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An elementary proof of the generalized form of Poincaré's inequality

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§ 1. Introduction and statement of the theorems.

In the recent paper [2] we have obtained the generalized form of Poincaré's inequality

(1.1) $$\|v\| \leqq C(\zeta^{-\tau}\|v\|_{\tau}+\zeta^{|\alpha_0|}\|gv\|) \quad for \quad v\in C_0^{\infty}(\omega), \zeta>0,$$

by using Hörmander's theorem proved by the Campbell-Hausdorff formula (Theorem 4.3 in [1]), and, as an application, we have studied the hypoellipticity for the operator $L=a(X, D_x)+g(X)b(X, Y, D_y)$ (see also [3]).

In the present paper we shall give an elementary proof of the inequality (1.1) only by using the Taylor expansion.

Now we state the theorems to be proved.

THEOREM 1.1 (cf. (1.3) in [2]). *Let $g(x)$ be a real-valued function in $\mathcal{B}(R_x^n)$. Assume that there exists a multi-index α_0 such that*

(1.2) $$\partial_x^{\alpha_0}g(0)\neq 0,$$

(1.3) $$\partial_x^{\alpha}g(0)=0 \quad for \quad |\alpha|<|\alpha_0|.$$

Then there exists a neighborhood ω of the origin in R_x^n, such that the following estimate holds for $0<\tau\leqq 1$

(1.4) $$\sup_{0<|t|<T}|t|^{-\tau_1}\|u(x, y+t)-u(x, y)\|\leqq C(\|u\|_{\langle\xi\rangle,\tau}+\|g(x)\partial_y u\|)$$
$$for \quad u\in C_0^{\infty}(\omega\times R_y^1),$$

where $\tau_1=(1+|\alpha_0|/\tau)^{-1}$ and $0<T\leqq\infty$.

Here we used the following usual notations: $\partial_{x_j}=\partial/\partial x_j$, $\partial_x^{\alpha}=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}$ for multi-index $\alpha=(\alpha_1, \cdots, \alpha_n)$, $\partial_y=\partial/\partial y$, $\mathcal{B}(R_x^n)=\{g\in C^{\infty}(R_x^n); \sup_x|\partial_x^{\alpha}g(x)|<\infty$ for any $\alpha\}$, $\langle\xi\rangle=\sqrt{1+|\xi|^2}$, and $|u|_{\langle\xi\rangle,\tau}^2=\int\langle\xi\rangle^{2\tau}|\hat{u}(\xi, \eta)|^2 d\xi d\eta$ ($d\xi=(2\pi)^{-n}d\xi$, $d\eta=(2\pi)^{-1}d\eta$), where $\hat{u}(\xi, \eta)=\int e^{-i(x\cdot\xi+y\eta)}u(x,y)dxdy$ ($x\cdot\xi=x_1\xi_1+\cdots+x_n\xi_n$) is the Fourier transform of $u\in C_0^{\infty}(R_x^n\times R_y^1)$.

THEOREM 1.2. *Let $g(x)$ be a function as in Theorem 1.1. Then we can* *find a neighborhood ω of the origin in R_x^n such that for $\tau>0$ we have* (1.1), *and, equivalently, we have*

$$\|v\|\leqq C\|v\|_\tau^p\|gv\|^q \quad \text{for} \quad v\in C_0^\infty(\omega), \tag{1.5}$$

where $p=|\alpha_0|/(|\alpha_0|+\tau)$ and $q=\tau/(|\alpha_0|+\tau)$.

This theorem can be easily proved, as in [2], by taking $u(x, y)=\chi(y)e^{i\zeta y}v(x)$ ($\chi\in C_0^\infty((-1, 1))$, $\chi\geqq 0, \int\chi(y)^2dy=1$) for u in Theorem 1.1 in case $0<\tau\leqq 1$ and $\zeta\geqq C_0$ for some $C_0>0$, and by using interpolation in case $\tau>1$ and $\zeta\geqq C_0$. Note that (1.1) is trivial when $0<\zeta<C_0$.

The formula (1.5) leads us to the following inequality

$$\|v\|\leqq C\delta^\tau\|v\|_\tau \quad \text{for} \quad v\in C_0^\infty(B_\delta), \tag{1.6}$$

where $B_\delta=\{x;\ |x|<\delta\}$ is an open ball contained in ω. In fact, noting that $|g(x)|\leqq C_1|x|^{|\alpha_0|}\leqq C_2\delta^{|\alpha_0|}$ for $x\in B_\delta$, we get

$$\|v\|\leqq C_3\delta^{|\alpha_0|q}\|v\|_\tau^p\|v\|^q=C_3\delta^{\tau(1-q)}\|v\|_\tau^{1-q}\|v\|^q \quad \text{for} \quad v\in C_0^\infty(B_\delta),$$

and hence we have (1.6). From this fact we can regard inequalities (1.1) and (1.5) as generalized forms of Poincaré's inequality.

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§ 2. Proof of Theorem 1.1.

First we shall consider Theorem 1.1 in case $n=1$, and show the following theorem.

THEOREM 2.1. *Let δ be a positive number, and $g(x)$ be a real-valued function in $\mathscr{B}(R_x^1)$, which satisfies for some integer N_0*

$$|\partial_x^{N_0}g(x)|\geqq c_0>0 \quad \text{in } [-\delta, \delta]. \tag{2.1}$$

Then we have for $0<\tau\leqq 1$

$$\sup_{0<|t|<T}|t|^{-\tau_1}\|u(x, y+t)-u(x, y)\|\leqq C(\|u\|_{\langle\xi\rangle,\tau}+\|g(x)\partial_y u\|) \quad \text{for} \quad u\in C_0^\infty(\Omega), \tag{2.2}$$

where $\tau_1=(1+N_0/\tau)^{-1}$, $\Omega=(-\delta, \delta)\times R_y^1$, and $0<T\leqq\infty$. Moreover we can find the constant C depending only on $\sup_x|\partial^{N_0+1}g(x)|$, c_0 and τ.

REMARK. For any $0<T<\infty$ there exists a constant $C(T)$ such that the following estimate holds:

$$\sup_{0<|t|<T}|t|^{-s}\|u(x, y+t)-u(x, y)\|\leqq \sup_{0<|t|<\infty}|t|^{-s}\|u(x, y+t)-u(x, y)\|$$

$$\leqq \sup_{0<|t|<T} |t|^{-s}\|u(x, y+t)-u(x, y)\|+C(T)\|u\|.$$

For the proof of Theorem 2.1 we need several lemmas.

LEMMA 2.2. *For* $0<s\leqq 1$ *we have*

$$\sup_{0<|t|<\infty} |t|^{-s}\|u(x+t, y)-u(x, y)\|\leqq C\|u\|_{\langle\xi\rangle,s} \quad \text{for} \quad u\in C_0^\infty(R^2_{x,y}), \tag{2.3}$$

$$\sup_{0<|t|<\infty} |t|^{-s}\|u(x, y+t)-u(x, y)\|\leqq C\|u\|_{\langle\eta\rangle,s} \quad \text{for} \quad u\in C_0^\infty(R^2_{x,y}), \tag{2.4}$$

where $\|u\|^2_{\langle\eta\rangle,s}=\int(1+|\eta|^2)^s|\hat{u}(\xi, \eta)|^2d\xi d\eta.$

PROOF. For any t we have

$$|t|^{-2s}\|u(x+t, y)-u(x, y)\|^2=|t|^{-2s}\int|u(x+t, y)-u(x, y)|^2dxdy$$

$$=\int\{|t|^{-2s}|e^{it\xi}-1|^2\}|\hat{u}(\xi, \eta)|^2d\xi d\eta\leqq C\int\langle\xi\rangle^{2s}|\hat{u}(\xi, \eta)|^2d\xi d\eta$$

$$=C\|u\|^2_{\langle\xi\rangle,s}.$$

Consequently we have (2.3). Similarly we can prove (2.4). Q.E.D.

LEMMA 2.3. *Let* $0<s'<s\leqq 1$. *Then we have*

$$\|u\|_{\langle\eta\rangle,s'}\leqq C\{\sup_{0<|t|<\infty} |t|^{-s}\|u(x, y+t)-u(x, y)\|+\|u\|\} \quad \text{for} \quad u\in C_0^\infty(R^2_{x,y}). \tag{2.5}$$

PROOF. To simplify the statements below, we introduce the notation $|u|_{y,s}=\sup_{0<|t|<\infty} |t|^{-s}\|u(x, y+t)-u(x, y)\|$. Noting that we can write $|\eta|^{2s'}=C_0^{-1}\int|t|^{-(2s'+1)}|e^{it\eta}-1|^2dt$ for some C_0, we have

$$\|u\|^2_{\langle\eta\rangle,s'}\leqq C_1\int(1+C_0|\eta|^{2s'})|\hat{u}(\xi, \eta)|^2d\xi d\eta$$

$$=C_1\{\iint|t|^{-(2s'+1)}|e^{it\eta}-1|^2|\hat{u}(\xi, \eta)|^2d\xi d\eta dt+\|u\|^2\}$$

$$=C_1\{\iint|t|^{-(2s'+1)}|u(x, y+t)-u(x, y)|^2dx\,dy\,dt+\|u\|^2\}$$

$$=C_1\{\int_{|t|\leqq 1}|t|^{-(2s'+1)}\|u(x, y+t)-u(x, y)\|^2dt$$

$$+\int_{|t|>1}|t|^{-(2s'+1)}\|u(x, y+t)-u(x, x)\|^2dt+\|u\|^2\}$$

$$\leqq C_1\{|u|^2_{y,s}\int_{|t|\leqq 1}|t|^{-(2(s'-s)+1)}dt$$

$$+4\|u\|^2\int_{|t|>1}|t|^{-(2s'+1)}dt+\|u\|^2\}$$

$$\leqq C_2\{|u|_{y,s}^2+\|u\|^2\}.$$

Consequently we have (2.5). Q.E.D.

LEMMA 2.4 (cf. Lemma 4.1 in [1]). *Let $h(x, t)$ be a continuous and real-valued function such that*

$$|h(x, t)|\leqq C_0.$$

Then we have for $0<s\leqq 1$

$$(2.6)\quad |t|^{-s}\|u(x, y+th(x, t))-u(x, y)\|\leqq C(1+C_0)^{s+\frac{1}{2}}|u|_{y,s} \quad \textit{for} \quad u\in C_0^\infty(R_{x,y}^2).$$

PROOF. We have for any σ

$$|u(x, y+th(x, t))-u(x, y)|^2$$
$$\leqq 2|u(x, y+th(x, t))-u(x, y+\sigma)|^2+2|u(x, y+\sigma)-u(x, y)|^2.$$

Integrating with respect to x and y, averaging over σ for $|\sigma|<|t|$, and multiplying by $|t|^{-2s}$, we have

$$|t|^{-2s}\|u(x, y+th(x, t))-u(x, y)\|^2$$
$$\leqq |t|^{-(2s+1)}\int_{|\sigma|<|t|}\int |u(x, y+th(x, t))-u(x, y+\sigma)|^2 dxdyd\sigma+C_1|u|_{y,s}^2.$$

Put $I=|t|^{-(2s+1)}\int_{|\sigma|<|t|}|u(x, y+th(x, t))-u(x, y+\sigma)|^2 dxdyd\sigma$, and change the variables by $z=y+th(x, t)$, $w=\sigma-th(x, t)$. Then we have

$$I=|t|^{-(2s+1)}\iint_{|w+th(x,t)|<|t|}|u(x, z)-u(x, z+w)|^2 dxdzdw$$
$$\leqq |t|^{-(2s+1)}\int_{|w|<(1+C_0)|t|}\int |u(x, z)-u(x, z+w)|^2 dxdzdw$$
$$\leqq C_2(1+C_0)^{2s+1}|u|_{y,s}^2.$$

Consequently we have (2.6). Q.E.D.

LEMMA 2.5. *Suppose that $h(x)$ be a continuous and real-valued function such that for a constant $c_0>0$*

$$(2.7)\qquad |h(x)|\geqq c_0 \quad \text{in} \quad [-\delta, \delta].$$

Then we have for $0<s\leqq 1$

$$(2.8)\quad \sup_{0<|t|<\infty}|t|^{-s}\|u(x, y+t)-u(x, y)\|$$
$$\leqq C(1+c_0^{-1})^{s+\frac{1}{2}}\sup_{0<|t|<\infty}|t|^{-s}\|u(x, y+th(x))-u(x, y)\| \quad \textit{for} \quad u\in C_0^\infty(\Omega).$$

PROOF. Since the proof is parallel to that of the preceding lemma, we mention only the outline of the proof. First we have

$$|t|^{-2s}\|u(x,y+t)-u(x,y)\|^2 \leqq C\{|t|^{-(2s+1)}\int_{|\sigma|<|t|}\int_{\Omega}|u(x,y+t)$$
$$-u(x,y+\sigma h(x))|^2 dx\,dy\,d\sigma + \sup_{0<|t'|<\infty}|t'|^{-2s}\|u(x,y+t'h(x))-u(x,y)\|^2\}\,.$$

In the integral we change the variables by $z=y+t$, $w=(\sigma h(x)-t)/h(x)$. Then the integral yields

$$|t|^{-(2s+1)}\int_{\Omega}\int_{|w+t/h(x)|<|t|}|u(x,z)-u(x,z+wh(x))|^2 dx\,dz\,dw$$
$$\leqq |t|^{-(2s+1)}\int_{|w|<(1+c_0^{-1})|t|}\|u(x,z)-u(x,z+wh(x))\|^2 dw$$
$$\leqq C(1+c_0^{-1})^{2s+1}\sup_{0<|t'|<\infty}|t'|^{-2s}\|u(x,y+t'h(x))-u(x,y)\|^2\,.$$

Consequently we have (2.8). Q.E.D.

For τ in Theorem 2.1 we set $\mu(m)=1+m/\tau$ and define for $m=0,1,\cdots$

$$A_m(u)=\sup_{0<|t|<\infty}|t|^{-1/\mu(m)}\|u(x,y+t\partial_x^m g(x))-u(x,y)\|.$$

Then we have

THEOREM 2.6. *For any ρ satisfying $0<\rho\leqq 1$ we take an integer N such that $\mu(N+1)\geqq 1/\rho$. Then for any $\varepsilon>0$ and m satisfying $1\leqq m\leqq N$ there exists a constant $C_{m,\varepsilon}$ such that the following estimate holds*

$$A_m(u)\leqq \varepsilon A_{m-1}(u)+C_{m,\varepsilon}\Big(\sum_{k=m+1}^{N}A_k(u)+\|u\|_{\langle\xi\rangle,\tau} \tag{2.9}$$
$$+(1+|g|_{N+1})^{\rho+\frac{1}{2}}\|u\|_{\langle\eta\rangle,\rho}\Big)\quad \text{for } u\in C_0^\infty(R^2_{x,y})\,,$$

where $|g|_N=\sup_x|\partial_x^N g(x)|$. *Moreover we have for any $\varepsilon>0$*

$$\sum_{m=1}^{N}A_m(u)\leqq \varepsilon A_0(u)+C_\varepsilon(\|u\|_{\langle\xi\rangle,\tau}+(1+|g|_{N+1})^{\rho+\frac{1}{2}}\|u\|_{\langle\eta\rangle,\rho}) \tag{2.10}$$
$$\text{for } u\in C_0^\infty(R^2_{x,y})\,.$$

PROOF. For the proof we introduce unitary operators in $L^2(R^2_{x,y})$ as follows:

$$(T(t)u)(x,y)=u(x+t,y)$$
$$(G_l(t)u)(x,y)=u(x,y+t\partial_x^l g(x))\quad \text{for any integer } l\,.$$

We define for any $\varepsilon_1>0$ $(Fu)(x,y)=u(x,y+\varepsilon_1 t^{\mu(m-1)}\{\partial_x^{m-1}g(x+\varepsilon_1^{-1}t^{1/\tau})-\partial_x^{m-1}g(x)\})$. Then we get

$$F=G_{m-1}(-\varepsilon_1 t^{\mu(m-1)})T(\varepsilon_1^{-1}t^{1/\tau})G_{m-1}(\varepsilon_1 t^{\mu(m-1)})T(-\varepsilon_1^{-1}t^{1/\tau})\,.$$

In fact we have

$$\begin{aligned} Fu(x, y) &= u(x, y+\varepsilon_1 t^{\mu(m-1)}\{\partial_x^{m-1}g(x+\varepsilon_1^{-1}t^{1/\tau})-\partial_x^{m-1}g(x)\}) \\ &= G_{m-1}(-\varepsilon_1 t^{\mu(m-1)})u(x, y+\varepsilon_1 t^{\mu(m-1)}\partial_x^{m-1}g(x+\varepsilon_1^{-1}t^{1/\tau})) \\ &= G_{m-1}(-\varepsilon_1 t^{\mu(m-1)})T(\varepsilon_1^{-1}t^{1/\tau})u(x-\varepsilon_1^{-1}t^{1/\tau}, y+\varepsilon_1 t^{\mu(m-1)}\partial_x^{m-1}g(x)) \\ &= G_{m-1}(-\varepsilon_1 t^{\mu(m-1)})T(\varepsilon_1^{-1}t^{1/\tau})G_{m-1}(\varepsilon_1 t^{\mu(m-1)})u(x-\varepsilon_1^{-1}t^{1/\tau}, y) \\ &= G_{m-1}(-\varepsilon_1 t^{\mu(m-1)})T(\varepsilon_1^{-1}t^{1/\tau})G_{m-1}(\varepsilon_1 t^{\mu(m-1)})T(-\varepsilon_1^{-1}t^{1/\tau})u(x, y). \end{aligned}$$

On the other hand, noting that we have

$$\begin{aligned} &\varepsilon_1 t^{\mu(m-1)}\{\partial_x^{m-1}g(x+\varepsilon_1^{-1}t^{1/\tau})-\partial_x^{m-1}g(x)\} \\ &\quad = t^{\mu(m)}\partial_x^m g(x)+\sum_{k=1}^{N-m}\varepsilon_1^{-k}t^{\mu(m+k)}\partial_x^{m+k}g(x)+\varepsilon_1^{-(N-m+1)}t^{\mu(N+1)}g_{N+1}(x, t) \end{aligned}$$

by using the Taylor expansion and $\mu(m)+k/\tau=\mu(m+k)$, we can write

$$F=G_m(t^{\mu(m)})\prod_{k=1}^{N-m}G_{m+k}(\varepsilon_1^{-k}t^{\mu(m+k)})F_{N+1}$$

with $F_{N+1}u(x, y)=u(x, y+\varepsilon_1^{-(N-m+1)}t^{\mu(N+1)}g_{N+1}(x, t))$.

By using Lemma 2.2 and Lemma 2.4 we have

$$\begin{aligned} &|t|^{-1}\|u(x, y+t^{\mu(m)}\partial_x^m g(x))-u(x, y)\|=|t|^{-1}\|G_m(t^{\mu(m)})u-u\| \\ &\leqq |t|^{-1}\{\|Fu-u\|+\|G_m(t^{\mu(m)})u-Fu\|\} \\ &\leqq \{|t|^{-1}\|G_{m-1}(-\varepsilon_1 t^{\mu(m-1)})u-u\|+|t|^{-1}\|T(\varepsilon_1^{-1}t^{1/\tau})u-u\| \\ &\quad +|t|^{-1}\|G_{m-1}(\varepsilon_1 t^{\mu(m-1)})u-u\|+|t|^{-1}\|T(-\varepsilon_1^{-1}t^{1/\tau})u-u\|\} \\ &\quad +\{\sum_{k=1}^{N-m}|t|^{-1}\|G_{m+k}(\varepsilon_1^{-k}t^{\mu(m+k)})u-u\|+|t|^{-1}\|F_{N+1}u-u\|\} \\ &\leqq 2\varepsilon_1^{1/\mu(m-1)}A_{m-1}(u)+\sum_{k=m+1}^{N}\varepsilon_1^{-(k-m)/\mu(k)}A_k(u) \\ &\quad +C_{\varepsilon_1}(\|u\|_{\langle\xi\rangle,\tau}+(1+|g|_{N+1})^{\rho+\frac{1}{2}}\|u\|_{\langle\eta\rangle,\rho}). \end{aligned}$$

Consequently we have (2.9) with $\varepsilon=2\varepsilon_1^{1/\mu(m-1)}$.

For the proof of (2.10) we take sequences $\{\varepsilon_j\}_{j=1}^N$ and $\{c_j\}_{j=1}^N$ which satisfy $\varepsilon_1=\varepsilon/2$, $\varepsilon_j=1/c_j$ $(j\geqq 2)$, $c_1=2$ and $c_j=2+\sum_{k=1}^{j-1}c_k c_{k,\varepsilon_k}(j\geqq 2)$. Then we can get (2.10) since we have by (2.9)

$$\begin{aligned} \sum_{m=1}^N c_m A_m(u) \leqq \sum_{m=1}^N c_m\{\varepsilon_m A_{m-1}(u)+C_{m,\varepsilon_m}(\sum_{k=m+1}^N A_k(u)+\|u\|_{\langle\xi\rangle,\tau} \\ +(1+|g|_{N+1})^{\rho+\frac{1}{2}}\|u\|_{\langle\eta\rangle,\rho})\} \end{aligned}$$

$$=\varepsilon A_0(u)+\sum_{m=1}^{N-1}A_m(u)+\sum_{m=2}^{N}(\sum_{k=1}^{m-1}c_kC_{k,\varepsilon_k})A_m(u)$$

$$+\sum_{m=1}^{N}c_mC_{m,\varepsilon_m}(\|u\|_{\langle\xi\rangle,\tau}+(1+|g|_{N+1})^{\rho+\frac{1}{2}}\|u\|_{\langle\eta\rangle,\rho}). \qquad \text{Q.E.D.}$$

REMARK. Let $\exp X$ be the one parameter group of transformations in $R^2_{x,y}$ defined by a real vector field X. Then we have $T(t)=\exp(t\partial_x)$, $G_l(t)=\exp(t\partial_x^l g(x)\partial_y)$, and $F=\exp Y\exp X\exp(-Y)\exp(-X)=\exp([Y,X])\cdots =\exp(t^{\mu(m)}\partial_x^m g(x)\partial_y)\cdots$, for $Y=-\varepsilon_1 t^{\mu(m-1)}\partial_x^{m-1}g(x)\partial_y$ and $X=\varepsilon_1^{-1}t^{1/\tau}\partial_x$.

PROOF OF THEOREM 2.1. Take $\rho=1/\mu(N_0+1)$ for ρ in theorem 2.6, then $N=N_0$. From (2.10) we have

$$A_N(u)\leqq C_1(A_0(u)+\|u\|_{\langle\xi\rangle,\tau}+(1+|g|_{N+1})^{\rho+\frac{1}{2}}\|u\|_{\langle\eta\rangle,\rho}) \quad \textit{for} \quad u\in C_0^\infty(R^2_{x,y}).$$

Since we have

$$A_0(u)=\sup_{0<|t|<\infty}|t|^{-1}\|u(x,y+tg(x))-u(x,y)\|\leqq\|g(x)\partial_y u(x,y)\|,$$

we get

$$\text{(2.11)} \quad A_N(u)\leqq C(\|u\|_{\langle\xi\rangle,\tau}+|g\partial_y u|+(1+|g|_{N+1})^{\rho+\frac{1}{2}}\|u\|_{\langle\eta\rangle,\rho}) \quad \textit{for} \quad u\in C_0^\infty(R^2_{x,y}).$$

Now using (2.1) we have from Lemma 2.5

$$\sup_{0<|t|<\infty}|t|^{-\tau_1}\|u(x,y+t)-u(x,y)\|$$

$$\leqq C'(1+c_0^{-1})^{\tau_1+\frac{1}{2}}\sup_{0<|t|<\infty}|t|^{-\tau_1}\|u(x,y+t\partial_x^N g(x))-u(x,y)\|$$

$$=C'(1+c_0^{-1})^{\tau_1+\frac{1}{2}}A_N(u) \quad \textit{for} \quad u\in C_0^\infty(\Omega).$$

Combining this with (2.11), we get

$$\text{(2.12)} \quad \sup_{0<|t|<\infty}|t|^{-\tau_1}\|u(x,y+t)-u(x,y)\|$$

$$\leqq C_1(g)(\|u\|_{\langle\xi\rangle,\tau}+\|g\partial_y u\|+\|u\|_{\langle\eta\rangle,\rho}) \quad \textit{for} \quad u\in C_0^\infty(\Omega).$$

Here and in what follows we denote by $C_j(g)$ the constants depending only on $|g|_{N+1}$, c_0 and τ. Taking ρ' with $\rho<\rho'<\tau_1$, we have from Lemma 2.3

$$\|u\|_{\langle\eta\rangle,\rho'}\leqq C''(\sup_{0<|t|<\infty}|t|^{-\tau_1}\|u(x,y+t)-u(x,y)\|+\|u\|)$$

$$\leqq C_2(g)(\|u\|_{\langle\xi\rangle,\tau}+\|g\partial_y u\|+\|u\|_{\langle\eta\rangle,\rho}).$$

By using interpolation we have

$$\text{(2.13)} \quad \|u\|_{\langle\eta\rangle,\rho'}\leqq C_3(g)(\|u\|_{\langle\xi\rangle,\tau}+\|g\partial_y u\|).$$

Then we have (2.2) from (2.12) and (2.13). Q.E.D.

PROOF OF THEOREM 1.1. We may assume that $\alpha_{01}\geqq\alpha_{02}\geqq\cdots\geqq\alpha_{0n}$, and for any β $(\partial_x^\beta g(0)\neq 0,\ |\beta|=|\alpha_0|,\ \beta\neq\alpha_0)$ we have $\alpha_{0j}=\beta_j$ $(j<k)$, $\alpha_{0k}>\beta_k$ for some k. Introducing new variables $z=(z_1, \cdots, z_n)$ by setting

$$x_1=\lambda^{(2N_0)^{n+1}}z_1\,,\quad x_2=\lambda^{(2N_0)^n}z_1+z_2, \cdots,\quad x_n=\lambda^{2N_0}z_1+z_n\ \ (N_0=|\alpha_0|)$$

with sufficiently large positive number λ, we can get for $N_0=|\alpha_0|$

$$\partial_{z_1}^{N_0}(0)\neq 0\,.$$

Then there exists a neighborhood $\omega=\{z_1\,;\ |z_1|<\delta\}\times\{z'\,;\ |z'|<\delta'\}$ $(z'=(z_2, \cdots, z_n))$ of the origin such that

$$|\partial_{z_1}^{N_0}g(z)|\geqq c_0>0 \quad in \quad \bar{\omega}\,.$$

Considering z' as a parameter we get by using Theorem 2.1

$$\sup_{0<|t|<\infty}|t|^{-2\tau_1}\|u(z, y+t)-u(z, y)\|^2_{L^2(R^2_{z_1,y})}$$
$$\leqq C\Big(\int(1+|\zeta_1|^2)^\tau|\tilde{u}(\zeta_1, z', y)|^2 d\zeta_1 dy$$
$$+\|g(z)\partial_y u(z, y)\|^2_{L^2(R^2_{z_1,y})}\Big) \qquad for \quad u\in C_0^\infty(\omega\times R_y^1)\,, \tag{2.14}$$

where $\tilde{u}(\zeta_1, z', y)=\int e^{-iz_1\zeta_1}u(z, y)dz_1$ is the partial Fourier transform of u. Then we get the inequality (1.4) by integrating (2.14) with respect to z' and rechanging the variables from z to x. Q.E.D.

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