J. Math. Soc. Japan
Vol. 29, No. 1, 1977

On an integer associated with an algebraic group

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(Received Feb. 16, 1976)

§1. Introduction.

Let G be a connected (real or complex) Lie group with Lie algebra $\mathfrak{g}$. In general, the exponential map $\exp:\mathfrak{g}\rightarrow G$ is not onto. But recently Markus in [3] and Lai in [2] pointed out that for some algebraic Lie groups G we can associate a natural number q such that for any g in G, the q-th power g^{q} of g lies in $\exp\mathfrak{g}$. In this note, we shall consider an algebraic group theoretic version of these results.

Throughout the paper, k will denote an algebraically closed field (of characteristic 0 or prime). By an algebraic group, we shall mean a linear algebraic group, i.e. a (Zariski) closed subgroup of $GL(m, k)$. The purpose of this note is to prove the following theorem.

THEOREM. *For a given algebraic group G over an algebraically closed field k, we can associate a natural number q such that for any g in G there exists a connected abelian subgroup of G containing g^{q}.*

As a general reference we will presume that the reader is familiar with Borel [1]. The author is pleased to acknowledge his gratitude to F. Grosshans for valuable suggestions and discussions during the preparation of the present paper.

§2. char $k=p>0$.

Let G be an algebraic group in $GL(m, k)$. Let g be in G, and let $g=xy=yx$, where x is semisimple and y unipotent, be the Jordan decomposition of g. Let r be the smallest natural number with $p^{r}\geqq m$. We set $q=p^{r}$, and we have $y^{q}=1$, see p. 142 in Borel [1], and so $g^{q}=x^{q}$. Since g^{q} is semisimple, it is contained in some maximal torus, which is connected and abelian.

§3. char $k=0$.

Let G_{0} denote the connected component of G containing the identity 1. Then G_{0} is of finite index, say i, in G, and for every $g\in G$ the i-th power g^{i} of g is contained in G_{0}. Hence it suffices to consider connected groups G.

Let G be a connected algebraic group and let B be a Borel subgroup of G. Every element of G is conjugate to some element of B. Since B is solvable, the proof reduces to the solvable case.

After this, let us suppose that G is a closed connected solvable subgroup of $GL(m, k)$. Let N be the unipotent radical and H a maximal torus of G. Then we have a semidirect product decomposition (Levi decomposition)

$$G=HN, \qquad H\cap N=1.$$

Let $\mathfrak{n}$ denote the Lie algebra of N. Since N is a closed normal subgroup, we have $Ad(g)\mathfrak{n}=\mathfrak{n}$ for $g\in G$. For $x\in H$, let $f(x)$ denote the restriction of $Ad(x)$ to $\mathfrak{n}$. Then

$$H\ni x\longmapsto f(x)\in GL(\mathfrak{n})$$

is a morphism and the image $f(H)$ is a torus. Hence we can find distinct characters $\chi_1, \cdots, \chi_l$ of H and a direct sum decomposition $\mathfrak{n}=\mathfrak{n}_1\oplus\cdots\oplus\mathfrak{n}_l$ such that

$$(f(x)-\chi_j(x))\mathfrak{n}_j=0 \qquad \text{for} \quad x\in H \qquad j=1, \cdots, l.$$

For any set of integers $J=\{j_1, \cdots, j_s\}$ with

$$1\leqq j_1<j_2<\cdots<j_s\leqq l,$$

we put $H(J)=\{x\in H;\ \chi_{j_1}(x)=\cdots=\chi_{j_s}(x)=1\}$. Then $H(J)$ is a closed subgroup of H. Let $q(J)$ denote the index of the identity component $H(J)_0$ in $H(J)$, and q the least common multiple of all $q(J)$. We shall prove that this q satisfies the requirement.

Let g be in G. We can find a semisimple x and a unipotent y such that $g=xy=yx$. There exists $z\in N$ with $zxz^{-1}\in H$, and $zgz^{-1}=(zxz^{-1})(zyz^{-1})$, where zxz^{-1} is semisimple, zyz^{-1} is unipotent, and they commute with each other. Therefore, without changing the notations, let us suopose that $x\in H$.

Let $C(x)$ denote the centralizer of x in G. Then $C(x)$ is connected and $C(x)=HY$, $Y\subset N$. Since $y\in N$ and $xy=yx$, we have $y\in Y$.

Let u be in N. Then $u-1$ is nilpotent and the power series $\log u=\sum_{j=1}(-1)^{j+1}(u-1)^j/j$ is a polynomial in u, and the Lie algebra $\mathfrak{n}$ is given by $\mathfrak{n}=\log N=\{\log u\,;\ u\in N\}$. Since $\log u$ is nilpotent, $\exp(\log u)$ is a polynomial in $\log u$ and reduces to $u: \exp(\log u)=u$. We have that $\exp(k\log u)$ is a closed connected one-dimensional subgroup of N containing u. Since $xy=yx$, we have that $x\cdot\log y=\log y\cdot x$ and $f(x)\log y=\log y$. Let us put $J=\{j\,;\ \chi_j(x)=1\}$. Then the Lie algebra of Y is given by $\mathfrak{y}=\sum_{j\in J}\mathfrak{n}_j$. We have that $\log y\in\mathfrak{y}$ and $Y_1=\exp(k\log y)\subset Y$.

On the other hand, since $f(H(J))=id.$ on $\mathfrak{y}$, we have that $H(J)$ and Y are elementwise commutative. Hence $H(J)\cdot Y_1$ is an abelian group. Since Y_1 is connected and $H(J)_0$ is the identity component of $H(J)$, we have that $H(J)_0 \cdot Y_1$ is connected, and $g^{q(J)} = x^{q(J)} y^{q(J)}$ is in $H(J)_0 \cdot Y_1$.

§ 4. An alternate proof for reductive groups.

Let G be a reductive algebraic group. Suppose that the root system R of G has an indecomposable decomposition $R = R_1 \cup \cdots \cup R_p$ with each R_j being one of the following forms

A_n	p arbitrary
B_n, C_n, D_n	$p \neq 2$
G_2, F_4, E_6, E_7	$p \neq 2, 3$
E_8	$p \neq 2, 3, 5$.

In this case the number of conjugacy classes of centralizers of elements of G is finite, see p. 107 in Steinberg [4]. This means that there exist closed subgroups $C_1, \cdots, C_t$ such that for any g in G, the centralizer $C(g)$ can be written as $C(g) = zC_iz^{-1}$ for some $z \in G$ and some $i = 1, \cdots, t$. Let Z_i denote the center of C_i. Then Z_i is closed. Let q_i be the index of the identity component $(Z_i)_0$ in Z_i. Since g is in the center of $C(g)$, which coincides with zZ_iz^{-1}, we have $g^{q_i} \in z(Z_i)_0 z^{-1}$, which is connected and abelian. Hence we can take the least common multiple of $q_1, \cdots, q_t$ as q.

Added May 27, 1976.

The author learned from D. A. Kajdan that the theorem he used in 4 was proved by G. Lustig recently only assuming the field is algebraically closed.

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