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On an optimal stopping problem and a variational inequality

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A. Bensoussan and J.L. Lions ([1]) has revealed a relation between an optimal stopping problem of an additive functional of a diffusion process and a certain variational inequality. More specifically let $y(t)$ be the solution of the stochastic differential equation:

$$(0.1)\qquad \begin{cases} dy(t)=\sigma(y(t), t)dB_t+g(y(t), t)dt \\ y(t_0)=y_0 . \end{cases}$$

Then they showed that the continuous and strong solution of the following variational inequality (0.2) is the solution of the optimal stopping problem:

$$u(x, s)\equiv E_{x\cdot s}\left[\int_s^{\tau_B^s} e^{-\alpha(t-s)}C(y(t), t)dt+e^{-\alpha(\tau_B^s-s)}D(y(\tau_B^s), \tau_B^s)\right]$$

$$=\inf_{\tau_s} E_{x,s}\left[\int_s^{\tau_s} e^{-\alpha(t-s)}C(y(t), t)+e^{-\alpha(\tau_s-s)}D(y(\tau_s), \tau_s)\right].$$

$$(0.2)\qquad \begin{cases} -\left(\dfrac{\partial u}{\partial t}, v-u\right)+\mathcal{E}_t(u, v-u)+\alpha(u, v-u)\geqq(C, v-u) \\ \qquad\qquad \text{for all} \quad v\in\mathcal{D}[\mathcal{E}_t] \text{ such that } v\leqq D \\ u\in\mathcal{D}[\mathcal{E}_t] \text{ such that } v\leqq D . \end{cases}$$

Here $A(t)$ is the generator of the diffusion process $y(t)$, $\mathcal{E}_t$ is the bilinear form associated with $A(t)$ and $\mathcal{D}[\mathcal{E}_t]$ is the domain of $\mathcal{E}_t$.

However it is in general not easy to show that the (weak) solution of (0.2) is the continuous and strong one, namely, a continuous solution of

$$(0.3)\qquad \begin{cases} -\dfrac{\partial u}{\partial t}+(\alpha-A(t))u-C\leqq 0 \\ \left\{-\dfrac{\partial u}{\partial t}+(\alpha-A(t))u-C\right\}(u-D)=0 \\ u\leqq D . \end{cases}$$

Some smoothness condition on σ, D are required in order to derive (0.3) from (0.2) (see [2]).

In this paper, we take up a general symmetric, temporary homogeneous Markov process and formulate an optimal stopping problem of a general additive functional. We then show that the weak solution of the variational inequality corresponding to (0.2) is just the solution of the optimal stopping problem of the additive functional. By virtue of the potential theory of the Markov process and associated Dirichlet space ([3], [4], [5]) we can dispense with difficult argument on the regularities of the weak solution.

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§ 1. Preliminaries.

Let X be a locally compact Hausdorff space with countable base and m be a positive Radon everywhere dense measure on X. We denote a m-symmetric standard process on X by $\boldsymbol{M}=\{\Omega, \mathcal{M}, \mathcal{M}_t, X_t, P_x, \theta_t, \zeta\}$ and its transition semi-group and resolvent by P_t and R_α respectively. We introduce a function space

$$\mathcal{F}=\left\{u\in L^2(X\,;\,m)\,;\ \lim_{t\downarrow 0}\frac{1}{t}(u-T_t u,\, u)<\infty\right\}$$

where T_t is the L^2-semi-group induced by the transition semi-group P_t of $\boldsymbol{M}$ and $(\ ,\ )$ is a inner product in $L^2(X, m)$:

$$(u,\, v)=\int u(x)v(x)m(dx)\,.$$

$\mathcal{F}$ is also described by the L^2-Resolvent G_α induced by R_α as follows:

$$\mathcal{F}=\{u\in L^2\,;\ \lim_{\alpha\to\infty}\alpha(u-\alpha G_\alpha u,\, u)<\infty\}\,.$$

Then we can define a symmetric bilinear form $\mathcal{E}$ on $\mathcal{F}$ by the relation

$$\mathcal{E}(u,\, v)=\lim_{t\downarrow 0}\frac{1}{t}(u-T_t u,\, v)\,.$$

In general a nonnegative symmetric bilinear form $\mathcal{E}^0$ defined on the product $\mathcal{F}^0\times\mathcal{F}^0$ of a linear subspace $\mathcal{F}^0$ of L^2 is called *a Dirichlet space* if the following $(\mathcal{E}.1)\sim(\mathcal{E}.3)$ hold:

($\mathcal{E}$.1) *$\mathcal{F}^0$ is dense in L^2*

($\mathcal{E}$.2) *$\mathcal{F}^0$ is a closed linear subspace of L^2 with respect to the norm*

$$\sqrt{\mathcal{E}_1^0(u,\, u)}=\sqrt{\mathcal{E}^0(u,\, u)+(u,\, u)}$$

($\mathcal{E}$.3) *if for* $u\in\mathcal{F}^0$ *and* $v\in L^2$ *there exists Borel functions* $\tilde{u}$, $\tilde{v}$ *such that* $u=\tilde{u}$ m-a.e., $v=\tilde{v}$ m-a.e., $|\tilde{v}(x)|\leqq|\tilde{u}(x)|$ *for each* $x\in X$ *and* $|\tilde{v}(x)-\tilde{v}(y)|\leqq|\tilde{u}(x)-\tilde{u}(y)|$ *for each* $x, y\in X$, *then* $v\in\mathcal{F}^0$ *and* $\mathcal{E}^0(v, v)\leqq\mathcal{E}^0(u, u)$.

In the present case $(\mathcal{F}, \mathcal{E})$ satisfies ($\mathcal{E}$.1)$\sim$($\mathcal{E}$.3), so it called the Dirichlet space associated with $\boldsymbol{M}$.

In this paper we assume that $(\mathcal{F}, \mathcal{E})$ is C_0-regular, that is, $\mathcal{F}\cap C_0$ is uniformly dense in $C_0(X)$ and $\mathcal{E}_1$ dense in $\mathcal{F}$. Here $C_0(X)$ is a family of all continuous functions on X with compact support.

We now introduce some related notions and summarize the known results necessary for the proof of our theorem.

DEFINITION 1. The *capacity* of a subset of X is defined as follows: For open $A\subset X$

$$\mathrm{Cap}\,(A)=\begin{cases}\inf\{\mathcal{E}_1(u, u)\,;\ u\in\mathcal{L}_A\} & \mathcal{L}_A\neq\phi\\ \infty & \mathcal{L}_A=\phi\,,\end{cases}$$

where $\mathcal{L}_A=\{u\in\mathcal{F}\,;\ u\geqq 1\ m\text{-a.e.}\ (A)\}$. For general $B\subset X$

$$\mathrm{Cap}\,(B)=\inf\{\mathrm{Cap}\,(A)\,;\ B\subset A,\ A \text{ is open}\}\,.$$

DEFINITION 2. A subset B of X with $\mathrm{Cap}\,(B)=0$ is called *almost polar.* "*Quasi-everywhere*" or "*q.e.*" will mean "except on an almost polar set".

1°) Following three statements are equivalent for a subset B of X:

1°. 1) $\mathrm{Cap}\,(B)=0$,

1°. 2) there exists a Borel set B' such that $B\subset B'$ and $P_x(\tau_{B'}<\infty)=0$ m-a.e. x,

1°. 3) there exists a Borel set $B'\supset B$ with $m(B')=0$ satisfying

(1°.3.1) $\quad P_x(X_{t-}$ or $X_t\in B'$, for some $t\in[0, \infty))=0$ for each $x\in X-B'$.

Here $\tau_B=\inf\{t: X_t\in B\}$ (cf. [3], [4]).

DEFINITION 3. A Borel almost polar set B' satisfying (1°.3) is called *a proper exceptional set.*

Obviously $X-B'$ is finely open if B' is proper exceptional.

DEFINITION 4. A function $f(x)$ on X is said to be *quasi-continuous* provided that for any $\varepsilon>0$ there exists an open subset G_ε of X such that $\mathrm{Cap}\,(G_\varepsilon)<\varepsilon$ and $f|_{X_\Delta-G_\varepsilon}$ is continuous, where Δ is a one point compactification of X and f is considered as a function extended to X_Δ with $f(\Delta)=0$.

2°) Any function $f\in\mathcal{F}$ has its quasi-continuous version $\tilde{f}$: $f(x)=\tilde{f}(x)$ m-a.e. and $\tilde{f}$ is quasi-continuous (cf. [3]).

From now on we use the expression $\tilde{f}$ for a quasi-continuous version of f.

3°) If a function $f(x)$ is quasi-continuous, then there exists a proper exceptional set N such that $f(x)$ is finely continuous on $X-N$ (cf. [3]).

4°) For each $\alpha>0$, following five statements are equivalent for $u\in\mathcal{F}$:

4°. 1) $u\geqq 0$, $e^{-\alpha t}T_t u\leqq u$ m-a. e.,

4°. 2) $u\geqq 0$, $\beta G_{\beta+\alpha}u\leqq u$ m-a. e., for each $\beta>0$,

4°. 3) $\mathcal{E}_\alpha(u, v)\geqq 0$ for all $v\geqq 0$, $\in\mathcal{F}$,

4°. 4) there exists a unique positive Radon measure μ such that $\mathcal{E}_\alpha(u, v)$ $=\int v(x)\mu(dx)$ for all $v\in\mathcal{F}\cap C_0(X)$,

4°. 5) there exists a unique positive Radon measure μ such that $\mathcal{E}_\alpha(u, v)$ $=\int \tilde{v}(x)\mu(dx)$ for all $v\in\mathcal{F}$ (cf. [3], [4]).

DEFINITION 5. If one of (4°.1)$\sim$(4°.5) holds we call u *an almost α-excessive function.* The positive Radon measure μ in 4°. 4) is called the measure of finite energy. u is denoted by $U_\alpha\mu$.

We denote by $\mathfrak{M}$ a family of all positive Radon measure with finite energy:

DEFINITION 6. A function $A_t(\omega)$: $[0, \infty)\times\Omega \rightsquigarrow (-\infty, +\infty]$ is called *an additive functional* provided that:

(A.1) *$A_t(\omega)$ is $\mathcal{M}_t$-adapted*

and there exist $\Lambda\in\mathcal{M}$ and an almost polar set N such that $P_x(\Lambda)=1$ for each $x\in X-N$ and following properties are satisfied for each $\omega\in\Lambda$;

(A.2) *$A_t(\omega)$ is finite and right continuous in $t\geqq 0$ and has a left limit in $0<t<\zeta$,*

(A.3) $A_t(\omega)=A_{\zeta-}(\omega)$, $\quad t\geqq\zeta$,

(A.4) $A_{t+s}(\omega)=A_t(\omega)+A_s(\theta_t\omega)$, $\quad t, s\geqq 0$

(A.5) $A_0(\omega)=0$.

When we assume the following (A.2)′ in place of (A.2), A is said to be *continuous additive functional.*

(A.2)′ *$A_t(\omega)$ is finite and continuous in $t\geqq 0$.*

Additive functionals A and B are said to be *equivalent* if for each t, $P_x(A_t=B_t)=1$, q. e. x.

We denote by $\mathcal{A}$ a family of all additive functionals and by $\mathcal{A}_c^+$ a family of all non-negative continuous additive functionals.

5°) i) For each $\mu\in\mathfrak{M}$ there exists $A\in\mathcal{A}_c^+$ such that $E_x\left[\int_0^\infty e^{-\alpha t}dA_t\right]$ is a quasi-continuous version of $U_\alpha\mu$.

ii) When μ and A are related as above $E_x\left[\int_0^\infty e^{-\alpha t}f(X_t)dA_t\right]$ is a quasi-continuous version of $U_\alpha(f\mu)$ for each bounded Borel function f.

iii) A in (i) is uniquely determined by $\mu\in\mathfrak{M}$ up to the above mentioned equivalence (cf. [5]).

§ 2. Statement of Theorem.

Let ν be a Radon measure such that

$$\nu=\nu_1-\nu_2\,, \qquad \nu_1,\ \nu_2\in\mathfrak{M}\,,$$

then there exists an additive functional

$$A_t=A_t^1-A_t^2\,, \qquad A_t^1,\ A_t^2\in\mathcal{A}_c^+\,,$$

where A_t (resp. A_t^2) is the unique non-negative continuous additive functional whose α-potential is a quasi-continuous version of $U_\alpha\nu_1$ (resp. $U_\alpha\nu_2$):

$$E_x\left[\int_0^\infty e^{-\alpha t}dA_t^i\right]=U_\alpha\nu_i \qquad m\text{-a. e. } i=1,\ 2\,.$$

We put $U_\alpha\nu=U_\alpha\nu_1-U_\alpha\nu_2$, then $E_x\left[\int_0^\infty e^{-\alpha t}dA_t\right]$ is a quasi-continuous version of $U_\alpha\nu$:

$$E_x\left[\int_0^\infty e^{-\alpha t}dA_t\right]=U_\alpha\nu \qquad m\text{-a. e.} \tag{2.1}$$

Now our problem is the following.

(O.S.P.) "Find a function u such that there exists a proper exceptional set N and a Borel finely closed subset B of $X-N$ for which

$$u(x)\equiv E_x\left[\int_0^{\tau_B} e^{-\alpha t}dA_t\right]=\inf_\tau E_x\left[\int_0^{\tau} e^{-\alpha t}dA_t\right] \qquad \text{for each } x\in X-N\,, \tag{2.2}$$

where α is a fixed positive constant, τ ranges over all stopping times of $\boldsymbol{M}$ and $\tau_B=\inf\{t\,;\,X_t\in B\}$." We call B in (2.2) an optimal region. The solution u_1 and u_2 of (O.S.P.) are said to be equivalent if $u_1=u_2$ q. e.

Our concern is the relation between the above (O.S.P.) and the following variational inequality (2.3).

$$\begin{cases} u\in\mathcal{R}\,, \\ \mathcal{E}_\alpha(u,\ v-u)\geqq\langle\nu,\ \tilde{v}-\tilde{u}\rangle \qquad \text{for all } v\in\mathcal{R}\,, \end{cases} \tag{2.3}$$

where $\mathcal{R}=\{v\in\mathcal{F}\,;\ v\leqq 0\ m\text{-a. e.}\}$, and

$$\langle\nu,\ \tilde{v}-\tilde{u}\rangle=\int\nu(dx)\{\tilde{v}(x)-\tilde{u}(x)\}\ .$$

We note that (2.3) has a unique solution. Indeed, (2.3) is rewritten as follows:

$$\begin{cases} \mathcal{E}_\alpha(u-U_\alpha\nu,\ v-u)\geqq 0 \qquad \text{for all } v\in\mathfrak{R}\,, \\ u\in\mathfrak{R} \end{cases}$$

which is also equivalent to

$$\begin{cases} \mathcal{E}_\alpha(u-U_\alpha\nu,\ u-U_\alpha\nu)\leqq\mathcal{E}_\alpha(v-U_\alpha\nu,\ v-U_\alpha v) \qquad \text{for all } v\in\mathfrak{R}\,, \\ u\in\mathfrak{R}\,, \end{cases} \tag{2.4}$$

because $\mathfrak{R}$ is convex. Now we can see that there exists a unique solution of (2.4) by making use of the Parallelogram Law and the closedness of the convex set $\mathfrak{R}$.

THEOREM. *There exists a unique (up to the equivalence specified above) solution of (O.S.P.) and any quasi-continuous version of the solution of the variational inequality* (2.3) *is the solution.*

§ 3. Proof of Theorem.

We prepare a lemma which is essentially due to A. Bensoussan and J. L. Lions ([1]).

LEMMA. *If there exists a proper exceptional set N and some Borel finely closed subset B of $X-N$ for which a function $u(x)$ satisfies the following* (3.1)~(3.4), *then u is a solution of (O.S.P.):*

$$u(x)\leqq 0 \qquad \textit{for any } x\in X-N \tag{3.1}$$

$$u(x)=0 \qquad \textit{for any } x\in B \tag{3.2}$$

$$u(x)\leqq E_x\left[\int_0^\tau e^{-\alpha t}dA_t+e^{-\alpha\tau}u(X_\tau)\right] \qquad \textit{for any } x\in X-N \tag{3.3}$$

and for any stopping time τ

$$u(x)=E_x\left[\int_0^\tau e^{-\alpha t}dA_t+e^{-\alpha\tau}u(X_\tau)\right] \qquad \textit{for any } x\in X-N \tag{3.4}$$

and for any stopping time $\tau\leqq\tau_B$.

PROOF. It follows from (3.1) and (3.3) that

$$u(x)\leqq E_x\left[\int_0^\tau e^{-\alpha t}dA_t\right] \qquad \text{for any } x\in X-N \text{ and for any } \tau\,. \tag{3.5}$$

Since B is finely closed

$$P_x(X_{\tau_B}\in B\,;\ \tau_B<\infty)=P_x(\tau_B<\infty) \qquad \text{for any } x\in X-N\,.$$

Then by (3.2)

$$E_x[e^{-\alpha\tau_B}u(X_{\tau_B})]=0\,.$$

Therefore we obtain by (3.4)

$$u(x)=E_x\Big[\int_0^{\tau_B}e^{-\alpha t}dA_t\Big] \qquad \text{for any } x\in X-N\,. \tag{3.6}$$

(3.5) and (3.6) show that $u(x)$ is a solution of (O.S.P.).

Now we proceed to the proof of our theorem.

At first we note that if $u_1(x)=E_x\Big[\int_0^{\tau_{B_1}}e^{-\alpha t}dA_t\Big]$ and $u_2(x)=E_x\Big[\int_0^{\tau_{B_2}}e^{-\alpha t}dA_t\Big]$ are solutions of (O.S.P.), then $u_1(x)=u_2(x)$ q.e. because $u_1(x)\leqq E_x\Big[\int_0^{\tau_{B_2}}e^{-\alpha t}dA_t\Big]$ $=u_2(x)$ q.e. and $u_2(x)\leqq E_x\Big[\int_0^{\tau_{B_1}}e^{-\alpha t}dA_t\Big]=u_1(x)$ q.e.

We further note that if $u(x)=E_x\Big[\int_0^{\tau_B}e^{-\alpha t}dA_t\Big]$ is a solution of (O.S.P.) and $\tilde{u}(x)=u(x)$ q.e. then $\tilde{u}(x)$ is also a solution of (O.S.P.). To prove this we denote by N_1 a proper exceptional set of $u(x)$ and put $N_2=\{x\,;\,u(x)\neq\tilde{u}(x)\}$. Since N_2 is almost polar, there exists a proper exceptional set N_2' such that $N_2'\supset N_1\cup N_2$. Then for each $x\in X-N_2'$, $\tilde{u}(x)=E_x\Big[\int_0^{\tau_B}e^{-\alpha t}dA_t\Big]$. On the other hand $B'=B-N_2'$ is a Borel finely closed subset of $X-N_2'$ and $E_x\Big[\int_0^{\tau_B}e^{-\alpha t}dA_t\Big]$ $=E_x\Big[\int_0^{\tau_{B'}}e^{-\alpha t}dA_t\Big]$ for each $x\in X-N_2'$. Therefore $\tilde{u}(x)$ is a solution of (O.S.P.).

Now it suffices to show that some quasi-continuous version of the solution of the variational inequality (2.3) is a solution of (O.S.P.). Take the solution u of (2.3) which is equivalent to the following double relations;

$$\begin{cases} \mathcal{E}_\alpha(u,\, v)\geqq\langle\nu,\, \tilde{v}\rangle & \text{for all } v\in\mathfrak{R} \\ \mathcal{E}_\alpha(u,\, u)=\langle\nu,\, \tilde{u}\rangle\,. & \end{cases}$$

Then we have

$$\begin{cases} \mathcal{E}_\alpha(U_\alpha\nu-u,\, -v)\geqq 0 \quad \text{for all } v\in\mathfrak{R} & (3.7) \\ \mathcal{E}_\alpha(U_\alpha\nu-u,\, u)=0\,. & (3.8) \end{cases}$$

(3.7) shows that $g\equiv U_\alpha\nu-u$ is almost α-excessive. Therefore there exists uniquely a non-negative continuous additive functional A_t^g such that its α-potential $E_x=\Big[\int_0^\infty e^{-\alpha t}dA_t^g\Big]$ is a quasi-continuous version of g. Now we put

$$u^*(x)=E_x\Big[\int_0^\infty e^{-\alpha t}dA_t\Big]-E_x\Big[\int_0^\infty e^{-\alpha t}dA_t^g\Big]\,, \qquad \text{q.e. } x\,, \tag{3.9}$$

and we will show that u^* satisfies the conditions of Lemma.

Since $u^*(x)$ is a quasi-continuous version of u and $u(x) \leqq 0$ m-a.e., we have

$$u^*(x) \leqq 0 \qquad \text{q.e.} \tag{3.10}$$

On the other hand, for all stopping time τ,

$$\begin{aligned} u^*(x) &= E_x\left[\int_0^\tau e^{-\alpha t} dA_t\right] + E_x\left[e^{-\alpha\tau} E_{x_\tau}\left[\int_0^\infty e^{-\alpha t} dA_t\right]\right] \\ &\quad - E_x\left[\int_0^\tau e^{-\alpha t} dA_t^g\right] - E_x\left[e^{-\alpha\tau} E_{x_\tau}\left[\int_0^\infty e^{-\alpha t} dA_t^g\right]\right] \\ &= E_x\left[\int_0^\tau e^{-\alpha t} dA_t + e^{-\alpha\tau} u^*(X_\tau)\right] - E_x\left[\int_0^\tau e^{-\alpha t} dA_t^g\right] \qquad \text{q.e.} \end{aligned} \tag{3.11}$$

Here A_t^g is a non-negative additive functional, so

$$E_x\left[\int_0^\tau e^{-\alpha t} dA_t^g\right] \geqq 0 \qquad \text{q.e.}$$

Therefore we get

$$u^*(x) \leqq E_x\left[\int_0^\tau e^{-\alpha t} dA_t + e^{-\alpha\tau} u^*(X_\tau)\right] \qquad \text{q.e.} \tag{3.12}$$

Denote by μ a positive Radon measure with finite energy corresponding to the excessive function g:

$$\mathcal{E}_\alpha(g, v) = \int \mu(dx) \tilde{v}(x) \qquad \text{for all } v \in \mathcal{F}.$$

Then from (3.8) it follows that

$$\int \mu(dx) u^*(x) = 0. \tag{3.13}$$

Since $u^*(x)$ is quasi-continuous, there exists a proper exceptional set N^0 such that $u^*(x)$ is Borel finely continuous on $X - N^0$. Therefore the set

$$B = \{x \in X - N^0 \,;\, u^*(x) = 0\}$$

is a Borel finely closed subset of $X - N^0$. From (3.10) and (3.13) it follows that $\mu(B^c) = 0$ and so $U_\alpha(I_{B^c}\mu) = 0$ m-a.e. On the other hand

$$E_x\left[\int_0^\infty e^{-\alpha t} I_{B^c}(X_t) dA_t^g\right] = U_\alpha(I_{B^c}\mu) \qquad m\text{-a.e.}$$

The left hand side being quasi-continuous, we get

$$E_x\left[\int_0^\infty e^{-\alpha t} I_{B^c}(X_t)\, dA_t^g\right]=0 \qquad \text{q.e.}$$

For all stopping time τ with $\tau \leqq \tau_B$,

$$0 \leqq E_x\left[\int_0^\tau e^{-\alpha t} dA_t^g\right] \leqq E_x\left[\int_0^{\tau_B} e^{-\alpha t} dA_t^g\right]$$

$$\leqq E_x\left[\int_0^\infty e^{-\alpha t} I_{B^c}(X_t)\, dA_t^g\right] \qquad \text{q.e.}$$

Hence we get

$$E_x\left[\int_0^\tau e^{-\alpha t} dA_t^g\right]=0 \qquad \text{q.e.}$$

Thus we obtain by (3.11)

$$u^*(x)=E_x\left[\int_0^\tau e^{-\alpha t} dA_t + e^{-\alpha\tau} u^*(X_\tau)\right] \qquad \text{q.e.} \tag{3.14}$$

for all stopping time $\tau \leqq \tau_B$.

Denote by N^* a common proper exceptional set for (3.10), (3.12) and (3.14) including N_0. Put

$$B^* = \{x \in X-N^*,\ u^*(x)=0\}.$$

Obviously B^* is a Borel finely closed subset of $X-N^*$ and (3.10) and (3.12) hold for each $x \in X-N^*$, so it only remains to show for each stopping time τ with $\tau \leqq \tau_{B^*}$

$$u^*(x)=E_x\left[\int_0^\tau e^{-\alpha t} dA_t + e^{-\alpha\tau} u^*(X_\tau)\right] \tag{3.14$'$}$$

for each $x \in X-N^*$. To see this, put $\tau' = \tau \wedge \tau_B$. Then $P_x(\tau'=\tau)=1$ for all $x \in X-N^*$ because $B-B^* \subset N^*$ and N^* is proper exceptional. Therefore (3.14)$'$ follows from (3.14). In view of our lemma, (3.10), (3.12) and (3.14)$'$ show that $u^*(x)$ is a solution of O.S.P. with a proper exceptional set N^* and an optimal region B^*. q.e.d.

REMARK. The same procedure as above applies to the following generalized (O.S.P.). "Find $\rho(x)$ such that

$$\rho(x) \equiv E_x\left[\int_0^{\tau_B} e^{-\alpha t} dA_t + e^{-\alpha\tau_B} \phi(X_{\tau_B})\right] = \inf_\tau E_x\left[\int_0^\tau e^{-\alpha t} dA_t + e^{-\alpha\tau} \phi(X_\tau)\right]$$

for each $x \in X-N$, with a proper exceptional set N and some finely closed subset B of $X-N$." Here ϕ is a finely continuous function belonging to $\mathcal{F}$. In the same way as the above lemma we can assert that if ρ satisfies the following (R.1)$\sim$(R.4) for a proper exceptional set N and some finely closed subset

B of $X-N$, then ρ is the solution of the present problem:

(R.1) $$\rho(x)\leqq\psi(x) \quad \text{for each } x\in X-N$$

(R.2) $$\rho(x)=\psi(x) \quad \text{for each } x\in B$$

(R.3) $$\rho(x)\leqq E_x\left[\int_0^\tau e^{-\alpha t}dA_t+e^{-\alpha\tau}\rho(X_\tau)\right]$$

for each $x\in X-N$ and for all stopping time τ

(R.4) $$\rho(x)=E_x\left[\int_0^\tau e^{-\alpha t}dA_t+e^{-\alpha\tau}\rho(X_\tau)\right]$$

for each $x\in X-N$ and for all stopping time $\tau\leqq\tau_B$. The present counterpart of (2.3) is the following variational inequality

(R.5) $$\begin{cases} \mathcal{E}_\alpha(u+\psi,\ v-u)\geqq\langle\nu,\ \tilde{v}-\tilde{u}\rangle & \text{for all } v\in\mathfrak{R} \\ u\in\mathfrak{R}\,. \end{cases}$$

Then we can see by the similar argument as the above proof of our theorem that $\tilde{\rho}=\tilde{u}+\psi$ is the solution of the present problem. Here $\tilde{u}$ is any quasi-continuous version of the solution u of (R.5).

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