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The curvatures of the analytic capacity

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§ 1. Introduction.

In [4] Suita has shown that the analytic capacity $c(z)$ of a plane region $D\notin 0_{AB}$ is real analytic and that the curvature of the metric $c(z)|dz|$ is $\leqq -4$. He also raised the conjecture that the curvature is equal to -4 at one point $z\in D$ if and only if $D\in\mathscr{D}_B$. D is said to belong to $\mathscr{D}_B$ if it is conformally equivalent to the unit disc less (possibly) a closed set expressed as a countable union of compact N_B sets. The papers [5] and [2] provide a different proof for Suita's result and actually resolve the conjecture of Suita in case $D\in\mathscr{D}_p$, $1\leqq p<\infty$. Here $\mathscr{D}_p$ denotes the class of all p-connected regions with no degenerate boundary component. In the present paper we generalize the results of [2] and [5] to higher order curvatures (Theorem 1). Specifically we show that, for any point z in $D\notin 0_{AB}$, $c^{(n+1)^2}\leqq(\prod_{k=1}^{n}k\,!)^{-2}\det\|c_{j\bar{k}}\|_{j,k=0}^{n}$, where $c=c(z)$ and $c_{j\bar{k}}=\dfrac{\partial^{j+k}c}{\partial z^j\partial\bar{z}^k}$. For $n=1$, we obtain the result of [4]. Moreover, if $D\in\mathscr{D}_B$ then we have equality in the above inequality for each $z\in D$ and every $n=0, 1,\cdots$. If $D\in\mathscr{D}_p$ then equality at one point $z\in D$ holds if and only if $p=1$. Several other properties related to the analytic capacity are proved. Our proofs are based on the "method of minimum integral" with respect to the Szegö kernel function. As in [2] we also show that the above inequality is strict if the Ahlfors function with respect to z has a zero in D other than z.

§ 2. Analytic capacity.

Let D be a plane region $\notin 0_{AB}$ and let $H(D:\Delta)$ designate the class of all analytic functions from D into the unit disc Δ. Let $\zeta\in D$ and set $H_\zeta(D:\Delta)=\{f\in H(D:\Delta):f(\zeta)=0\}$. The analytic capacity $c(\zeta)=c_D(\zeta)$ is given by $c(\zeta)=\sup\{|f'(\zeta)|:f\in H_\zeta(D:\Delta)\}$. There exists (cf. [3]) a unique function F in $H_\zeta(D:\Delta)$, called the Ahlfors function $F(z)=F(z:\zeta)$, such that $F'(\zeta)=c(\zeta)$. Clearly, $c(z)|dz|$ is a conformal invariant metric. Using a canonical exhaustion process (cf. [4]) it can be shown that $c(z)$ is real analytic and hence we can introduce

$$J_n(z) = \det \|c_{j\bar{k}}\|_{j,k=0}^n, \qquad n = 0, 1, \cdots,$$

where $c_{j\bar{k}} = c_{j\bar{k}}(z)$.

PROPOSITION 1. *$J_n(z)/[c(z)]^{(n+1)^2}$ is conformally invariant ($n=0, 1, \cdots$).*

PROOF. Let $w: D \to D^*$, $w = w(z)$, be a conformal mapping of D onto D^*. Then $c_D(z) = c_{D^*}(w)|w'(z)|$. Using the properties of Wronskians one can show that $J_n^D(z) = J_n^{D^*}(w)|w'(z)|^{(n+1)^2}$ and the assertion follows.

§ 3. The Szegö kernel.

Let $D \in \mathfrak{D}_p$, $1 \leqq p < \infty$. In what follows we can assume that D is bounded by p analytic curves. As usual, $H_2 = H_2(\partial D)$ stands for the Hardy-Szegö space of D. It is a Hilbert space of analytic functions in D with the scalar product $(f, g) = \int_{\partial D} f(z)\overline{g(z)}|dz|$ and $\|f\|^2 = (f, f)$. The integration is carried over the boundary values of the analytic functions f and g (this refers to an arbitrary non-tangential approach). H_2 admits a reproducing kernel $K(z, \bar{\zeta})$ which is the classical Szegö kernel for D. In this case (cf. [1, pp. 117-118]) $c(\zeta) = 2\pi K(\zeta, \bar{\zeta})$ and $F(z) = F(z : \zeta) = K(z, \bar{\zeta})/L(z, \zeta)$. Here $F'(\zeta) = c(\zeta)$ and $L(z, \zeta)$ is the adjoint of $K(z, \bar{\zeta})$ satisfying the boundary relation

$$\text{(3.1)} \qquad \overline{iK(z, \bar{\zeta})}|dz| = L(z, \zeta)dz; \; z \in \partial D, \; \zeta \in D.$$

Therefore $|F(z)| \equiv 1$ for $z \in \partial D$ and $|F(z)| \leqq 1$ throughout D. Moreover, the divisor of $L(z, \zeta)$ is exactly ζ^{-1} with residue $(2\pi)^{-1}$ and the analytic function $L(z, \zeta) - (2\pi)^{-1}(z-\zeta)^{-1}$ vanishes at $z = \zeta$. The divisor of $F(z)$ is therefore $\zeta, \overline{b_1(\zeta)}, \cdots, \overline{b_{p-1}(\zeta)}$ where $\overline{b_j(\zeta)}, j = 1, \cdots, p-1$, are the $p-1$ (possibly repeated) zeros of $K(z, \bar{\zeta})$ (none of which is on ∂D). The functions $b_j(\zeta)$ are analytic in ζ, if they are simple.

For fixed $\zeta \in D$, let $A_n(\zeta) = \{f \in H_2 : f^{(k)}(\zeta) = \delta_{kn}, \; k = 0, 1, \cdots, n\}$, $n = 0, 1, \cdots$. $A_n(\zeta)$ is a closed convex subset of H_2 and it is not empty for, the function $\varphi_n(z) = 2\pi F(z)^n K(z, \bar{\zeta})/n![c(\zeta)]^{n+1}$ is in $A_n(\zeta)$ for each $n = 0, 1 \cdots$. Let ψ_n be the unique solution of the minimal problem $\lambda_n(\zeta) = \min\{\|f\|^2 : f \in A_n(\zeta)\}$. Then (cf. Bergman [1, p. 26])

$$\lambda_n(\zeta) = I_{n-1}(\zeta)/I_n(\zeta); \quad I_n(\zeta) = \det \|K_{j\bar{k}}\|_{j,k=0}^n,$$

$(I_{-1}(\zeta) \equiv 1)$, where $K_{j\bar{k}} = (\partial^{j+k}/\partial\zeta^j \partial\bar{\zeta}^k)K$, $K = K(\zeta, \bar{\zeta})$. Also

$$\psi_n(z) = \frac{(-1)^n}{I_n(\zeta)} \begin{vmatrix} K_{0\bar{0}}(z, \bar{\zeta}) & \cdots\cdots & K_{0\bar{n}}(z, \bar{\zeta}) \\ K_{0\bar{0}} & \cdots\cdots & K_{0\bar{n}} \\ \vdots & & \\ K_{n-1\bar{0}} & \cdots\cdots & K_{n-1\bar{n}} \end{vmatrix}, \quad n = 0, 1, \cdots.$$

Here $\psi_0(z) = K(z, \bar{\zeta})/K(\zeta, \bar{\zeta})$. Clearly, $I_n(\zeta) = [\lambda_0(\zeta) \cdots \lambda_n(\zeta)]^{-1}$ and $\lambda_0(\zeta) = 1/K(\zeta, \bar{\zeta})$.

It follows that

$$\lambda_n(\zeta)=\|\psi_n\|^2\leqq\|\varphi_n\|^2=\frac{2\pi}{(n\,!)^2c(\zeta)^{2n+1}}$$

and equality, for $n\geqq 1$, holds if and only if $\psi_n(z)=\varphi_n(z)$. This is equivalent to $(f, \psi_n)=(f, \varphi_n)$ for all $f\in H_2$ or that

(3.2) $$(f, \varphi_n)=\frac{(-1)^n}{I_n(\zeta)}\begin{vmatrix} f(\zeta) & \cdots\cdots f^{(n)}(\zeta) \\ K_{0\bar{0}} & \cdots\cdots K_{n\bar{0}} \\ \vdots & \\ K_{0\overline{n-1}} & \cdots\cdots K_{n\overline{n-1}} \end{vmatrix}, \quad f\in H_2.$$

However, for $p>1$ (3.2) does not hold. Indeed, by (3.1)

$$\begin{aligned} n\,!\,c(\zeta)^{n+1}(f, \varphi_n) &= 2\pi\int_{\partial D} f(z)\overline{F(z)}^n\overline{K(z, \bar{\zeta})}\,|dz| \\ &= 2\pi\int_{\partial D}\frac{f(z)}{F(z)^n}\overline{K(z, \bar{\zeta})}\,|dz| \\ &= \frac{2\pi}{i}\int_{\partial D}\frac{f(z)}{F(z)^n}L(z, \zeta)dz \\ &= \frac{2\pi}{i}\int_{\partial D}f(z)\frac{L(z, \zeta)^{n+1}}{K(z, \bar{\zeta})^n}dz\,. \end{aligned}$$

The function $H(z)=L(z, \zeta)(z-\zeta)$ does not vanish in D also $K(z, \bar{\zeta})=(z-\overline{b_1(\zeta)})\cdots(z-\overline{b_{p-1}(\zeta)})h(z)$, where $h(z)$ does not vanish in D. Consider the function $f_0(z)=(z-\overline{b_1(\zeta)})^{n-1}(z-\overline{b_2(\zeta)})^n\cdots(z-\overline{b_{p-1}(\zeta)})^n(z-\zeta)^{n+1}$. Then $f_0\in H_2$ and

$$\begin{aligned} \frac{2\pi}{i}\int_{\partial D}f_0(z)\frac{L(z, \zeta)^{n+1}}{K(z, \bar{\zeta})^n}dz &= \frac{2\pi}{i}\int_{\partial D}\frac{H(z)^{n+1}}{h(z)^n(z-\overline{b_1(\zeta)})}dz \\ &= 4\pi^2\frac{H(\overline{b_1(\zeta)})^{n+1}}{h(\overline{b_1(\zeta)})^n}\neq 0\,. \end{aligned}$$

On the other hand the right hand side of (3.2) is zero for f_0. Therefore

(3.3) $$c(\zeta)^{2n+1}\leqq\frac{2\pi}{(n\,!)^2}\frac{1}{\lambda_n(\zeta)}$$

where, for $n\geqq 1$, we have a strict inequality if $p>1$. Upon multiplying the inequalities of (3.3) for $k=1, 2, \cdots, n$, using the fact that $I_n(\zeta)^{-1}=\lambda_0(\zeta)\cdots\lambda_n(\zeta)$, we obtain

$$c(\zeta)^{n(n+2)}\leqq\Big(\prod_{k=1}^{n}k\,!\Big)^{-2}(2\pi)^n\lambda_0(\zeta)I_n(\zeta)\,, \qquad n\geqq 1\,.$$

Since $c(\zeta)=2\pi K(\zeta, \bar{\zeta})=2\pi[\lambda_0(\zeta)]^{-1}$ we get at once

$$(3.4) \qquad c(\zeta)^{(n+1)^2} \leqq \left(\prod_{k=1}^{n} k\,!\right)^{-2} J_n(\zeta)\,, \qquad n \geqq 1\,.$$

Here $J_n(\zeta)=(2\pi)^{n+1}I_n(\zeta)=\det \|c_{j\bar{k}}\|_{j,k=0}^{n}$, $c=c(\zeta)$.

§ 4. The n-th order curvature.

The n-th order curvature of $c(\zeta)|d\zeta|$ is defined by

$$\kappa_n(\zeta : D) = -4c(\zeta)^{-(n+1)^2} J_n(\zeta)\,, \qquad n \geqq 1\,.$$

By Proposition 1 this curvature is conformally invariant. Note also that, for $n=1$ $\kappa_1(\zeta : D)=-c^{-2}\Delta \log c$, $c=c(\zeta)$, which is the usual curvature of $c(\zeta)|d\zeta|$.

PROPOSITION 2. *$\kappa_n(\zeta : D)=-4(\prod_{k=1}^{n} k\,!)^2$ for each $D\in \mathcal{D}_B$.*

PROOF. Due to the conformal invariance and the definition of $\mathcal{D}_B$ we only have to establish this identity for $D=\Delta$, the unit disc, and $\zeta=0$. In this case $\psi_k^{(\Delta)}(z)=z^k/k\,!$, $k=0, 1, \cdots$, and so $\lambda_k^{(\Delta)}(0)=2\pi/(k\,!)^2$. Therefore $J_n^{(\Delta)}(0)=(\prod_{k=1}^{n} k\,!)^2$ and the proposition follows.

Combining the things we said in § 3 we arrive at the following generalization of a result in [2] and [5].

THEOREM 1. *Let $D\in \mathcal{D}_p$, $1\leqq p<\infty$. Then $\kappa_n(\zeta : D)\leqq -4(\prod_{k=1}^{n} k\,!)^2$ for each $\zeta\in D$ and each $n\geqq 1$. Equality holds for one point ζ and any $n\geqq 1$ if and only if $D\in \mathcal{D}_1$. Moreover, the identity*

$$\frac{1}{i}\int_{\partial D} f(z)\,\frac{L(z,\zeta)^{n+1}}{K(z,\bar{\zeta})^n}\,dz = \frac{(-1)^n n\,!}{(\prod_{k=1}^{n} k\,!)^2 c^{n(n+1)}} \begin{vmatrix} f(\zeta) & \cdots\cdots & f^{(n)}(\zeta) \\ c_{0\bar{0}} & \cdots\cdots & c_{n\bar{0}} \\ \vdots & & \\ c_{0\,\overline{n-1}} & \cdots\cdots & c_{n\,\overline{n-1}} \end{vmatrix},$$

$c=c(\zeta)$, $f\in H_2$, holds for any $n\geqq 1$ and some $\zeta\in D$ if and only if $D\in \mathcal{D}_1$.

COROLLARY 1. *Let $D\notin 0_{AB}$. Then $\kappa_n(\zeta : D)\leqq -4(\prod_{k=1}^{n} k\,!)^2$, $n=1, 2, \cdots$.*

PROOF. Let $\{D_m\}$ be a canonical exhaustion of D such that ∂D_m consists of a finite number of analytic curves. Then, for each $n\geqq 1$, $\kappa_n(\zeta : D)=\lim_{m\to\infty} \kappa_n(\zeta : D_m)$ and since $\kappa_n(\zeta : D_m)\leqq -4(\prod_{k=1}^{n} k\,!)^2$ for each m the corollary follows.

The case $n=1$ of Corollary 1 is the main result of Suita [4]. Following Suita we conjecture that $\kappa_n(\zeta : D)=-4(\prod_{k=1}^{n} k\,!)^2$ at one point $\zeta\in D$, $D\notin 0_{AB}$, and any $n\geqq 1$ implies that $D\in \mathcal{D}_B$.

Actually, we have shown a little more. According to Proposition 1 the positive domain function

$$\mu_n=\mu_n(\zeta:D)=\frac{(n!)^{-2}}{c(\zeta)^{2n+1}}\frac{J_n(\zeta)}{J_{n-1}(\zeta)}$$

is conformally invariant. Let

$$\nu_n=\nu_n(\zeta:D)=(\prod_{k=1}^{n}k!)^{-2}\frac{J_n(\zeta)}{c(\zeta)^{(n+1)2}}, \qquad n\geqq 0,$$

so $\nu_n(\zeta:D)\geqq 1$ by (3.4). However, $\nu_n/\mu_n=\nu_{n-1}\geqq 1$ for $n\geqq 1$ and thus $\nu_n\geqq\mu_n\geqq 1$. Combining this with the previous results we obtain at once. (Note that $\nu_1=\mu_1$.)

THEOREM 2. *Let $D\in\mathfrak{D}_p$, $1\leqq p<\infty$. Then $\nu_n(\zeta:D)\geqq\mu_n(\zeta:D)\geqq 1$ for each $\zeta\in D$ and $n\geqq 1$. Equality, in any one of the two inequalities, holds for one point ζ and any $n\geqq 2$ if and only if $D\in\mathfrak{D}_1$.*

This theorem implies the main part of Theorem 1.

COROLLARY 2. *Let $D\notin 0_{AB}$. Then $\nu_n(\zeta:D)\geqq\mu_n(\zeta:D)\geqq 1$.*

COROLLARY 3. $\nu_n(\zeta:D)=\mu_n(\zeta:D)=1$ *for each $D\in\mathfrak{D}_B$.*

§ 5. Sharper results.

For $\zeta\in D$ we write $\delta_D(\zeta)=\sup_{z\in\partial D}|z-\zeta|$ whence if $\infty\in D$ and $\zeta\neq\infty$, $\delta_D(\zeta)\leqq\delta<\infty$. Designate by $\mathfrak{D}_p^{(a)}$, $1\leqq p<\infty$, the class of all plane regions bounded by p analytic Jordan curves. The following is an improvement on Theorem 1 (see also [2]).

THEOREM 3. *Let $D\in\mathfrak{D}_p^{(a)}$, $p>1$ and $\zeta\in D$. Then, for $n\geqq 1$,*

$$\kappa_n(\zeta:D)<-4(\prod_{k=1}^{n}k!)^2\Big[1+\frac{4\pi^2 n}{\delta_D^{2n}(\zeta)}\cdot\frac{|\zeta_j-\zeta|^{2n}|L(\zeta_j,\zeta)|^2}{c(\zeta)c(\zeta_j)}\Big],$$

where ζ_j is any one of the $(p-1)$ zeros of $K(z,\bar{\zeta})$, (i.e., $\zeta_j=\overline{b_j(\zeta)}$, $1\leqq j\leqq p-1$).

PROOF. Clearly, $\zeta_j\neq\zeta$. Let

$$g_j(z)=(z-\zeta)^n F(z)L(z,\zeta_j), \qquad 1\leqq j\leqq p-1.$$

Since $F(\zeta_j)=0$ it is clear that $g_j\in H_2$ and $g_j^{(k)}(\zeta)=0$, $k=0,1,\cdots,n$. Also, by (3.1), $\|g_j\|^2\leqq\delta_D^{2n}(\zeta)K(\zeta_j,\bar{\zeta}_j)$. Let

$$h_j=\varphi_n-\frac{(\varphi_n,g_j)}{\|g_j\|^2}g_j, \qquad 1\leqq j\leqq p-1.$$

Then $h_j\in A_n(\zeta)$ and therefore

$$\lambda_n(\zeta)\leqq\|h_j\|^2=\|\varphi_n\|^2-\frac{|(g_j,\varphi_n)|^2}{\|g_j\|^2}.$$

Here

$$\|\varphi_n\|^2=\frac{2\pi}{(n!)^2 c(\zeta)^{2n+1}}$$

and

$$(g_j, \varphi_n) = \frac{2\pi}{n!\, c(\zeta)^{n+1}} \frac{1}{i} \int_{\partial D} (z-\zeta)^n L(z, \zeta_j) L(z, \zeta) dz$$

$$= \frac{2\pi}{n!\, c(\zeta)^{n+1}} (\zeta_j - \zeta)^n L(\zeta_j, \zeta) .$$

Therefore

$$\lambda_n(\zeta) \leqq \frac{2\pi}{(n!)^2 c(\zeta)^{2n+1}} (1-A_j) ,$$

with

$$A_j = \frac{2\pi}{c(\zeta)} \frac{|\zeta_j - \zeta|^{2n} |L(\zeta_j, \zeta)|^2}{\delta_D^{2n}(\zeta) K(\zeta_j, \bar{\zeta}_j)} .$$

Clearly, $0 < A_j < 1$ and therefore $(1-A_j) < (1+A_j)^{-1}$. Consequently,

$$(1+A_j) c^{2n+1} < \frac{2\pi}{(n!)^2} \frac{1}{\lambda_n} ; \quad c = c(\zeta) , \quad \lambda_n = \lambda_n(\zeta) .$$

Upon multiplication of these inequalities (running from $k=1$ to $k=n$) we obtain

$$(1+A_j)^n c^{(n+1)^2} < (\prod_{k=1}^{n} k!)^{-2} J_n .$$

The assertion now follows from $(1+A_j)^n \geqq 1 + nA_j$.

The next theorem sharpens the assertions of Corollary 1. Let $D \notin 0_{AB}$ and let $\{D_m\}$ be a canonical exhaustion of D such that ∂D_m consists of a finite number of analytic curves. In every D_m we have the Szegö kernel $K_m(z, \bar{\zeta})$, its adjoint $L_m(z, \zeta)$ and the Ahlfors function $F_m(z) = F_m(z : \zeta)$. Then, the sequences $\{F_m(z)\}$ and $\{K_m(z, \bar{\zeta})\}$ converge uniformly on compacta of D to $F(z)$ and $K(z, \bar{\zeta})$ respectively [4]. Of course, $c(\zeta) = 2\pi K(\zeta, \bar{\zeta})$. Therefore, $\{L_m(z, \zeta)\}$ converges uniformly on compacta of $D - \{\zeta\}$ to $L(z, \zeta)$.

THEOREM 4. *Let $D \notin 0_{AB}$ and $\zeta \in D$. Assume the Ahlfors function $F(z) = F(z : \zeta)$ has a zero ζ_0 in D other than ζ. Then $\kappa_n(\zeta : D) < -4(\prod_{k=1}^{n} k!)^2$.*

PROOF. We may assume that $\infty \in D$ and $\zeta, \zeta_0 \neq \infty$. Let $\{D_m\}$ be a canonical exhaustion of D as before. Since $F(\zeta_0 : \zeta) = 0$, $\zeta \neq \zeta_0$, it follows from Hurwitz's theorem that, for a sufficiently large m, $F_m(z : \zeta)$ has a zero $\zeta_m \neq \zeta$ near ζ_0. This zero must be a zero of $K_m(z, \bar{\zeta})$ and thus $D_m \in \mathcal{D}_{p_m}^{(a)}$, $p_m > 1$, for such large m. Since $\delta_{D_m}(\zeta) \leqq \delta_{D_1}(\zeta) \leqq \delta < \infty$, it follows from Theorem 3 that

$$\kappa_n(\zeta : D_m) < -4(\prod_{k=1}^{n} k!)^2 \Big[1 + \frac{4\pi^2 n}{\delta^{2n}} \cdot \frac{|\zeta_m - \zeta|^{2n} |L_m(\zeta_m, \zeta)|^2}{c_m(\zeta) c_m(\zeta_m)} \Big]$$

for a sufficiently large m. Since $L_m(z, \zeta)$ has no zero in $D_m - \{\zeta\}$ it follows by another application of Hurwitz's theorem that $L_m(\zeta_m, \zeta) \to L(\zeta_0, \zeta) \neq 0$. Letting

$m \to \infty$ in the above inequality concludes the proof.

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