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Barycenters and extreme points

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§ 1. Introduction.

Let Y be a completely regular Hausdorff space, $M(Y)$ the set of all positive, regular, and finitely additive measures, each of total mass 1, on the field generated by zero sets in Y, and $M_{\sigma}(Y)$, $M_{\tau}(Y)$ the subsets of $M(Y)$ consisting of all σ-additive, and all τ-additive elements of $M(Y)$. Every element of $M_{\sigma}(Y)$ is a Baire measure on the σ-algebra generated by zero sets called Baire algebra of Y. Every μ of $M_{\tau}(Y)$ is uniquely extendible to a countably additive and regular Borel measure on Y which also is denoted by μ. Each element of $M(Y)$ can be identified with a positive linear functional L on $C_b(Y)$, the set of all real-valued bounded and continuous functions on Y, such that $L(1)=1$. By the weak topology on $M_{\sigma}(Y)$, we mean the topology $\sigma(M_{\sigma}, C_b)$. An element μ of $M_{\sigma}(Y)$ is said to be separable if for every continuous pseudometric d on Y, there is a d-closed subset Z of Y such that $\mu(Y\backslash Z)=0$ and such that Z is d-separable (see § 1 in [1] or [7]). We denote the set of all separable measures by $M_s(Y)$. Then we should remark that $M_{\tau}(Y)\subset M_s(Y)$ (see p. 267 in [1]), and $M_{\sigma}(Y)=M_s(Y)$ for any D-topological space (see p. 1 in [3] and p. 137 in [7]).

Let X be a non-void, closed, convex, and bounded subset of a locally convex Hausdorff space F over reals. A point a in X is called the barycenter of a $\mu\in M(X)$ if $\mu(f)=f(a)$ for any f of F^*, the topological dual of F, where $\mu(f)=\int f_{|X}d\mu$, and $f_{|X}$ is the restriction of f on X. We denote by ext X the set of all extreme points of X, by ε_a the point measure at a, and by clconv X the closed convex hull of X.

Let p be in S_F which denotes the set of all continuous seminorms on F. Define on F the equivalence relation $a\sim b$ if and only if $p(a-b)=0$. If $\dot{F}_p$ is the class of all such equivalence classes associated with p, $\dot{a}$ being that which contains a of F, then we can define the norm $\dot{p}$ on $\dot{F}_p$ by $\dot{p}(\dot{a})=p(a)$. In $\dot{F}_p$, sum of two elements and scalar multiplication can be defined as usual. Then $\dot{F}_p$ is a normed space with norm $\dot{p}$. This normed space is denoted by $\dot{F}_{\dot{p}}$.

Define the map $Q_p: F \to \dot{F}_{\dot{p}}$ by $Q_p(a)=\dot{a}$, which is continuous and linear. Putting $\dot{X}_p=Q_p(X)$, $\dot{X}_p$ is a bounded subset of $\dot{F}_{\dot{p}}$, and a metric space with metric induced by $\dot{p}$. This metric space is denoted by $\dot{X}_{\dot{p}}$. For $\mu\in M_\sigma(X)$, define $(\dot{\mu})_p\in M_\sigma(\dot{X}_{\dot{p}})$ by $(\dot{\mu})_p(\dot{B})=\mu(Q_p^{-1}(\dot{B})\cap X)$ for any Baire set $\dot{B}\subset\dot{X}_{\dot{p}}$. $(\varepsilon_{\dot{a}})_p$ denotes the point measure at $\dot{a}$ in $\dot{X}_{\dot{p}}$. Then we consider the following set of measures on X associated with F and $p\in S_F$, which is denoted by $\dot{M}^p_{\tau F}(X)$.

$$\dot{M}^p_{\tau F}(X)=\{\mu\in M_\sigma(X):\ (\dot{\mu})_p\in M_\tau(\dot{X}_{\dot{p}})\}\ .$$

On this set of measures on X, we should remark two following facts. One is that $M_s(X)\subset\dot{M}^p_{\tau F}(X)$, since $(\dot{\mu})_p\in M_\tau(\dot{X}_{\dot{p}})$ for any $\mu\in M_s(X)$, which is easily checked by Varadarajan's theorem (Corollary of Theorem 27 in part 1 in [**9**]). The other is that if F is separable, $M_\sigma(X)=\dot{M}^p_{\tau F}(X)$, which also is easily checked by Varadarajan's theorem (Corollary 4 of Theorem 25 in part 1 in [**9**]). From these sets $\dot{M}^p_{\tau F}(X)$, $p\in S_F$, we define the set of measures on X, which is denoted by $\dot{M}_{\tau F}(X)$ and associated with F, as follows.

$$\dot{M}_{\tau F}(X)=\bigcap_{p\in S_F}\dot{M}^p_{\tau F}(X)\,.$$

Then we have, from above properties of $\dot{M}^p_{\tau F}(X)$,

(1) $M_s(X)\subset\dot{M}_{\tau F}(X)$

and

(2) $M_\sigma(X)=\dot{M}_{\tau F}(X)$ if F is separable.

The purpose of this paper is to prove the following theorem, which gives a characterization of extreme points of X by $\dot{M}_{\tau F}(X)$, and also is a generalization of the results due to Khurana in the cases of $M_\tau(X)$ and $M_\sigma(X)$ (Theorems 2, 3 in [6] respectively).

THEOREM 1. *If X is complete, $a\in ext\ X$ if and only if ε_a is the only one element of $\dot{M}_{\tau F}(X)$ having a as its barycenter.*

In order to prove this theorem, we need the following theorem on the existence of barycenters of elements of $\dot{M}_{\tau F}(X)$, which is a generalization of the results obtained by Khurana in [5] and [6] in the cases of $M_\tau(X)$ and $M_\sigma(X)$.

THEOREM 2. *If X is complete, every element of $\dot{M}_{\tau F}(X)$ has a barycenter in X.*

In § 2, we give the proof of Theorem 2. Our method of the proof is based on the fact that $\mu(f)=\int f_{|X}d\mu$ is weak* continuous on every equicontinuous set of F_1^*, where μ is an element of $\dot{M}_{\tau F}(X)$ and F_1 is the completion of F.

In § 3, we give the proof of Theorem 1. Our method is based on the facts that $(\dot{\mu})_p$ is τ-additive for any $\mu \in \dot{M}^p_{\tau F}(X)$ and that every element of $\dot{M}_{\tau F}(X)$ has a barycenter in X.

Finally, in § 4, we prove the following theorem concerning the connection between the compactness of clconv K and barycenters of elements of $M(K)$ for any compact subset K of F. Then, the well-known theorem on the compactness of clconv K of a complete locally convex Hausdorff space F follows immediately from this theorem.

THEOREM 3. *Let K be a compact subset of a locally convex Hausdorff space F. Then, the following statements are equivalent.*

(a) *Every element of $M(K)$ has a barycenter in F.*

(b) *Clconv K is compact.*

(c) *Clconv K is complete.*

I express my hearty thanks to the referee for many valuable comments.

§ 2. Proof of Theorem 2.

For the proof, we prepare some lemmas.

LEMMA 1 (Theorem 17.7, chapter 5 in [4]). *A locally convex Hausdorff space F is complete if and only if each linear functional on F^* which is weak* continuous on every equicontinuous set of F^* is weak* continuous on F^*, (equivalently, if each such linear functional is evaluation at some point of F).*

LEMMA 2. *When $C_b(X)$ is topologized by the pointwise convergence topology, every element of $\dot{M}_{\tau F}(X)$ is continuous on $\mathcal{H}_X$ for any equicontinuous set $\mathcal{H}$ of F^*, where $\mathcal{H}_X = \{f_{|X} : f \in \mathcal{H}\}$.*

PROOF. Let $\mu \in \dot{M}_{\tau F}(X)$. To prove that μ is continuous on $\mathcal{H}_X$, take a net $f_{\alpha|X} \to f_{|X}$ pointwise in $\mathcal{H}_X$. Define the seminorm p on F by $p(a) = \sup\{|f(a)| : f \in \mathcal{H}\}$. The equicontinuity of $\mathcal{H}$ implies that p is in S_F. Define on F the equivalence relation $a \sim b$ if and only if $p(a-b)=0$. By the same argument as in introduction, we have a metric space $\dot{X}_{\dot{p}}$. We define $(\dot{f}_{|X})_{\dot{p}}$ on $\dot{X}_{\dot{p}}$ by $(\dot{f}_{|X})_{\dot{p}}(\dot{a}) = f_{|X}(a)$ for some a in $\dot{a}$ and $f_{|X} \in \mathcal{H}_X$. The class $\dot{\mathcal{H}}_{X\dot{p}} = \{(\dot{f}_{|X})_{\dot{p}} : f_{|X} \in \mathcal{H}_X\}$ is uniformly bounded and uniformly equicontinuous on $\dot{X}_{\dot{p}}$ since

$$|(\dot{f}_{|X})_{\dot{p}}(\dot{a})| = |f_{|X}(a)| \leqq p(a) = \dot{p}(\dot{a})$$

for any $f_{|X} \in \mathcal{H}_X$. For $\mu \in \dot{M}_{\tau F}(X)$, and so $\mu \in \dot{M}^p_{\tau F}(X)$, we have

$$\int \dot{g}(\dot{a}) d(\dot{\mu})_p(\dot{a}) = \int \dot{g}(Q_p(a)) d\mu(a)$$

for any $\dot{g} \in C_b(\dot{X}_{\dot{p}})$, particularly

$$\int (\dot{f}_{|X})_{\dot{p}}(\dot{a})\, d(\dot{\mu})_p(\dot{a}) = \int (\dot{f}_{|X})_{\dot{p}}(Q_p(a)) d\mu(a) = \int f_{|X}(a) d\mu(a)$$

for any $f_{|X}\in\mathcal{H}_X$. Since $f_{\alpha|X}\to f_{|X}$ pointwise, $(\dot{f}_{\alpha|X})_{\dot{p}}\to(\dot{f}_{|X})_{\dot{p}}$ pointwise. So we have $\inf_{\delta}\sup_{\alpha\geqq\delta}(\dot{f}_{\alpha|X})_{\dot{p}}(\dot{a})=(\dot{f}_{|X})_{\dot{p}}(\dot{a})=\sup_{\delta}\inf_{\alpha\geqq\delta}(\dot{f}_{\alpha|X})_{\dot{p}}(\dot{a})$. Putting $\dot{g}_{\delta}(\dot{a})=\sup_{\alpha\geqq\delta}(\dot{f}_{\alpha|X})_{\dot{p}}(\dot{a})$, and $\dot{h}_{\delta}(\dot{a})=\inf_{\alpha\geqq\delta}(\dot{f}_{\alpha|X})_{\dot{p}}(\dot{a})$, $\dot{g}_{\delta}$ and $\dot{h}_{\delta}$ are functions such that $\dot{g}_{\delta}\downarrow(\dot{f}_{|X})_{\dot{p}}$, $\dot{h}_{\delta}\uparrow(\dot{f}_{|X})_{\dot{p}}$, $(\dot{f}_{\delta|X})_{\dot{p}}\leqq\dot{g}_{\delta}$ and $\dot{h}_{\delta}\leqq(\dot{f}_{\delta|X})_{\dot{p}}$. The uniform boundedness and the equicontinuity of $\dot{\mathcal{H}}_{X\dot{p}}$ imply that $\dot{g}_{\delta}$ and $\dot{h}_{\delta}$ are bounded and continuous. $(\dot{\mu})_p$ being in $M_{\tau}(\dot{X}_{\dot{p}})$, we have

$$(\dot{\mu})_p(\dot{f}_{|X})_{\dot{p}}=\varliminf_{\delta}(\dot{\mu})_p(\dot{h}_{\delta})\leqq\varliminf_{\delta}(\dot{\mu})_p(\dot{f}_{\delta|X})_{\dot{p}}\leqq\varlimsup_{\delta}(\dot{\mu})_p(\dot{f}_{\delta|X})_{\dot{p}}$$

$$\leqq\varlimsup_{\delta}(\dot{\mu})_p(\dot{g}_{\delta})=(\dot{\mu})_p(\dot{f}_{|X})_{\dot{p}}.$$

Hence we have $\lim_{\alpha}\mu(f_{\alpha|X})=\mu(f_{|X})$. Thus the proof is completed.

PROOF OF THEOREM 2. Let F_1 be the completion of F. We know that $F_1^*=F^*$. Consider the linear functional

$\mu:(F_1^*,\sigma(F_1^*,F_1))\to R$, $\mu(f)=\int f_{|X}d\mu$, $f\in F_1^*$, where R is the set of all real numbers. By Lemma 1 and Lemma 2, we have that μ is continuous, that is, $\mu\in(F_1^*,\sigma(F_1^*,F_1))^*=F_1$, and so there exists a point a in F_1 such that $f(a)=\mu(f)=\int f_{|X}d\mu$ for any $f\in F_1^*=F^*$. It easily follows from the separation theorem that a lies in X, since X is closed in F_1.

COROLLARY 1. *If X is complete, every element of $M_s(X)$ has a barycenter in X.*

PROOF. This follows from the fact that $M_s(X)\subset\dot{M}_{\tau F}(X)$.

REMARK. This corollary contains a theorem of Khurana (Theorem 1 in [**6**]) asserting that if F is complete, every element of $M_{\tau}(X)$ has a barycenter in X, since $M_{\tau}(X)\subset M_s(X)$.

COROLLARY 2. *If X is a complete D-topological space, every element of $M_{\sigma}(X)$ has a barycenter in X.*

PROOF. Since X is a D-topological space, we have $M_{\sigma}(X)=M_s(X)\subset\dot{M}_{\tau F}(X)$. Hence the corollary holds.

REMARK. This corollary contains a theorem of Khurana (Theorem 2.2 in [**5**]) asserting that if X is complete and separable, every element of $M_{\sigma}(X)$ has a barycenter in X.

COROLLARY 3 (Theorem 2.2 in [**5**]). *If X is complete and F is separable, every element of $M_{\sigma}(X)$ has a barycenter in X.*

PROOF. This follows from the fact that $M_{\sigma}(X)=\dot{M}_{\tau F}(X)$ if F is separable.

REMARK. Concerning the vector integration, we obtain the following result. Suppose $f:Y\to F$ is continuous and bounded, $\mu\in M_s(Y)$, and $f(Y)$ is contained in a complete convex subset of F. Then the weak integral $\int f\,d\mu$ is in F. This is a generalization of a result due to Khurana (Proposition 1.5 in [**5**]). We give a sketch of the proof. Consider the linear functional

$$L:(F_1^*,\sigma(F_1^*,F_1))\to R,\ L(g)=\int(g,f(y))d\mu(y),\ g\in F_1^*=F^*.$$

$\mathcal{H}_f=\{h(y)=(g,f(y)):g\in\mathcal{H}\}$ is uniformly bounded and equicontinuous on Y for every equicontinuous set $\mathcal{H}$ of F_1^*. Define on Y the continuous pseudometric $d(x,y)=\sup\{|h(x)-h(y)|:h\in\mathcal{H}_f\}$. Define on Y the equivalence relation $x\sim y$ if and only if $d(x,y)=0$. If $\dot{Y}$ is the class of all such equivalence classes, $\dot{y}$ being that which contains y of Y, then we can define the metric $\dot{d}$ on $\dot{Y}$ by $\dot{d}(\dot{x},\dot{y})=d(x,y)$. $(\dot{Y},\dot{d})$ is a metric space and $Q:Y\to\dot{Y}$ defined by $Q(y)=\dot{y}$ is continuous onto. We define $\dot{h}$ on $\dot{Y}$ by $\dot{h}(\dot{y})=h(y)$ for some y in $\dot{y}$ and $h\in\mathcal{H}_f$. The class $\dot{\mathcal{H}}_f=\{\dot{h}:h\in\mathcal{H}_f\}$ is uniformly bounded and equicontinuous on $(\dot{Y},\dot{d})$. For $\mu\in M_s(Y)$, define $\dot{\mu}\in M_\sigma(\dot{Y})$ by $\dot{\mu}(\dot{B})=\mu(Q^{-1}(\dot{B}))$ for any Baire set $\dot{B}\subset\dot{Y}$. Then $\dot{\mu}\in M_s(\dot{Y})$, and so $\dot{\mu}\in M_\tau(\dot{Y})$ by Varadarajan's theorem (Corollary of Theorem 27 in part 1 in [**9**]). The rest is analogous to Theorem 2.

§ 3. Proof of Theorem 1.

For the proof, we prepare some lemmas.

LEMMA 3. *Let μ be an element of $\dot{M}_{\tau F}(X)$. If $0<\nu\leqq\mu$, we have $\nu/\nu(X)\in\dot{M}_{\tau F}(X)$.*

PROOF. We prove that $(\dot{\nu}/\nu(X))_p\in M_\tau(\dot{X}_p)$ for any $p\in S_F$. Take a net $\dot{g}_\alpha\in C_b(\dot{X}_p)$ with $\dot{g}_\alpha\downarrow 0$. Then we have

$$\int\dot{g}_\alpha(\dot{a})d(\dot{\mu})_p(\dot{a})=\int\dot{g}_\alpha(Q_p(a))d\mu(a)\geqq\int\dot{g}_\alpha(Q_p(a))d\nu(a)$$

$$=\int\dot{g}_\alpha(\dot{a})d(\dot{\nu})_p(\dot{a})\geqq 0\,.$$

$(\dot{\mu})_p$ being in $M_\tau(\dot{X}_p)$, we have $(\dot{\nu})_p(\dot{g}_\alpha)\to 0$, and so $(\dot{\nu}/\nu(X))_p\in M_\tau(\dot{X}_p)$. p being arbitrary, we have $\nu/\nu(X)\in\dot{M}_{\tau F}(X)$.

LEMMA 4. *Let μ be an element of $\dot{M}_{\tau F}(X)$ having a as its barycenter. If there exists $p\in S_F$ such that $(\dot{\mu})_p\neq(\varepsilon_{\dot{a}})_p$, there exists a convex zero set $A\subset X$ with $0<\mu(A)<1$ and $a\overline{\in}A$.*

PROOF. Suppose that there exists $p\in S_F$ such that $(\dot{\mu})_p\neq(\varepsilon_{\dot{a}})_p$. Let $\dot{b}\in$Support of $(\dot{\mu})_p$, $\dot{b}\neq\dot{a}$, and take $\dot{f}\in(\dot{F}_p)^*$ such that $\dot{f}(\dot{b})<c<\dot{f}(\dot{a})$, for some real c. Putting $A=\{x\in X;\dot{f}(Q_p(x))\leqq c\}$, A is a convex zero set with $0<\mu(A)<1$, and $a\overline{\in}A$, which are proved as follows. If $\mu(A)=1$, we have

$$\dot{f}(\dot{a})=\dot{f}(Q_p(a))=\int_X \dot{f}(Q_p(x))d\mu(x)=\int_A \dot{f}(Q_p(x))d\mu(x)\leqq c\,,$$

which is a contradiction. Putting $\dot{B}=\{\dot{x}\in\dot{X}_p:\dot{f}(\dot{x})\leqq c\}$, we have $A=Q_p^{-1}(\dot{B})\cap X$ and $(\dot{\mu})_p(\dot{B})>0$, which can be easily checked. Hence we have $\mu(A)=\mu(Q_p^{-1}(\dot{B})\cap X)=(\dot{\mu})_p(\dot{B})>0$. It is clear that $a\bar{\in}A$.

LEMMA 5. *If* $(\dot{\mu})_p=(\varepsilon_{\dot{a}})_p$ *for any* $p\in S_F$, $\mu(\{x:p(x-a)=0\})=1$ *for any* $p\in S_F$.

PROOF. $\mu(\{x:p(x-a)=0\})=\mu(Q_p^{-1}(\{\dot{a}\})\cap X)=(\dot{\mu})_p(\{\dot{a}\})=(\varepsilon_{\dot{a}})_p(\{\dot{a}\})=1$.

LEMMA 6. *If* $\mu(\{x:p(x-a)=0\})=1$ *for any* $p\in S_F$, *we have* $\mu=\varepsilon_a$.

PROOF. Let g be a real-valued continuous function on X with $\|g\|<1$ and $g(a)=0$. For every positive number ε, there exists $p\in S_F$ such that $\{x:p(x-a)=0\}\subset\{x:|g(x)|<\varepsilon\}$. Putting $Z=\{x:p(x-a)=0\}$, Z is a zero set of X and $\mu(Z)=1$. Hence we have

$$\left|\int g(x)d\mu(x)\right|\leqq\int_Z|g(x)|d\mu(x)+\int_{X\setminus Z}|g(x)|d\mu(x)<\varepsilon\mu(Z)=\varepsilon\,.$$

ε being arbitrary, $\int g(x)d\mu(x)=0$. This implies that $\mu=\varepsilon_a$.

PROOF OF THEOREM 1. Let μ be an element of $\dot{M}_{\tau F}(X)$ having $a\in\text{ext }X$ as its barycenter. If there exists $p\in S_F$ such that $(\dot{\mu})_p\neq(\varepsilon_{\dot{a}})_p$, there exists a convex zero set A with $0<\mu(A)<1$ and $a\bar{\in}A$ by Lemma 4. Define $\mu_1(B)=t^{-1}\cdot\mu(A\cap B)$, and $\mu_2(B)=(1-t)^{-1}\cdot\mu((X\setminus A))\cap B)$ for any Baire set $B\subset X$, where $t=\mu(A)$. Then we have by Lemma 3 and a simple verification that $\mu_1\in\dot{M}_{\tau F}(X)$, $\mu_2\in\dot{M}_{\tau F}(X)$ and $\mu=t\cdot\mu_1+(1-t)\cdot\mu_2$, which implies that $a=ta_1+(1-t)a_2$, a_1, a_2, being barycenters of μ_1, μ_2, respectively (Theorem 2), and $a_1\neq a$ (Lemma 4). Since a is an extreme point, this is a contradiction, and so $(\dot{\mu})_p=(\varepsilon_{\dot{a}})_p$ for any $p\in S_F$. By Lemma 5 and Lemma 6, we have $\mu=\varepsilon_a$. The converse is trivial. Thus the proof is completed.

COROLLARY 4. *If* X *is complete,* $a\in$ *ext* X *if and only if* ε_a *is the only one element of* $M_s(X)$ *having* a *as its barycenter.*

REMARK. This corollary contains a theorem of Khurana (Theorem 2 in [6]) asserting that if F is complete, $a\in$ *ext* X if and only if ε_a is the only one element of $M_\tau(X)$ having a as its barycenter.

COROLLARY 5. *If* X *is a complete D-topological space,* $a\in$ *ext* X *if and only if* ε_a *is the only one element of* $M_\sigma(X)$ *having* a *as its barycenter.*

COROLLARY 6 (Theorem 3 in [6]). *If* X *is complete and* F *is separable,* $a\in$ *ext* X *if and only if* ε_a *is the only one element of* $M_\sigma(X)$ *having* a *as its barycenter.*

§ 4. Proof of Theorem 3.

$(b) \Rightarrow (c)$. This is trivial.

$(c) \Rightarrow (a)$. This follows from Corollary 1 of Theorem 2, since $M(K)=M_s(K)$ for any compact K.

$(a) \Rightarrow (b)$. Every element of $M(K)$ having a barycenter in F, we can define the map $r: M(K) \rightarrow F$ by $r(\mu)=$barycenter of μ. Since it is proved in [8] (Proposition 1.2, section 1) that a point x lies in clconv K if and only if there exists $\mu \in M(K)$ having x as its barycenter, the image $r(M(K))$ coincides with clconv K. Hence we have only to show that the map r is continuous on $M(K)$ with the weak topology, since $M(K)$ with the weak topology is compact. Take a net $\mu_\alpha \rightarrow \mu$ in $M(K)$. For any equicontinuous set $\mathscr{H}$ of F^*, we put $\mathscr{H}_K=\{f_{|K}: f \in \mathscr{H}\}$. Then $\mathscr{H}_K$ is a uniformly bounded and equicontinuous subset of $C(K)$, the set of all real-valued continuous functions on K. When $C(K)$ is topologized by the sup norm topology, $\mathscr{H}_K$ is a totally bounded subset of $C(K)$ by Arzelà's theorem (Theorem 6.7, chapter 4 in [2]). Hence, putting $x_\alpha=r(\mu_\alpha)$ and $x=r(\mu)$, we have

$$\sup\{|f(x_\alpha)-f(x)|: f \in \mathscr{H}\}=\sup\{|\mu_\alpha(f)-\mu(f)|: f \in \mathscr{H}_K\} \rightarrow 0,$$

since $M(K)$ is equicontinuous on $C(K)$ with the sup norm topology. This proves that $x_\alpha \rightarrow x$ with respect to the locally convex topology. Hence the map r is continuous. Thus the proof is completed.

COROLLARY 7. *Let K be a compact subset of a complete locally convex Hausdorff space F. Then clconv K is compact.*

REMARK. In [5], Khurana has obtained some results concerning the connection between the weak compactness of X and barycenters of elements of $M(X)$.

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