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On stability of proper leaves of codimension one foliations

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Introduction.

In this note we will try to extend the Reeb stability theorem for proper leaves of codimension one foliations, using the concept of a locally infinite holonomy pseudogroup which was defined by R. Sacksteder and A. J. Schwartz [7] for the purpose of studying the limit sets of foliations.

Let M be a smooth manifold with a transversely oriented, C^1, codimension one foliation $\mathcal{F}$. We will assume throughout that a dimension one foliation $\mathcal{T}$ transverse to $\mathcal{F}$ has been fixed. If x is a point of M, we let L_x and T_x denote the leaves of $\mathcal{F}$ and $\mathcal{T}$ which contain x respectively. We say that the holonomy pseudogroup of L_x at x is *locally infinite* if for every neighborhood N of x on T_x, there exists an element of the holonomy pseudogroup of L_x at x whose domain is contained in N and which is not a restriction of the identity. (See [7] for the definition of a holonomy pseudogroup.) We assume that M has a Riemannian metric and define the r-neighborhood of L_x to be $\{y \in M \mid d(y, L_x) < r\}$, where d is the distance function derived from the metric. A subset of M is called *saturated* if it is a union of leaves of $\mathcal{F}$. The *saturation* of a subset X of M is the smallest saturated subset of M which contains X.

THEOREM 1. *Let L_x be a proper, relatively compact leaf of $\mathcal{F}$. Then L_x does not have a locally infinite holonomy pseudogroup if and only if for every $r>0$, there exists a saturated neighborhood N of L_x contained in the r-neighborhood of L_x such that $\mathcal{F}|N$ is a product foliation.*

THEOREM 2. *Suppose that M is a compact* 3*-dimensional manifold and that L_x is a proper, simply connected leaf of $\mathcal{F}$. Then for every $r>0$, L_x has a saturated neighborhood N contained in the r-neighborhood of L_x such that $\mathcal{F}|N$ is a product foliation.*

These theorems can be viewed as generalizations of the Reeb stability theorem for proper leaves of codimension one foliations. The author does not know whether Theorem 2 can be extended for compact manifolds of dimension greater than three. On the other hand, we easily see that these results cannot be generalized for foliations of codimension greater than one

(See § 4).

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§ 1. Holonomy groups and locally infinite holonomy pseudogroups.

H. Imanishi [3] has constructed an example of a codimension one foliation on a closed 3-dimensional manifold with a proper leaf which does not have an infinite holonomy group, but has a locally infinite holonomy pseudogroup. So in general there is a gap between an infinite holonomy group and a locally infinite holonomy pseudogroup. However if we impose some particular conditions upon $\mathcal{F}$ or L_x, we can stop the gap.

PROPOSITION 1. *Let L_x be a compact leaf. Then L_x has a locally infinite holonomy pseudogroup if and only if it has an infinite holonomy group.*

PROOF. Immediate from the Reeb stability theorem for compact leaves without holonomy [1].

PROPOSITION 2. *Let the dimension of L_x be one. Then L_x has a locally infinite holonomy pseudogroup if and only if it has an infinite holonomy group.*

PROOF. From the assumption, L_x is homeomorphic either to R^1 or to S^1. Therefore the result follows from Proposition 1 above and Proposition 3.5 of [7].

PROPOSITION 3. *Let $\mathcal{F}$ be real analytic and let L_x be a leaf of $\mathcal{F}$. Then L_x has a locally infinite holonomy pseudogroup if and only if it has an infinite holonomy group.*

PROOF. Suppose L_x has a trivial holonomy group. Then every element of the holonomy pseudogroup of L_x is a restriction of the identity, because the element is a real analytic local diffeomorphism of T_x whose germ at x is a germ of the identity. Hence L_x does not have a locally infinite holonomy pseudogroup.

§ 2. Proof and corollaries of Theorem 1.

A continuous map $P: [0, b]\times[0, S]\to M$, $(b>0,\ S>0)$, is called a *projector* if the following conditions are satisfied.

1) $P(t, s)\in L_{P(0,s)}\cap T_{P(t,0)}$ for $t\in[0, b]$ and $s\in[0, S]$,
2) $s\mapsto P(0, s)$ is an isometry.

We need the following two lemmas which was proved by Sacksteder and Schwartz in [7].

LEMMA 1. *Let l be a positive number and let $g: [0, b]\to M$ be a path contained in a single leaf such that $g(t)$ is the midpoint of an interval of length*

$2l$ in $T_{g(t)}$ for each $t\in[0, b]$. Let J be an interval in $T_{g(0)}$ with length m $(<l)$ with one end point at $g(0)$. Let c be the largest number in $[0, b]$ such that there exists a projector $P: [0, c]\times[0, m]\to M$ satisfying

1) $P(t, 0)=g(t)$ *for* $0\leqq t\leqq c$,
2) $P(0\times[0, m])=J$,
3) $P(t\times[0, m])$ *has length less than or equal to* l.

Then either $c=b$ or $P(c\times[0, m])$ is of length l.

LEMMA 2. *Let $x\in M$ and let L_x be a relatively compact leaf. Let P_i: $[0, b_i]\times[0, S_i]\to M$ be a sequence of projectors such that*

1) $P_i(0, 0)=x$ *for all* i,
2) $P_i(0\times(0, S_i])\cap L_x=\emptyset$,
3) $\lim_{i\to\infty} S_i=0$,
4) $\lim_{i\to\infty}$ (*length of* $P_i(b_i\times[0, S_i]))=l>0$.

Then L_x has a locally infinite holonomy pseudogroup.

PROOF OF THEOREM 1. Since the "if" part of the theorem is obvious, we prove the "only if" part.

Suppose that L_x does not have a locally infinite holonomy pseudogroup. For each $y\in L_x$, let h_y be an injective path which satisfies the following conditions and which has the biggest image among all pathes: $[0, 1)\to T_y$ satisfying the same conditions as those of h_y.

1) h_y starts from y in the positive direction. (Remark that each T_y has been oriented by the transverse orientation of $\mathcal{F}$.)
2) $S_y\cap L_x=\{y\}$, where S_y denotes $h_y([0, 1))$.

Such h_y's exist because L_x is proper. Let J be $h_x([0, \varepsilon))$ for some $0<\varepsilon<1$. Note that J is diffeomorphic to $R_+(=[0, \infty))$. Since $\bar{L}_x$ is compact, we can take a positive number l sufficiently small that the length of J is greater than l and that each $y\in\bar{L}_x$ is the midpoint of an interval in T_y of length $2l$.

ASSERTION 1. *There is a connected open neighborhood N of x in J such that for each leaf L, if $N\cap L\neq\emptyset$, then $S_x\cap L$=one point.*

PROOF. Suppose that there are infinite sequences $\{x_i\}$, $\{y_i\}$ such that

1) $x_i\in J$, $y_i\in L_{x_i}\cap S_x$, and $x_i\neq y_i$ for all i,
2) $\lim_{i\to\infty} x_i=x$ and $\overline{xx_i}$ is of length less than l, where $\overline{xx_i}$ is the unique interval in J with end points x and x_i.

Let $g_i: [0, 1]\to L_{x_i}$ be a sequence of paths satisfying $g_i(0)=x_i$, $g_i(1)=y_i$. By Lemma 1, there is a sequence of projectors satisfying the following condition C.

$$C\begin{cases} 1) & P_i:[0, t_i]\times[0, s_i]\to M \text{ where } 0<t_i\leqq 1, \\ 2) & P_i(t, s_i)=g_i(t) \text{ for } 0\leqq t\leqq t_i, \\ 3) & P_i(0, 0)=x, \\ 4) & P_i(\{t\}\times[0, s_i]) \text{ has length less than } l \text{ for } 0\leqq t<t_i, \\ 5) & P_i(\{t\}\times[0, s_i]) \text{ has length equal to } l \text{ if } t_i<1. \end{cases}$$

In the case where $t_i<1$ for all but a finite number of i, by Lemma 2, L_x has a locally infinite holonomy pseudogroup. This contradicts the supposition on L_x. In the case where $t_i=1$ for infinitely many i, taking a subsequence if necessary, we can assume $t_i=1$ for all i. Then the element of the holonomy pseudogroup of L_x at x along the loop $t\mapsto P_i(t, 0)$ whose domain contains $P_i(0\times[0, s_i])$ is not a restriction of the identity because $x_i\neq y_i$. This fact and that $\lim_{i\to\infty} s_i=0$ imply that L_x has a locally infinite holonomy pseudogroup, which is also a contradiction. This proves Assertion 1.

The above method of proof is due to Sacksteder and Schwartz [7].

ASSERTION 2. *There is a connected open neighborhood N' of x in N such that for each leaf L, if $N'\cap L\neq\emptyset$, then $S_y\cap L=$at most one point for each $y\in L_x$.*

PROOF. Suppose that there is a sequence $\{x_i\}$ such that

1) $x_i\in N$ for all i, and $\lim_{i\to\infty} x_i=x$,
2) there is a point y_i in L_x such that $L_{x_i}\cap S_{y_i}$ contains at least two points.

We take a point a_i in $h_{y_i}^{-1}(L_{x_i}\cap S_{y_i})$ which is not the smallest point in it. Let $g_i:[0, 1]\to L_{x_i}$ be a sequence of paths such that $g_i(0)=x_i$ and $g_i(1)=h_{y_i}(a_i)$. By Lemma 1, there exists a sequence of projectors with the property C. The condition on g_i implies that $t_i<1$. Therefore by Lemma 2, L_x has a locally infinite holonomy pseudogroup. But this is a contradiction and Assertion 2 is proved.

ASSERTION 3. *There is a connected open neighborhood N'' of x in N' such that for each leaf L, if $N''\cap L\neq\emptyset$, then $S_y\cap L=$just one point for each $y\in L_x$.*

PROOF. Suppose that there is a sequence $\{x_i\}$ such that

1) $x_i\in N'$ for all i, and $\lim_{i\to\infty} x_i=x$,
2) there is a point y_i in L_x such that $L_{x_i}\cap S_{y_i}=\emptyset$.

Let $g_i:[0, 1]\to L_x$ be a sequence of paths satisfying $g_i(0)=x$, $g_i(1)=y_i$. By Lemma 1, there exists a sequence of projectors satisfying the following conditions.

1) $P_i:[0, t_i]\times[0, s_i]\to M$ where $0<t_i\leqq 1$,
2) $P_i(t, 0)=g_i(t)$ for $0\leqq t\leqq t_i$,
3) $P_i(0, s_i)=x_i$,
4) $P_i(t\times[0, s_i])$ has length less than l for $0\leqq t<t_i$,

5) $P_i(t_i\times[0, s_i])$ has length l if $t_i<1$.

In the present case, the condition on g_i implies that $t_i<1$. Therefore by Lemma 2, L_x has a locally infinite holonomy pseudogroup. But this contradicts the supposition on L_x and Assertion 3 is proved.

ASSERTION 4. *There is a connected open neighborhood N''' of x in N'' such that for each leaf L, if $N'''\cap L\neq\emptyset$, then for each $z\in L$, a path starting from z in the negative direction on T_z intersects L_x and the first intersecting point exists.*

PROOF. Suppose that there is a sequence $\{x_i\}$ such that

1) $x_i\in N''$ for all i, and $\lim_{i\to\infty} x_i=x$,

2) there is a point z_i in L_{x_i} such that a path starting from z_i into the negative direction on T_{z_i} either doesn't intersect L_x or intersects L_x but the first intersection point doesn't exist.

Let $g_i: [0, 1]\to L_{x_i}$ be a sequence of paths satisfying $g_i(0)=x_i$, $g_i(1)=z_i$. Then by Lemma 1, there is a sequence of projectors with the property C. The condition on g_i implies that $t_i<1$. Therefore by Lemma 2, L_x has a locally infinite holonomy pseudogroup. This is a contradiction and Assertion 4 is proved.

We remark that (N''', x) is diffeomorphic to $(R_+, 0)$. Let $\widehat{N'''}$ denote the saturation of N'''. If we define a foliation of $N'''\times L_x$ so that its leaves are $\{t\}\times L_x$ $(t\in N''')$, then there exist foliation preserving homeomorphisms between $\widehat{N'''}$ and $N'''\times L_x$ which are the inverses of each other. In fact the homeomorphisms $\varphi: N'''\times L_x\to\widehat{N'''}$, $\psi: \widehat{N'''}\to N'''\times L_x$ are given by $\varphi(t, y)=S_y\cap L_t$, $\psi(z)=(N'''\cap L_z, f(z))$ for $t\in N'''$, $y\in L_x$, $z\in\widehat{N'''}$, where $f(z)$ is the first intersecting point when a path starting from z into the negative direction on T_z intersects L_x.

In the entirely same way, we can find a set homeomorphic (foliation-preservingly) to $R_+\times L_x$ in the negative side of L_x. If we choose both sets sufficiently small that their intersection is L_x, then their union is the saturated, trivially foliated neighborhood of L_x homeomorphic to $R\times L_x$, as desired.

Finally, let $r>0$ and suppose that there is a sequence $\{x_i\}$ satisfying the following conditions.

1) $x_i\in N'''$ for all i, and $\lim_{i\to\infty} x_i=x$.

2) There is a point y_i in the saturation of $\overline{xx_i}$ such that $d(y_i, L_x)\geqq r$.

Define z_i by $z_i=L_{y_i}\cap\overline{xx_i}$. Then $\lim_{i\to\infty} z_i=x$. Let $g_i: [0, 1]\to L_{z_i}$ be a sequence of paths satisfying $g_i(0)=z_i$, $g_i(1)=y_i$. If we take r instead of l in Lemmas 1 and 2, we are led to the contradiction as in the proof of the preceding assertions. Thus we can find a saturated trivial neighborhood in the r-neighborhood of L_x. This completes the proof of Theorem 1.

The following corollary is Theorem 1 of [7].

COROLLARY 1. *Let L_x be a proper, relatively compact leaf. If there is a leaf L such that $L \neq L_x$ and $\bar{L} \supset L_x$, then L_x has a locally infinite holonomy pseudogroup.*

The following two corollaries are generalizations of the Reeb stability theorem for proper leaves of codimension one foliations under some particular conditions.

COROLLARY 2. *Let $\mathcal{F}$ be a codimension one foliation of a 2-dimensional manifold. Then every proper, relatively compact leaf of $\mathcal{F}$ with a trivial holonomy group has a saturated tubular neighborhood with a product foliation.*

PROOF. Immediate from Proposition 2 and Theorem 1.

COLOLLARY 3. *Let $\mathcal{F}$ be a C^ω codimension one foliation. Then every proper, relatively compact leaf of $\mathcal{F}$ with a trivial holonomy group has a saturated tubular neighborhood with a product foliation.*

PROOF. Immediate from Proposition 3 and Theorem 1.

§ 3. Proof of Theorem 2.

Theorem 2 is a direct corollary of Theorem 1 and the following lemma.

LEMMA 3. *Let $\mathcal{F}$ be a C^1 transversely oriented codimension one foliation of a compact 3-dimensional manifold M. If a leaf of $\mathcal{F}$ is simply connected, it does not have a locally infinite holonomy pseudogroup.*

PROOF. Suppose that there exists a simply connected leaf L_x of $\mathcal{F}$ which has a locally infinite holonomy pseudogroup. Then for an arbitrary neighborhood N of x on T_x, there exists a non-trivial element γ of the holonomy pseudogroup of L_x at x whose domain is contained in N. Let D denote the domain of γ. We may assume that D is diffeomorphic to the open interval $(0, 1)$. Let $P: D \times [0, 1] \to M$ be a projector which induces γ.

Since γ is non-trivial, there exists a point $y \in D$ such that $y \neq \gamma(y)$, which, we may assume without loss of generality, lies on the positive side of x. If we define X by

$$X = \{y \in D \mid y \neq \gamma(y) \text{ and } y \text{ lies on the positive side of } x\},$$

X is not empty and bounded below. Hence $\inf_{y \in X} y$ exists and we denote it by z. Then $z \neq x$, $z = \gamma(z)$, and the loop $g_z: [0, 1] \to L_z$ defined by $g_z(t) = P(z, t)$ is not homotopic to zero on L_z, because g_z induces a non-trivial element of the holonomy group of L_z. If we define Y by

$$Y = \left\{ y \in D \;\middle|\; \begin{array}{l} y = \gamma(y),\ y \text{ lies on the positive side of } x, \text{ and} \\ \text{the loop } g_y \text{ is not homotopic to zero on } L_y \end{array} \right\},$$

$Y \ni z$ and Y is bounded below. Hence $\inf_{y \in Y} y$ exists and we denote it by w. Then $w \neq x$ and the loop g_w is not homotopic to zero on L_w because of the well-known holonomy lemma [2]. Therefore g_w is a vanishing cycle in the sense of S. P. Novikov [2], [5].

The following result is proved by Novikov [2], [5].

THEOREM 3. *If M is compact and of dimension 3, then every leaf that contains a vanishing cycle is compact.*

By this theorem, L_w is a compact leaf. Since N can be chosen arbitrarily small, there exists a family of compact leaves L_λ $(\lambda \in \Lambda)$ such that $\overline{\bigcup_{\lambda \in \Lambda} L_\lambda} \supset L_x$.

The following theorem is well known ([2], [4], [6]).

THEOREM 4. *Let L_λ $(\lambda \in \Lambda)$ be a family of compact leaves. Then every leaf contained in $\overline{\bigcup_{\lambda \in \Lambda} L_\lambda}$ is compact.*

By this theorem, L_x itself is a compact leaf. But a simply connected compact leaf does not have a locally infinite holonomy pseudogroup by Proposition 1. This is a contradiction and Lemma 3 is proved.

REMARK. A simply connected leaf in a 3-manifold is either S^2 or R^2.

§ 4. An example of a codimension two foliation.

Here is an example of a codimension two foliation of T^3 (c. f. [1], p. 113) which shows that Theorem 2 cannot be generalized for foliations of codimension greater than one.

Let $T^3 = \{(x, y, \varphi, \theta) \in R^2 \times (R/2\pi Z)^2 \mid x^2 + y^2 = 1\}$. $\mathfrak{F}$ is defined by the differential system $\omega_1 = \omega_2 = 0$, where $\omega_1 = d\theta$ and $\omega_2 = \{(1 - \sin\theta)^2 + x^2\} d\varphi + \sin\theta dx$.

Every leaf of $\mathfrak{F}$ is proper. And every leaf diffeomorphic to R^1 does not have any saturated tubular neighborhood with a product foliation, because the union of all the compact leaves is dense in T^3.

§ 5. Addendum.

Recently we have generalized Theorem 2 as follows.

THEOREM 2′. *Let M, $\mathfrak{F}$ be as in Theorem 2 and let L be a proper leaf of $\mathfrak{F}$ such that $\pi_1(L)$ is finitely generated and that the holonomy group of L is trivial. Then L has a saturated tubular neighborhood with a product foliation.*

From Corollary 3 and the result of W. Thurston [8], we have obtained the following.

COROLLARY 3′. *Let M, $\mathfrak{F}$ be as in Corollary 3 and let L be a proper, relatively compact leaf of $\mathfrak{F}$ such that $H^1(L; R) = 0$ and that the holonomy group of L is finitely generated. Then the holonomy group of L is trivial and L has a saturated tubular neighborhood with a product foliation.*

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