p_j)>0\}$ and $E=\tau^{-1}(q)-F$. In view of the relation (11) and $\mu_{\mathfrak{R}}(q)\leqq\mu_{\mathfrak{R}}(\Delta_{\mathfrak{R}}(R))=1$, we see that F is a countable set. If $F=\phi$, then $\tau^{-1}(q)$ is of type II. Suppose $F\neq\phi$ and $F=\{p_j\,;\,1\leqq j<n+1\}(n\in\bar{\boldsymbol{N}})$. If moreover $\mu_{\mathfrak{W}}(E)=0$, then $\tau^{-1}(q)$ is of type I_n. If $\mu_{\mathfrak{W}}(E)>0$, then $\tau^{-1}(q)$ is of type III_n. Thus merely classifying fibers $\tau^{-1}(q)$ into three types is trivial and really nontrivial part is to show the existence of $(R,\,q)$ such that $\tau^{-1}(q)$ is of any type I, II, and III in which the existence of Tôki covering surface of any open Riemann surface is very conveniently made use of.

4.3. Take a hyperbolic Riemann surface R and a Tôki covering surface $R^{\sim}$ of R. Then $\Delta_{\mathfrak{R}}(R^{\sim})$ consists of a single point q with $\mu_{\mathfrak{R}}(q)>0$. Then $\tau^{-1}(q)=\tau_{R^{\sim}}^{-1}(q)\supseteqq\Delta_{\mathfrak{W}}(R^{\sim})$. By (2) we see that the measure spaces $(\Delta_{\mathfrak{W}}(R^{\sim}),\,\mu_{\mathfrak{W},R^{\sim}})$ and $(\Delta_{\mathfrak{W}}(R),\,\mu_{\mathfrak{W},R})$ can be identified, viz. we have the following relation for a Tôki covering surface $R^{\sim}$ of a hyperbolic Riemann surface R:

$$(\tau^{-1}(\Delta_{\mathfrak{R}}(R^{\sim})),\,\mu_{\mathfrak{W},R^{\sim}})=(\Delta_{\mathfrak{W}}(R^{\sim}),\,\mu_{\mathfrak{W},R^{\sim}})\approx(\Delta_{\mathfrak{W}}(R),\,\mu_{\mathfrak{W},R}) \tag{12}$$

where $\approx$ means an isomorphism as topological measure spaces. Thus we can produce fibers $\tau^{-1}(q)=\tau^{-1}(\Delta_{\mathfrak{R}}(R^{\sim}))$ as $\Delta_{\mathfrak{W}}(R)$ quite arbitrarily by choosing R suitably. For example, take R as the open unit disk $|z|<1$. Then each point of $\Delta_{\mathfrak{W}}(R)$ has $\mu_{\mathfrak{W}}$-measure zero and therefore $\tau^{-1}(q)$ is of type II. It is known that there exists an R in the class O_{HB}^{n} (cf. e. g. [6]) which may be characterized by that $\Delta_{\mathfrak{W}}(R)=\{p_j\,;\,1\leqq j<n+1\}\cup E$, where $p_i\neq p_j\,(i\neq j)$, $\mu_{\mathfrak{W}}(p_j)>0$, and $\mu_{\mathfrak{W}}(E)=0\,(n\in\bar{\boldsymbol{N}})$. Then $\tau^{-1}(q)$ is of type I_n $(n\in\bar{\boldsymbol{N}})$. Remove a closed parametric disk from the above surface and let R be the resulting surface. Then $\tau^{-1}(q)$ is of type III_n $(n\in\bar{\boldsymbol{N}})$. Thus we have obtained the following

THEOREM. *The fiber $\tau^{-1}(q)$ over a point $q\in\Delta_{\mathfrak{R}}(R)$ of positive $\mu_{\mathfrak{R}}$-measure can be classified into three types* I_n, II, *and* III_n, *and there really exist an R and $q\in\Delta_{\mathfrak{R}}(R)$ of positive $\mu_{\mathfrak{R}}$-measure such that the fiber $\tau^{-1}(q)$ is of any given type* I_n, II, *and* III_n $(n\in\bar{\boldsymbol{N}})$.

Surfaces with given harmonic dimensions.

5.1. The cardinal number $x(R)\,(x=b,\,d,\,d^{\sim})$ (cf. no. 2.2) is also called the *X-harmonic dimension* $(X=B,\,D,\,D^{\sim})$ of R. We denote by $\boldsymbol{R}$ the class of open Riemann surfaces and consider a mapping $\delta:\boldsymbol{R}\to\bar{\boldsymbol{N}}_0^3=\bar{\boldsymbol{N}}_0\times\bar{\boldsymbol{N}}_0\times\bar{\boldsymbol{N}}_0$ such that $\delta(R)=(b(R),\,d(R),\,d^{\sim}(R))$ where $\bar{\boldsymbol{N}}_0=\{0\}\cup\bar{\boldsymbol{N}}=\boldsymbol{N}\cup\{0,\,\infty\}$. We wish to determine the range $\delta(\boldsymbol{R})$ in $\bar{\boldsymbol{N}}_0^3$. In other words we are interested in the following problem: *Find an open Riemann surface R such that $x(R)=x\,(x=b,\,d,\,d^{\sim})$ for a* *given triple* $(b, d, d^{\sim})$ *of countable cardinal numbers.* We will give a necessary and sufficient condition on the triple $(b, d, d^{\sim})$ such that the above problem has a solution.

5.2. As a preparation we consider a countable family $\{R_k\}(1 \leqq k < N)$ $(N \in \bar{N}, N > 1)$ of hyperbolic Riemann surfaces R_k. Let U_k be a parametric disk in R_k. For convenience we represent U_k as the 'disk' $1/4 < |z-(3k-2)| \leqq \infty$ about the point at infinity ∞ of $\hat{C} = C \cup \{\infty\}$, where C is the finite complex plane. We denote by V_k the concentric parametric 'disk' $1 < |z-(3k-2)| \leqq \infty$ and α_k the curve $|z-(3k-2)| = 1/2$ in U_k. Let w_k be the harmonic measure of the ideal boundary of R_k with respect to $R_k - \bar{V}_k$. We extend w_k to R_k so as to be in $C(R_k)$ by setting $w_k \equiv 0$ on $\bar{V}_k$. By choosing U_k sufficiently small in R_k we may assume that

$$(13) \qquad \begin{cases} D_{R_k}(w_k) < 1/2^k \\ \inf_{\alpha_k} w_k > 1/2 \,. \end{cases}$$

Let $W = \hat{C} - \overline{\bigcup_{1 \leq k < N} \{|z-(3k-2)| < 1\}}$. Weld each $R_k - \bar{V}_k$ to W by identifying $|z-(3k-2)| = 1$ in $R_k - V_k$ and $\bar{W}$. The resulting Riemann surface will be denoted by $\bigoplus_{1 \leq k < N} R_k$. As a consequence of (13) we have the following identity:

$$(14) \qquad x\Big(\bigoplus_{1 \leq k < N} R_k\Big) = \sum_{1 \leq k < N} x(R_k) \qquad (x = b, d, d^{\sim}).$$

This relation is trivial for $N < \infty$ and the condition (13) is redundant for the validity of (14) for $N < \infty$. The relation must be well known even for the case $N = \infty$ but we cannot locate the exact reference except for [4].

5.3. A triple $(b, d, d^{\sim})$ of countable cardinal numbers (i.e. $(b, d, d^{\sim}) \in \bar{N}_0^3$) will be referred to as being *solvable* if the following condition is satisfied:

$$(15) \qquad \begin{cases} \textit{If } d^{\sim} \geqq 1, \textit{ then } b \textit{ is arbitrary and } d \leqq d^{\sim}; \\ \textit{If } d^{\sim} = 0, \textit{ then } b = d = 0 \,. \end{cases}$$

We will prove that the image set $\delta(\boldsymbol{R}) \subset \bar{N}_0^3$ is the set of solvable triples, i.e. we will prove the following

THEOREM. *There exists a Riemann surface R such that $x(R) = x$ $(x = b, d, d^{\sim})$ if and only if the triple $(b, d, d^{\sim})$ is solvable.*

For convenience we denote by $R_{bdd^{\sim}}$ a Riemann surface such that $x(R_{bdd^{\sim}}) = x$ $(x = b, d, d^{\sim})$. Observe that an HD-minimal function is always an $HD^{\sim}$-minimal function, i.e. $d(R) \leqq d^{\sim}(R)$. Suppose that there exists an HB-minimal function on R. Then $\Delta_{\mathcal{W}}(R)$ contains a point p with $\mu_{\mathcal{W}}(p) > 0$ and thus, by

(11), $\mu_{\mathfrak{R}}(q)>0$ with $q=\tau(p)$ which implies the existence of an $HD^{\sim}$-minimal function (cf. e.g. [6]). Therefore $b(R)\geqq 1$ implies $d^{\sim}(R)\geqq 1$, or equivalently, $d^{\sim}(R)=0$ implies $b(R)=0$. From these observations it follows that the existence of an $R_{bdd^{\sim}}$ assures the solvability of the triple $(b, d, d^{\sim})$. Conversely assume that $(b, d, d^{\sim})$ is a solvable triple. We will prove the existence of an $R_{bdd^{\sim}}$. Any (hyperbolic) subregion of $\hat{C}$ is an R_{000}, and the nontrivial case is when $d^{\sim}\geqq 1$. Let $n\in\bar{N}_0$ be arbitrarily given. There exists a hyperbolic Riemann surface $R(n)$ belonging to the class O_{HB}^{n} for the case $n\geqq 1$ (cf. e.g. [6]) and, e.g. $R(0)=\{|z|<1\}$, so that $b(R(n))=n$. Then an even Tôki covering surface $R(n)^{\sim}$ of $R(n)$ is an R_{n11}. By Theorem 3.1 an admissible subsurface $S^{\sim}$ of $R(n)^{\sim}$ is an R_{n01}. Thus surfaces R_{n11} and R_{n01} exist for any $n\in\bar{N}_0$. Assume first that $d=d^{\sim}$. There exists a sequence $\{b_k\}\subset\bar{N}_0$ such that $\sum_{1\leq k<d+1} b_k=b$. Let $R_k=R_{b_k 11}$ and consider $\bigoplus_{1\leq k<d+1} R_k$. By (14) we see that $\bigoplus_{1\leq k<d+1} R_k$ is an $R_{bdd^{\sim}}$. Next consider the case $d<d^{\sim}$. We choose a sequence $\{b_k\}\subset\bar{N}_0$ such that $b=\sum_{1\leq k<d^{\sim}+1} b_k$. If $d=0$, then, by (14), $\bigoplus_{1\leq k<d^{\sim}+1} R_k$ with $R_k=R_{b_k 01}$ is an $R_{b0d^{\sim}}$. If $d>0$, then let $R_k=R_{b_k 11}$ $(1\leqq k<d+1)$ and $R_k=R_{b_k 01}$ $(d<k<d^{\sim}+1)$. Once more by (14) we see that $\bigoplus_{1\leq k<d^{\sim}+1} R_k$ is an $R_{bdd^{\sim}}$.

References

[1] C. Constantinescu and A. Cornea, Ideale Ränder Riemannscher Flächen, Springer-Verlag, 1963.
[2] M. Nakai, Order comparisons on canonical isomorphism, Nagoya Math. J., **50** (1973), 67-87.
[3] M. Nakai and S. Segawa, On a subsurface of Tôki surface, Tôhoku Math. J., **28** (1976), 591-600.
[4] M. Nakai and S. Segawa, Harmonic dimensions related to Dirichlet integrals, J. Math. Soc. Japan, **29** (1977), 107-121.
[5] K. Noshiro, Cluster Sets, Springer-Verlag, 1960.
[6] L. Sario and M. Nakai, Classification Theory of Riemann Surfaces, Springer-Verlag, 1970.
[7] Y. Tôki, On the examples in the classification of open Riemann surfaces I, Osaka Math. J., **5** (1953), 267-280.

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