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On compact complex affine manifolds

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Introduction.

In this paper we study compact complex affine manifolds. Let $A(n, \boldsymbol{C})$ be the group of the affine transformations on $\boldsymbol{C}^n$ and let Γ be a subgroup of $A(n, \boldsymbol{C})$ such that 1) Γ acts on $\boldsymbol{C}^n$ properly discontinuously and freely 2) $\boldsymbol{C}^n/\Gamma$ is compact. A compact complex manifold $\boldsymbol{C}^n/\Gamma$ is called a compact complex affine manifold. For $n=2$, such manifolds have been classified by Vitter [**6**], Fillmore and Scheuneman [**2**] and Suwa [**5**]. The purpose of this paper is to study the complex manifold $\boldsymbol{C}^n/\Gamma$ under certain conditions. Put

$$N(n, \boldsymbol{C})=\left\{A\in A(n, \boldsymbol{C}) \,\middle|\, A=\begin{pmatrix} a & \alpha \\ 0 & 1\end{pmatrix},\ a=\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1\end{pmatrix},\ \alpha\in \boldsymbol{C}^n\right\}.$$

In section 1 we show that if Γ is contained in $N(n, \boldsymbol{C})$, then every non-zero holomorphic vector field on $\boldsymbol{C}^n/\Gamma$ has no zero point and the Lie algebra $\mathfrak{a}$ of all holomorphic vector fields on $\boldsymbol{C}^n/\Gamma$ is solvable and of dimension $\leqq n$. In section 2 we study the case when Γ is contained in $N(n, \boldsymbol{C})$ and the Lie algebra $\mathfrak{a}$ is of n-dimension. In this case we show that there exist a simply connected complex nilpotent Lie subgroup G of $N(n, \boldsymbol{C})$ which contains Γ and a biholomorphic map $\phi: \boldsymbol{C}^n \to G$ such that $\phi(\gamma(z))=\gamma\phi(z)$ for any $\gamma\in\Gamma$ and any $z\in \boldsymbol{C}^n$. In particular, we see that there is a biholomorphic map $\bar{\phi}: \boldsymbol{C}^n/\Gamma \to \Gamma\backslash G$. In section 3 we show that if Γ is contained in $N(n, \boldsymbol{C})$ and $\boldsymbol{C}^n/\Gamma$ has a Kähler metric, then $\boldsymbol{C}^n/\Gamma$ is biholomorphic to a complex torus. In section 4 we consider the case when Γ is an abelian subgroup of $A(n, \boldsymbol{C})$ and prove that $\boldsymbol{C}^n/\Gamma$ is biholomorphic to a complex torus.

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§ 1. Preliminaries.

Let $A(n, \boldsymbol{C})$ be the group of all affine transformations on $\boldsymbol{C}^n$. The group $A(n, \boldsymbol{C})$ is represented by the group of all matrices of the form $A=\begin{pmatrix} a & \alpha \\ 0 & 1\end{pmatrix}$ where $a=(a_{ij})\in GL(n, \boldsymbol{C})$ and $\alpha=(\alpha_i)\in \boldsymbol{C}^n$ is a column vector. Let $N(n, \boldsymbol{C})$

denote the subgroup of all unipotent elements:

$$N(n, \boldsymbol{C})=\left\{A\in A(n, \boldsymbol{C})\,\middle|\,A=\begin{pmatrix} a & \alpha \\ 0 & 1\end{pmatrix},\ a=\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1\end{pmatrix}\right\}.$$

Let Γ be a subgroup of $N(n, \boldsymbol{C})$ such that 1) the action of Γ on $\boldsymbol{C}^n$ is properly discontinuous 2) $\boldsymbol{C}^n/\Gamma$ is compact. Since $N(n, \boldsymbol{C})$ has no torsion, Γ acts freely on $\boldsymbol{C}^n$, so that $\boldsymbol{C}^n/\Gamma$ is a compact complex manifold.

Note that Γ is finitely generated, since the fundamental group $\pi_1(\boldsymbol{C}^n/\Gamma)$ of $\boldsymbol{C}^n/\Gamma$ is isomorphic to Γ and $\boldsymbol{C}^n/\Gamma$ is compact.

Let M be a connected compact complex manifold and Aut(M) denote the group of all holomorphic automorphisms of M. Then Aut(M) is a complex Lie group and the Lie algebra $\mathfrak{a}$ of Aut(M) can be identified with the Lie algebra of all holomorphic vector fields on M.

PROPOSITION 1.1. *Every non-zero holomorphic vector field on* $\boldsymbol{C}^n/\Gamma$ *is non-vanishing.*

PROOF. Let $\pi: \boldsymbol{C}^n \rightarrow \boldsymbol{C}^n/\Gamma$ be the canonical map. Note that π is the covering map. Take a non-zero holomorphic vector field X on $\boldsymbol{C}^n/\Gamma$. Let Y be the lift of X on $\boldsymbol{C}^n$, that is, the holomorphic vector field Y on $\boldsymbol{C}^n$ such that $\pi_* Y=X$. Then we have $\gamma_* Y=Y$ for any $\gamma\in\Gamma$. Conversely a holomorphic vector field Y on $\boldsymbol{C}^n$ which satisfies $\gamma_* Y=Y$ for any $\gamma\in\Gamma$ defines a holomorphic vector field X on $\boldsymbol{C}^n/\Gamma$ such that $\pi_* Y=X$.

Let $(z_1, \cdots, z_n)$ be the canonical coordinates on $\boldsymbol{C}^n$. The vector field Y can be written uniquely in the form

$$Y=\sum_{j=1}^{n} f_j \frac{\partial}{\partial z_j}$$

where f_j $(j=1, \cdots, n)$ are holomorphic functions on $\boldsymbol{C}^n$. We note that

$$\gamma_* \frac{\partial}{\partial z_j}=\sum_{k<j} a_{kj}(\gamma)\frac{\partial}{\partial z_k}+\frac{\partial}{\partial z_j}, \qquad j=1, \cdots, n \tag{1.1}$$

for $\gamma\in\Gamma$, where

$$\gamma=\begin{bmatrix} 1 & a_{12}(\gamma) & \cdots\cdots & a_{1n}(\gamma) & \alpha_1(\gamma) \\ & 1 & a_{23}(\gamma)\ \cdots & a_{2n}(\gamma) & \alpha_2(\gamma) \\ & & \ddots & \vdots & \vdots \\ & & 1 & a_{n-1n}(\gamma) & \alpha_{n-1}(\gamma) \\ & \mathbf{0} & & 1 & \alpha_n(\gamma) \\ & & & & 1 \end{bmatrix}.$$

Since $\gamma_* Y=Y$, we have

$$\sum_{j=1}^{n} f_j \gamma_* \frac{\partial}{\partial z_j}=\sum_{j=1}^{n}(f_j\circ\gamma)\frac{\partial}{\partial z_j} \tag{1.2}$$

where $f_j \circ \gamma$ denotes a holomorphic function on $\boldsymbol{C}^n$ defined by $(f_j \circ \gamma)(z) = f_j(\gamma(z))$. By (1.1) and (1.2), $\gamma_* Y = Y$ is equivalent to

$$(1.3) \qquad \begin{cases} f_1 \circ \gamma = f_1 + a_{12}(\gamma) f_2 + \cdots + a_{1n}(\gamma) f_n \\ f_2 \circ \gamma = f_2 + a_{23}(\gamma) f_3 + \cdots + a_{2n}(\gamma) f_n \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ f_{n-1} \circ \gamma = f_{n-1} + a_{n-1n}(\gamma) f_n \\ f_n \circ \gamma = f_n . \end{cases}$$

Since f_n is a holomorphic function on $\boldsymbol{C}^n$ and $f_n \circ \gamma = f_n$ for $\gamma \in \Gamma$, f_n defines a holomorphic function on a compact complex manifold $\boldsymbol{C}^n/\Gamma$, so that f_n is a constant function. If the constant f_n is not zero, the vector field $Y = \sum_{j=1}^{n-1} f_j \frac{\partial}{\partial z_j} + f_n \frac{\partial}{\partial z_n}$ has no zero point, so that X has no zero point. If the constant f_n is zero, we have $f_{n-1} \circ \gamma = f_{n-1}$ for any $\gamma \in \Gamma$, so that f_{n-1} is a constant function. If the constant f_{n-1} is not zero, the vector field $Y = \sum_{j=1}^{n-2} f_j \frac{\partial}{\partial z_j} + f_{n-1} \frac{\partial}{\partial z_{n-1}}$ has no zero point. Similarly if $f_n = f_{n-1} = \cdots = f_{j_0+1} = 0$ and $f_{j_0} \neq 0$, $f_{j_0} \circ \gamma = f_{j_0}$ for any $\gamma \in \Gamma$ by (1.3), so that f_{j_0} is a constant function and Y can be written as

$$Y = \sum_{j<j_0} f_j \frac{\partial}{\partial z_j} + f_{j_0} \frac{\partial}{\partial z_{j_0}}$$

where f_{j_0} is a non-zero constant. Therefore Y has no zero point, so that X has no zero point. q. e. d.

REMARK 1. The proof shows that if $Y = \sum_{j=1}^{j_0} f_j \frac{\partial}{\partial z_j}$ with $f_{j_0} \neq 0$, then f_{j_0} is constant.

COROLLARY 1.

$$1 \leqq \dim_C \operatorname{Aut}(\boldsymbol{C}^n/\Gamma) \leqq n .$$

PROOF. Since the holomorphic vector field $\frac{\partial}{\partial z_1}$ on $\boldsymbol{C}^n$ satisfies $\gamma_* \frac{\partial}{\partial z_1} = \frac{\partial}{\partial z_1}$ for any $\gamma \in \Gamma$, $1 \leqq \dim_C \mathfrak{a} = \dim_C \operatorname{Aut}(\boldsymbol{C}^n/\Gamma)$. Let $T_x(\boldsymbol{C}^n/\Gamma)$ denote the holomorphic tangent space at $x \in \boldsymbol{C}^n/\Gamma$, and consider the linear map $E_x : \mathfrak{a} \to T_x(\boldsymbol{C}^n/\Gamma)$ for $x \in \boldsymbol{C}^n/\Gamma$ defined by $E_x(X) = X_x$ for $X \in \mathfrak{a}$. Proposition 1.1 shows E_x is injective. Hence, $\dim_C \mathfrak{a} \leqq \dim_C T_x(\boldsymbol{C}^n/\Gamma) = n$. q. e. d.

A complex manifold M of dimension n is called *parallelisable* if there exist n holomorphic vector fields on M which are linearly independent at every point of M.

COROLLARY 2. *A compact complex manifold $\boldsymbol{C}^n/\Gamma$ is parallelisable if and only if* $\dim_C \operatorname{Aut}(\boldsymbol{C}^n/\Gamma) = n$.

PROOF. Obvious from the proof of Corollary 1.

PROPOSITION 1.2. *The Lie algebra* $\mathfrak{a}$ *of all holomorphic vector fields on* C^n/Γ *is solvable.*

PROOF. We identify a holomorphic vector field on C^n/Γ with the corresponding vector field on C^n. Let $(z_1, \cdots, z_n)$ be the canonical coordinates on C^n. Define *the length* $l(X)$ *of a holomorphic vector field* X on C^n/Γ by

$$l(X)=\begin{cases} \operatorname{Max}\left\{j \mid X=\sum_{j=1}^{n} f_j \frac{\partial}{\partial z_j}, f_j \neq 0\right\} & \text{for } X \neq 0 \\ 0 & \text{for } X=0 . \end{cases}$$

Let B be a subset of $\mathfrak{a}$. Define *the length* $l(B)$ *of* B by $l(B)=\operatorname{Max}\{l(X) \mid X \in B\}$. Let $[B, B]$ denote the subset defined by

$$[B, B]=\{\sum_{\text{finite sum}} a_{kl}[X_k, X_l] \mid a_{kl} \in C \text{ and } X_k, X_l \in B\} .$$

We claim that, for a subset $B \neq (0)$,

$$l([B, B]) \leqq l(B)-1 .$$

Take two elements $0 \neq X$, Y of B. Put $j_0=l(X)$ and $i_0=l(Y)$. Then

$$X=\sum_{j<j_0} f_j \frac{\partial}{\partial z_j}+f_{j_0} \frac{\partial}{\partial z_{j_0}}$$

$$Y=\sum_{i<i_0} g_i \frac{\partial}{\partial z_i}+g_{i_0} \frac{\partial}{\partial z_{i_0}}$$

where f_{j_0} and g_{i_0} are non-zero constants.

$$[X, Y]=\sum_l\left(\sum_k\left(f_k \frac{\partial g_l}{\partial z_k}-g_k \frac{\partial f_l}{\partial z_k}\right)\right) \frac{\partial}{\partial z_l}$$

$$=\sum_{l<\operatorname{Max}\{i_0, j_0\}}\left(\sum_k\left(f_k \frac{\partial g_l}{\partial z_k}-g_k \frac{\partial f_l}{\partial z_k}\right)\right) \frac{\partial}{\partial z_l} .$$

Thus $l([X, Y]) \leqq \operatorname{Max}\{l(X), l(Y)\}-1$ for $X, Y \in B$. Obviously $l(X+Y) \leqq \operatorname{Max}\{l(X), l(Y)\}$. Therefore $l([B, B]) \leqq l(B)-1$.

Define $D_k(\mathfrak{a})$ inductively by $D_0(\mathfrak{a})=\mathfrak{a}$, $D_k(\mathfrak{a})=[D_{k-1}(\mathfrak{a}), D_{k-1}(\mathfrak{a})]$. Then we have $l(D_k(\mathfrak{a})) \leqq l(D_{k-1}(\mathfrak{a}))-1 \leqq \cdots \leqq l(\mathfrak{a})-k \leqq n-k$ for $k=0, 1, 2, \cdots$. Hence, $l(D_n(\mathfrak{a}))=0$, that is, $D_n(\mathfrak{a})=(0)$. q. e. d.

By the same way, we can study holomorphic p-forms on C^n/Γ. Let $H^{p,0}(C^n/\Gamma)$ be the vector space of all holomorphic p-forms on C^n/Γ. Let $h^{p,0}$ denote the dimension $\dim_C H^{p,0}(C^n/\Gamma)$.

PROPOSITION 1.3. *Let* θ *be a holomorphic p-form on* C^n/Γ. *If* θ *is non-zero,* θ *has no zero point, that is,* $\theta_x \neq 0$ *for any* $x \in C^n/\Gamma$.

PROOF. Let $\pi: C^n \rightarrow C^n/\Gamma$ be the canonical map. Put $\eta=\pi^* \theta$. Then $\gamma^* \eta=\eta$ for any $\gamma \in \Gamma$. Let $(z_1, \cdots, z_n)$ be a canonical coordinate on C^n. η can be written uniquely as $\eta=\sum_{J} f_J dz_J$ where $J=(j_1, \cdots, j_p)$ $(1\leqq j_1< \cdots <j_p\leqq n)$, $dz_J=dz_{j_1}\wedge \cdots \wedge dz_{j_p}$ and $f_J=f_{j_1\cdots j_p}$ are holomorphic function on $\boldsymbol{C}^n$.

Define C_p by $C_p=\{J=(j_1, \cdots, j_p)\in \boldsymbol{N}^p \mid 1\leqq j_1< \cdots <j_p\leqq n\}$. Let us introduce a linear order $<$ on C_p by $I<J$ for $I, J\in C_p$, $I\neq J$, $I=(i_1, \cdots, i_p)$, $J=(j_1, \cdots, j_p)$ if $i_1=j_1, \cdots, i_{k-1}=j_{k-1}$, $i_k<j_k$ for some k $(1\leqq k\leqq p)$.

We have $\gamma^* dz_J=dz_J+\sum_{I\lneqq J} P_{JI}(\gamma)dz_I$ for $\gamma\in\Gamma$, where $P_{JI}(\gamma)$ is a polynomial of $a_{ij}(\gamma)$. We then have $\gamma^*\eta=\eta$ if and only if

$$f_K\circ\gamma+\sum_{J\lneqq K} P_{JK}(f_J\circ\gamma)=f_K \qquad \text{for all} \quad K\in C_p\,. \tag{1.4}$$

In particular, $f_{I_p}\circ\gamma=f_{I_p}$ for any $\gamma\in\Gamma$, where $I_p=(1, 2, \cdots, p)\in C_p$, so that f_{I_p} is a constant function. If $f_J=0$ for any $J\lneqq J_0$, then $f_{J_0}\circ\gamma=f_{J_0}$ for any $\gamma\in\Gamma$, so that f_{J_0} is constant. Thus, for a non-zero form η, there is a $J_0\in C_p$ such that

$$\eta=\sum_{J\gneqq J_0} f_J dz_J+f_{J_0}dz_{J_0}$$

where f_{J_0} is a non-zero constant. Hence, θ has no zero point. q. e. d.

COROLLARY. $1\leqq h^{p,0}\leqq\binom{n}{p}$ *for* $p=0, 1, \cdots, n$. *In particular*, $h^{n,0}=1$.

PROOF. Consider the largest element J_p of C_p, that is, $J_p=(n-p+1, \cdots, n-1, n)$. Then $\gamma^* dz_{J_p}=dz_{J_p}$ for any $\gamma\in\Gamma$. Hence, $1\leqq h^{p,0}$.

Take a point $x\in\boldsymbol{C}^n/\Gamma$. Let $\wedge^p T_x^*(\boldsymbol{C}^n/\Gamma)$ be the p-th exterior product of holomorphic cotangent bundle $T_x^*(\boldsymbol{C}^n/\Gamma)$ at x. Define a linear map ψ: $H^{p,0}(\boldsymbol{C}^n/\Gamma)\to\wedge^p T_x^*(\boldsymbol{C}^n/\Gamma)$ by $\psi(\theta)=\theta_x$ for $\theta\in H^{p,0}(\boldsymbol{C}^n/\Gamma)$. Then ψ is injective by Proposition 1.3, so that $h^{p,0}\leqq\binom{n}{p}$. q. e. d.

§ 2. The case when $\dim \mathrm{Aut}(\boldsymbol{C}^n/\Gamma)=n$.

In this section we prove the following theorem.

THEOREM 2.1. *Let Γ be a subgroup of $N(n, \boldsymbol{C})$ acting freely and properly discontinuously on $\boldsymbol{C}^n$ and such that $\boldsymbol{C}^n/\Gamma$ is compact. If* $\dim \mathrm{Aut}(\boldsymbol{C}^n/\Gamma)=n$, *there exist a simply connected complex nilpotent Lie subgroup G of $N(n, \boldsymbol{C})$ which contains Γ and a biholomorphic map $\phi:\boldsymbol{C}^n\to G$ such that $\phi(\gamma(z))=\gamma\cdot\phi(z)$ for any $\gamma\in\Gamma$ and $z\in\boldsymbol{C}^n$. In particular, ϕ induces a biholomorphic map $\bar\phi:\boldsymbol{C}^n/\Gamma\to\Gamma\backslash G$.*

Let $(z_1, \cdots, z_n)$ be the canonical coordinates on $\boldsymbol{C}^n$. We identify a holomorphic vector field on $\boldsymbol{C}^n/\Gamma$ with the corresponding vector field on $\boldsymbol{C}^n$ as in section 1. Then every holomorphic vector field X on $\boldsymbol{C}^n/\Gamma$ can be written as

$$X=\sum_{j<k} f_j\frac{\partial}{\partial z_j}+f_k\frac{\partial}{\partial z_k}$$

where f_k is a non-zero constant and f_j is holomorphic function on $\boldsymbol{C}^n$. Let $a_{ij}(\gamma)$ $(1 \leqq i < j \leqq n)$ and $\alpha_j(\gamma)$ $(1 \leqq j \leqq n)$ denote the matrix-components of $\gamma \in \Gamma$.

LEMMA 2.2. *Let $g_l(z)$ be polynomial functions of $z_1, \cdots, z_n$ and $P_l(\gamma)$ be polynomial functions of $a_{ij}(\gamma)$ and $\alpha_j(\gamma)$ $(1 \leqq l \leqq p)$. If f is a holomorphic function on $\boldsymbol{C}^n$ which satisfies the relations*

$$f(\gamma(z)) = f(z) + \sum_{l=1}^{p} P_l(\gamma) g_l(z) \tag{2.1}$$

for any $\gamma \in \Gamma$, then f is a polynomial function of $z_1, \cdots, z_n$.

PROOF. We prove our lemma by induction on the number of variables $z_1, \cdots, z_n$. Consider the case when f and g_l are functions depending only on z_n. We denote by m_l the degree of the polynomial g_l. Put $m = \mathrm{Max}\,\{m_j \mid j=1, \cdots, p\}$. Then we get $\frac{d^{m+1}f}{dz_n^{m+1}}(\gamma(z)) = \frac{d^{m+1}f}{dz_n^{m+1}}(z)$ for any $\gamma \in \Gamma$, since

$$\gamma(z) = \begin{pmatrix} z_1 + a_{12}(\gamma) z_2 + \cdots + a_{1n}(\gamma) z_n + \alpha_1(\gamma) \\ \cdots\cdots\cdots\cdots \\ z_k + a_{kk+1}(\gamma) z_k + \cdots + a_{kn}(\gamma) z_n + \alpha_k(\gamma) \\ \cdots\cdots\cdots\cdots \\ z_n + \alpha_n(\gamma) \end{pmatrix}. \tag{2.2}$$

Since $\boldsymbol{C}^n/\Gamma$ is a connected compact complex manifold, $\frac{d^{m+1}f}{dz_n^{m+1}}$ is constant and hence f is a polynomial function of z_n. We may assume that if f, g_l are functions depending only on $z_2, \cdots, z_n$ and f satisfies the relation (2.1) for some $P_l(\gamma)$ then f is a polynomial function of $z_2, \cdots, z_n$. Let m_l^1 denote the degree of the polynomial g_l with respect to z_1 and $m_1 = \mathrm{Max}\,\{m_j^1 \mid j=1, \cdots, p\}$. By (2.1) and (2.2), we get

$$\frac{\partial^{m_1+1} f}{\partial z_1^{m_1+1}}(\gamma(z)) = \frac{\partial^{m_1+1} f}{\partial z_1^{m_1+1}}(z)$$

for any $\gamma \in \Gamma$. Therefore

$$f(z) = a_{m_1+1} z_1^{m_1+1} + \sum_{j=0}^{m_1} a_j(z_2, \cdots, z_n) z_1^j \tag{2.3}$$

where $a_{m_1+1} \in \boldsymbol{C}$ and $a_j(z_2, \cdots, z_n)$ are holomorphic functions depending only on $z_2, \cdots, z_n$. By (2.1), (2.2) and (2.3), we see that $a_{m_1}(z_2, \cdots, z_n)$ satisfies the relation (2.1) for some polynomial functions $g_{m_1 l}(z)$ of $z_2, \cdots, z_n$ and some polynomial functions $P_{m_1 l}(\gamma)$. Thus $a_{m_1}(z_2, \cdots, z_n)$ is polynomial function. Considering the coefficient of $z_1^{m_1-1}$ of (2.1) and noting that $a_{m_1}(z_2, \cdots, z_n)$ is a polynomial function, we see that $a_{m_1-1}(z_2, \cdots, z_n)$ satisfies the relation (2.1) and hence $a_{m_1-1}(z_2, \cdots, z_n)$ is a polynomial function. By the same way, we see that $a_j(z_2, \cdots, z_n)$ are polynomial functions of $z_2, \cdots, z_n$ for $j=0, 1, \cdots, m_1$. q. e. d.

COROLLARY OF LEMMA 2.2. *Let $X = \sum_{j<k} f_j \frac{\partial}{\partial z_j} + \frac{\partial}{\partial z_k}$ be a holomorphic vec-*

tor field on C^n/Γ. *Then* f_j *are polynomial functions of* $z_1, \cdots, z_n$.

PROOF. By (1.3) in section 1, we can see that f_j satisfies the relation (2.1).

LEMMA 2.3. *Suppose that* $\dim \operatorname{Aut}(C^n/\Gamma)=n$. *Then there are holomorphic vector fields* $X_1, \cdots, X_n$ *on* C^n/Γ *such that*

$$X_j=\sum_{i<j} f_{ij}\frac{\partial}{\partial z_i}+\frac{\partial}{\partial z_j}, \tag{2.4}$$

where f_{ij} *is a polynomial function such that* $f_{ij}(0)=0$. *Moreover the matrix component* $a_{ij}(\gamma)$ *of* $\gamma\in\Gamma$ *satisfies*

$$a_{ij}(\gamma)=f_{ij}(\alpha_1(\gamma), \cdots, \alpha_n(\gamma)). \tag{2.5}$$

PROOF. It is easy to see that there are holomorphic vector fields $Y_1, \cdots, Y_n$ on C^n/Γ such that $Y_j=\sum_{i<j} g_{ij}\frac{\partial}{\partial z_i}+\frac{\partial}{\partial z_j}$ for some holomorphic functions g_{ij}. Put $X_j=Y_j-\sum_{k<j} g_{kj}(0)X_k$ for $j=1, \cdots, n$. Then, $X_1, \cdots, X_n$ satisfy the conditions of Lemma 2.3 by Corollary of Lemma 2.2. By (1.3) in section 1, f_{ij} satisfies the relations

$$f_{ij}\circ\gamma=f_{ij}+\sum_{i<k<j} a_{ik}(\gamma)f_{kj}+a_{ij}(\gamma). \tag{2.6}$$

By (2.2), we see that $a_{ij}(\gamma)=f_{ij}(\gamma(0))=f_{ij}(\alpha_1(\gamma), \cdots, \alpha_n(\gamma))$. q. e. d.

LEMMA 2.4. *Suppose that* Γ *is a subgroup of* $A(n, C)$ *acting on* C^n *freely and properly discontinuously and such that* C^n/Γ *is compact. Then the set of translational parts* α *of elements* $\begin{pmatrix} A & \alpha \\ 0 & 1\end{pmatrix}$ *of* Γ *contains a basis for* C^n *regarded as a real vector space.*

PROOF. See [1], [2].

LEMMA 2.5. *Let* G *be a subset of* $N(n, C)$ *defined by*

$$G=\left\{\left(\begin{array}{cccc|c} 1 & & & & z_1 \\ & \ddots & & f_{ij}(z) & \vdots \\ \mathbf{0} & & \ddots & 1 & z_n \\ \hline 0 & \cdots & \cdots & 0 & 1 \end{array}\right) \,\middle|\, z_j\in C,\ j=1, \cdots, n\right\}.$$

Then G *is a simply connected complex nilpotent Lie subgroup of* $N(n, C)$ *and contains* Γ.

PROOF. By Lemma 2.4 the set of translational parts of elements $\gamma\in\Gamma$ contains a basis for C^n regarded as a real vector space, a fortiori, it contains a basis for C^n as a complex vector space. In particular, we see that if f is a polynomial function on C^n such that $f(\alpha_1(\gamma), \cdots, \alpha_n(\gamma))=0$ for any $\gamma\in\Gamma$ then f is identically zero. We prove that G is a subgroup of $N(n, C)$. We denote $(z_1(g), \cdots, z_n(g))$ by $z(g)$. For elements

$$g=\left(\begin{array}{ccc|c} 1 & & f_{ij}(z(g)) & z_1(g) \\ & \ddots & & \cdots \\ \mathbf{0} & & 1 & z_n(g) \\ \hline 0 & \cdots\cdots\cdots & 0 & 1 \end{array}\right), \quad h=\left(\begin{array}{ccc|c} 1 & & f_{ij}(z(h)) & z_1(h) \\ & \ddots & & \vdots \\ \mathbf{0} & & 1 & z_n(h) \\ \hline 0 & \cdots\cdots\cdots & 0 & 1 \end{array}\right)$$

of G, the components $a_{ij}(gh)$, $\alpha_l(gh)$ of $gh \in N(n, \boldsymbol{C})$ can be written as

$$a_{ij}(gh)=f_{ij}(z(h))+\sum_{i<k<j} f_{ik}(z(g))f_{kj}(z(h))+f_{ij}(z(g))$$

$$\alpha_l(gh)=z_l(h)+\sum_{l<k} f_{lk}(z(g))z_k(h)+z_l(g)\,.$$

Since Γ is a subgroup of $N(n, \boldsymbol{C})$, we have

$$\text{(2.7)} \qquad \begin{cases} f_{ij}(z(\gamma\delta))=f_{ij}(z(\delta))+\sum_{i<k<j} f_{ik}(z(\gamma))f_{kj}(z(\delta))+f_{ij}(z(\gamma)) \\ z_l(\gamma\delta)=z_l(\delta)+\sum_{l<k} f_{lk}(z(\gamma))z_k(\delta)+z_l(\gamma) \end{cases}$$

for $1 \leqq i<j \leqq n$, $1 \leqq l \leqq n$, and any $\gamma, \delta \in \Gamma$. Define $z_l(z, y)$ for $1 \leqq l \leqq n$, and $z, y \in \boldsymbol{C}^n$.

$$z_l(z, y)=y_l+\sum_{l<k} f_{lk}(z)y_k+z_l\,.$$

We also define polynomial functions $P_{ij}(z, y)$ on $\boldsymbol{C}^n \times \boldsymbol{C}^n$ for $1 \leqq i<j \leqq n$ by

$$\text{(2.8)} \qquad P_{ij}(z, y)=f_{ij}(z_1(z, y), \cdots, z_n(z, y))-f_{ij}(y) - \sum_{i<k<j} f_{ik}(z)f_{kj}(y)-f_{ij}(z)\,.$$

By (2.7), we have $P_{ij}(z(\gamma), z(\delta))=0$ for any $\gamma, \delta \in \Gamma$. For a fixed $\delta \in \Gamma$, $P_{ij}(z, z(\delta))$ is a polynomial function on $\boldsymbol{C}^n$ which vanishes on the set of translational parts of elements of Γ. Thus $P_{ij}(z, z(\delta))=0$ for any $z \in \boldsymbol{C}^n$ and $\delta \in \Gamma$. Now for a fixed $z \in \boldsymbol{C}^n$, we can see that $P_{ij}(z, y)=0$ for any $y \in \boldsymbol{C}^n$ by the same way. This implies that $gh \in G$. Similarly we can see that if $g \in G$ then $g^{-1} \in G$. Thus G is a subgroup of $N(n, \boldsymbol{C})$. The other claim in Lemma 2.5 is obvious. q. e. d.

PROOF OF THEOREM 2.1. We define a biholomorphic map $\phi: \boldsymbol{C}^n \to G$ by

$$\phi(z)=\left(\begin{array}{ccc|c} 1 & & f_{ij}(z) & z_1 \\ & \ddots & & \vdots \\ \mathbf{0} & & 1 & z_n \\ \hline 0 & \cdots\cdots\cdots & 0 & 1 \end{array}\right) \quad \text{for} \quad z \in \boldsymbol{C}^n\,.$$

Then we can see that

$$\text{(2.9)} \qquad \phi(g(z))=g\cdot\phi(z)$$

for any $g \in G$ and any $z \in \boldsymbol{C}^n$. Since Γ acts on $\boldsymbol{C}^n$ properly discontinuously and freely, it follows from (2.9) that the same is true for the action of Γ on G by left-translations. Therefore Γ is a discrete subgroup of G and ϕ induces a biholomorphic map $\bar{\phi}: \boldsymbol{C}^n/\Gamma \to \Gamma\backslash G$. This proves Theorem 2.1.

EXAMPLE. Let Γ be a subgroup of $N(3, \boldsymbol{C})$ defined by

$$\Gamma = \left\{ \begin{pmatrix} 1 & 0 & \alpha_2 & \alpha_1 \\ & 1 & 0 & \alpha_2 \\ & & 1 & \alpha_3 \\ \boldsymbol{0} & & & 1 \end{pmatrix} \middle| \begin{array}{c} \alpha_i \in \boldsymbol{Z} + \sqrt{-1}\,\boldsymbol{Z} \\ i = 1, 2, 3 \end{array} \right\}.$$

Then it is easy to see that 1) Γ acts on $\boldsymbol{C}^3$ properly discontinuously and freely 2) $\boldsymbol{C}^3/\Gamma$ is compact. Moreover $\dim \mathrm{Aut}_0(\boldsymbol{C}^3/\Gamma) = 3$ and $\mathrm{Aut}_0(\boldsymbol{C}^3/\Gamma)$ is not abelian. In fact, $\boldsymbol{C}^3/\Gamma$ is biholomorphic to G/Γ_1 where

$$G = \left\{ \begin{pmatrix} 1 & z_3 & z_1 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \middle| z_i \in \boldsymbol{C},\ i = 1, 2, 3 \right\}$$

and

$$\Gamma_1 = \left\{ \begin{pmatrix} 1 & \alpha_3 & \alpha_1 \\ 0 & 1 & \alpha_2 \\ 0 & 0 & 1 \end{pmatrix} \middle| \begin{array}{c} \alpha_i \in \boldsymbol{Z} + \sqrt{-1}\,\boldsymbol{Z} \\ i = 1, 2, 3 \end{array} \right\}.$$

§ 3. The case of Kähler manifolds.

In this section we prove the following theorem.

THEOREM 3.1. *Let Γ be a subgroup of $N(n, \boldsymbol{C})$ satisfying the conditions* 1) *and* 2) *in section* 1. *If a compact complex manifold $\boldsymbol{C}^n/\Gamma$ has a Kähler metric, $\boldsymbol{C}^n/\Gamma$ is biholomorphic to a complex torus.*

We need some lemmas to prove this theorem.

LEMMA 3.2. *All Chern classes $c_i(\boldsymbol{C}^n/\Gamma) \in H^{2i}(\boldsymbol{C}^n/\Gamma, \boldsymbol{R})$ $(i=1, \cdots, n)$ of a compact complex manifold $\boldsymbol{C}^n/\Gamma$ are zero.*

PROOF. Since $\boldsymbol{C}^n/\Gamma$ has an affine structure, $\boldsymbol{C}^n/\Gamma$ has an affine connection with zero curvature and zero torsion (cf. Matsushima [3], Vitter [6]). In particular, all Chern classes $c_i(\boldsymbol{C}^n/\Gamma)$ are zero. q. e. d.

LEMMA 3.3. *If M is a compact Kähler manifold with $c_1(M) = 0$, then we have*

(1) *The Lie algebra $\mathfrak{a}$ of all holomorphic vector fields on M is abelian.*

(2) *Every non-zero holomorphic vector field and every non-zero holomorphic* 1*-form are nonvanishing.*

(3) *Let $H^{1,0}(M)$ be the vector space of all holomorphic* 1*-forms on M. The bilinear function $B: H^{1,0}(M)\times\mathfrak{a}\to\boldsymbol{C}$ defined by $B(\theta, X)=\theta(X)$ is non-degenerate.*

PROOF. See [4] § 9 Theorem 3.

PROOF OF THEOREM 3.1. By Corollary of Proposition 1.3, there is a holomorphic 1-form $\theta_n=dz_n$ on $\boldsymbol{C}^n/\Gamma$. By (3) of Lemma 3.3, there is a holomorphic vector field $X\in\mathfrak{a}$ such that $\theta_n(X)\neq 0$ for $\theta_n\in H^{1,0}(\boldsymbol{C}^n/\Gamma)$. Since $\theta_n(X)=f_n$ for $X=\sum_{i=1}^{n} f_i\frac{\partial}{\partial z_i}$, there exists a holomorphic vector field $X_n\in\mathfrak{a}$ of the form $X_n=\sum_{i<n} f_{in}\frac{\partial}{\partial z_i}+\frac{\partial}{\partial z_n}$.

We now claim that if there exist holomorphic vector fields $X_{k+1}, \cdots, X_n$ on $\boldsymbol{C}^n/\Gamma$ and holomorphic 1-forms $\theta_{k+1}, \cdots, \theta_n$ on $\boldsymbol{C}^n/\Gamma$ such that

$$X_j=\sum_{i<j} f_{ij}\frac{\partial}{\partial z_i}+\frac{\partial}{\partial z_j} \qquad (j=k+1, \cdots, n),$$

$$\theta_l=dz_l+\sum_{i>l} g_{li}dz_i \qquad (l=k+1, \cdots, n)$$

and

$$\theta_l(X_j)=\delta_{lj},$$

then there are a holomorphic vector field X_k and a holomorphic 1-form θ_k on $\boldsymbol{C}^n/\Gamma$ such that

$$X_k=\sum_{i<k} f_{ik}\frac{\partial}{\partial z_i}+\frac{\partial}{\partial z_k},$$

$$\theta_k=dz_k+\sum_{i>k} g_{ki}dz_i$$

and $\theta_l(X_j)=\delta_{lj}$ $(l, j=k, \cdots, n)$.

By (1.3), f_{ij} satisfies the relation

$$\begin{pmatrix} f_{1k+1}\circ\gamma & \cdots\cdots & f_{1n}\circ\gamma \\ \vdots & & \vdots \\ f_{kk+1}\circ\gamma & \cdots\cdots & f_{kn}\circ\gamma \\ 1 & \cdots\cdots & f_{k+1n}\circ\gamma \\ & \ddots & \vdots \\ \mathbf{0} & 1 & f_{n-1n}\circ\gamma \\ & & 1 \end{pmatrix}=\begin{pmatrix} 1 & a_{12} & \cdots\cdots & a_{1n} \\ & 1 & & \vdots \\ & & \ddots & \vdots \\ & & & \vdots \\ \mathbf{0} & & 1 & a_{n-1n} \\ & & & 1 \end{pmatrix}\cdot\begin{pmatrix} f_{1k+1} & \cdots\cdots & f_{1n} \\ \vdots & & \vdots \\ f_{kk+1} & \cdots\cdots & f_{kn} \\ 1 & \cdots\cdots & f_{k+1n} \\ & \ddots & \vdots \\ \mathbf{0} & 1 & f_{n-1n} \\ & & 1 \end{pmatrix}. \tag{3.1}$$

Define holomorphic functions g_{ki} $(k<i\leqq n)$ on $\boldsymbol{C}^n$ by $g_{ki}=-(f_{kk+1}g_{k+1i}+\cdots+f_{ki-1}g_{i-1i}+f_{ki})$ and the holomorphic 1-form θ_k by $\theta_k=dz_k+\sum_{i>k} g_{ki}dz_i$. Then f_{ij}, g_{ij} satisfy the relation

$$\begin{pmatrix} 1 & f_{kk+1} & \cdots\cdots & f_{kn} \\ & \ddots & & \vdots \\ & & 1 & f_{n-1n} \\ \mathbf{0} & & & 1 \end{pmatrix} \begin{pmatrix} 1 & g_{kk+1} & \cdots\cdots & g_{kn} \\ & \ddots & & \vdots \\ & & 1 & g_{n-1n} \\ \mathbf{0} & & & 1 \end{pmatrix} = 1 . \tag{3.2}$$

By (3.1) and (3.2), we get

$$\begin{pmatrix} 1 & g_{kk+1} & \cdots\cdots & g_{kn} \\ & \ddots & & \vdots \\ & & 1 & g_{n-1n} \\ \mathbf{0} & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & g_{kk+1}\circ\gamma & \cdots & g_{kn}\circ\gamma \\ & \ddots & & \vdots \\ & & 1 & g_{n-1n}\circ\gamma \\ \mathbf{0} & & & 1 \end{pmatrix} \begin{pmatrix} 1 & a_{kk+1} & \cdots\cdots & a_{kn} \\ & \ddots & & \vdots \\ & & 1 & a_{n-1n} \\ \mathbf{0} & & & 1 \end{pmatrix} .$$

Thus θ_k is invariant by any $\gamma\in\Gamma$ and $\theta_k\in H^{1,0}(\boldsymbol{C}^n/\Gamma)$. By (3) of Lemma 3.3, there is a holomorphic vector field $X\in\mathfrak{a}$ such that $\theta_k(X)\neq 0$. By Remark 1 in the section 1, there are constants C_j $(j=k+1, \cdots, n)$ such that

$$X-\sum_{j=k+1}^{n} C_j X_j=\sum_{i=1}^{k} h_i\frac{\partial}{\partial z_i},$$

where h_k is a constant. Since $\theta_k(X)=\theta_k(X-\sum_{j=k+1}^{n} C_j X_j)=h_k\neq 0$, there is a holomorphic vector field $X_k=\sum_{i=1}^{k-1} f_{ik}\frac{\partial}{\partial z_i}+\frac{\partial}{\partial z_k}$ in $\mathfrak{a}$. Obviously

$$\theta_k(X_k)=1, \qquad \theta_l(X_k)=0 \ (l>k) \quad \text{and} \quad \theta_k(X_l)=0 \ (l>k).$$

Therefore $\dim\mathfrak{a}=n$ and $\boldsymbol{C}^n/\Gamma$ is a complex parallelisable manifold. Since $\boldsymbol{C}^n/\Gamma$ is a Kähler manifold, $\boldsymbol{C}^n/\Gamma$ is biholomorphic to a complex torus by a theorem of Wang [7]. q. e. d.

§ 4. The case when Γ is abelian.

In this section we prove the following theorem.

THEOREM 4.1. *If Γ is an abelian subgroup of $A(n, \boldsymbol{C})$ acting freely and properly discontinuously on $\boldsymbol{C}^n$ and such that $\boldsymbol{C}^n/\Gamma$ is compact, then the compact complex manifold $\boldsymbol{C}^n/\Gamma$ is biholomorphic to a complex torus.*

PROOF. Let $A(\gamma)$ be the holonomy part and $\alpha(\gamma)$ the translation part of $\gamma\in A(n, \boldsymbol{C}^n)$. Since Γ is abelian, $\{A(\gamma)\in GL(n, \boldsymbol{C})\,|\,\gamma\in\Gamma\}$ is abelian. It is well-known that there is a basis $\{v_1, \cdots, v_n\}$ of $\boldsymbol{C}^n$ such that

$$A(\gamma)\in\left\{\begin{pmatrix} * & & * \\ & \ddots & \\ \mathbf{0} & & * \end{pmatrix}\right\} \qquad \text{for any } \gamma\in\Gamma .$$

Thus we can write each element γ of Γ as

$$\gamma=\begin{pmatrix} A(\gamma) & \alpha(\gamma) \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} a_{11}(\gamma) & \cdots\cdots & a_{1n}(\gamma) & \alpha_1(\gamma) \\ & \ddots & \vdots & \vdots \\ \mathbf{0} & & a_{nn}(\gamma) & \alpha_n(\gamma) \\ 0 & \cdots\cdots\cdots & 0 & 1 \end{pmatrix}.$$

By Lemma 2.4, there are elements $\delta_1, \cdots, \delta_n$ of Γ such that $\{\alpha(\delta_1), \cdots, \alpha(\delta_n)\}$ is a basis of $\boldsymbol{C}^n$. Since Γ is abelian, we get

$$(4.1)\qquad \begin{pmatrix} \sum_{j=1}^{n} a_{1i}(\delta_i)\alpha_j(\gamma)+\alpha_1(\delta_i) \\ \sum_{j=2}^{n} a_{2j}(\delta_i)\alpha_j(\gamma)+\alpha_2(\delta_i) \\ \cdots\cdots\cdots\cdots\cdots \\ a_{nn}(\delta_i)\alpha_n(\gamma)+\alpha_n(\delta_i) \end{pmatrix}=\begin{pmatrix} \sum_{j=1}^{n} a_{1j}(\gamma)\alpha_j(\delta_i)+\alpha_1(\gamma) \\ \sum_{j=2}^{n} a_{2j}(\gamma)\alpha_j(\delta_i)+\alpha_2(\gamma) \\ \cdots\cdots\cdots\cdots\cdots \\ a_{nn}(\gamma)\alpha_n(\delta_i)+\alpha_n(\gamma) \end{pmatrix}$$

for $i=1, \cdots, n$ and any $\gamma\in\Gamma$. By (4.1) we have

$$(4.2)\qquad \begin{pmatrix} \sum_{j=1}^{n} a_{1j}(\delta_i)\alpha_j(\gamma)-\alpha_1(\gamma) \\ \sum_{j=2}^{n} a_{2j}(\delta_i)\alpha_j(\gamma)-\alpha_2(\gamma) \\ \cdots\cdots\cdots\cdots\cdots \\ a_{nn}(\delta_i)\alpha_n(\gamma)-\alpha_n(\gamma) \end{pmatrix}$$

$$=\begin{pmatrix} a_{11}(\gamma)-1 & a_{12}(\gamma) & \cdots\cdots & a_{1n}(\gamma) \\ & a_{22}(\gamma)-1 & \cdots & a_{2n}(\gamma) \\ & & \ddots & \vdots \\ \mathbf{0} & & & a_{nn}(\gamma)-1 \end{pmatrix}\begin{pmatrix} \alpha_1(\delta_i) \\ \alpha_2(\delta_i) \\ \vdots \\ \alpha_n(\delta_i) \end{pmatrix}$$

for $i=1, \cdots, n$ and any $\gamma\in\Gamma$. Thus we have

$$(4.3)\qquad \begin{pmatrix} a_{11}(\gamma)-1 & a_{12}(\gamma) & \cdots\cdots & a_{1n}(\gamma) \\ & a_{22}(\gamma)-1 & \cdots & a_{2n}(\gamma) \\ & & \ddots & \vdots \\ \mathbf{0} & & & a_{nn}(\gamma)-1 \end{pmatrix}\begin{pmatrix} \alpha_1(\delta_1) & \cdots\cdots & \alpha_1(\delta_n) \\ \alpha_2(\delta_1) & \cdots\cdots & \alpha_2(\delta_n) \\ \vdots & & \vdots \\ \alpha_n(\delta_1) & \cdots\cdots & \alpha_n(\delta_n) \end{pmatrix}$$

$$=\begin{pmatrix} \cdots\cdots & (a_{11}(\delta_k)-1)\alpha_1(\gamma)+\sum_{j=2}^{n} a_{1j}(\delta_k)\alpha_j(\gamma) & \cdots\cdots \\ \cdots\cdots & (a_{22}(\delta_k)-1)\alpha_2(\gamma)+\sum_{j=3}^{n} a_{2j}(\delta_k)\alpha_j(\gamma) & \cdots\cdots \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \cdots\cdots\cdots\cdots & (a_{nn}(\delta_k)-1)\alpha_n(\gamma) & \cdots\cdots\cdots\cdots \end{pmatrix}$$
$$\underset{k}{\wedge}$$

Since $\{\alpha(\delta_1), \cdots, \alpha(\delta_n)\}$ is a basis of $\boldsymbol{C}^n$, we get

$$(4.4)\qquad \begin{cases} a_{ii}(\gamma)=1+\sum_{k=1}^{n} C_{ii}^{k}\alpha_k(\gamma) & \text{for} \quad i=1, \cdots, n \\ a_{ij}(\gamma)=\sum_{k=1}^{n} C_{ij}^{k}\alpha_k(\gamma) & \text{for} \quad 1\leqq i<j\leqq n \end{cases}$$

for any $\gamma\in\Gamma$, by (4.3).

Since Γ is a subgroup of $A(n, \boldsymbol{C})$,

$$(4.5)\qquad \begin{cases} (1+\sum_{k=1}^{n} C_{ii}^{k}\alpha_k(\gamma))(1+\sum_{l=1}^{n} C_{ii}^{l}\alpha_l(\delta))=1+\sum_{k=1}^{n} C_{ii}^{k}\alpha_k(\gamma\delta) \\ \alpha_t(\gamma\delta)=\sum_{j=t}^{n} a_{tj}(\gamma)\alpha_j(\delta)+\alpha_t(\gamma) \end{cases}$$

for $i, t=1, \cdots, n$ and any $\gamma, \delta\in\Gamma$.
By (4.4) and (4.5), we get

$$(4.6)\qquad \sum_{k,l=1}^{n} C_{ii}^{k}C_{ii}^{l}\alpha_k(\gamma)\alpha_l(\delta)$$

$$=\sum_{t,k=1}^{n} C_{ii}^{t}C_{ti}^{k}\alpha_k(\gamma)\alpha_t(\delta)+\sum_{k=1}^{n}(\sum_{t=1}^{n}\sum_{j>t}^{n} C_{ii}^{t}C_{tj}^{k}\alpha_j(\delta))\alpha_k(\gamma)$$

for $i=1, \cdots, n$ and any $\gamma, \delta\in\Gamma$.
Since $\{\alpha(\gamma)|\gamma\in\Gamma\}$ contains a basis of $\boldsymbol{C}^n$, we get

$$(4.7)\qquad C_{ii}^{k}C_{ii}^{l}=C_{ii}^{l}C_{li}^{k}+\sum_{t<l} C_{ii}^{t}C_{tl}^{k}$$

for $i, k, l=1, \cdots, n$.

We now claim that $C_{ii}^{k}=0$ for $i, k=1, \cdots, n$. Suppose that $C_{ii}^{1}\neq 0$ for some i. Then $C_{ii}^{k}=C_{11}^{k}$ by (4.7). In particular, $C_{11}^{1}\neq 0$. Define an element $g_0\in A(n, \boldsymbol{C})$ by

$$g_0=\left(\begin{array}{ccc|c} 1 & & \boldsymbol{0} & -\frac{1}{C_{11}^{1}} \\ & \ddots & & 0 \\ & & & \vdots \\ \boldsymbol{0} & & 1 & 0 \\ \hline 0 & \cdots\cdots & 0 & 1 \end{array}\right).$$

Then we have

$$(4.8)\qquad g_0\cdot\left(\begin{array}{ccc|c} 1+\sum_{i=1}^{n} C_{11}^{i}\alpha_i(\gamma) & & * & \alpha_1(\gamma) \\ & \ddots & & \alpha_2(\gamma) \\ \boldsymbol{0} & & 1+\sum_{i=1}^{n} C_{nn}^{i}\alpha_i(\gamma) & \vdots \\ & & & \alpha_n(\gamma) \\ \hline 0 & \cdots\cdots\cdots & 0 & 1 \end{array}\right)\cdot g_0^{-1}$$

$$= \left(\begin{array}{ccc|c} 1+\sum_{i=1}^{n} C_{11}^{i}\alpha_i(\gamma) & & * & \sum_{i\geq 2} C_{11}^{i}\alpha_i(\gamma) \\ & \ddots & & \alpha_2(\gamma) \\ \mathbf{0} & & 1+\sum_{i=1}^{n} C_{nn}^{i}\alpha_i(\gamma) & \vdots \\ & & & \alpha_n(\gamma) \\ \hline 0 & \cdots\cdots & 0 & 1 \end{array}\right)$$

for any $\gamma\in\Gamma$. It is easy to see that if Γ acts freely and properly discontinuously on C^n and C^n/Γ is compact, so does $g\Gamma g^{-1}$ for any $g\in A(n, C)$. By Lemma 2.4, the translational parts of $g\Gamma g^{-1}$ contains a basis of C^n. By (4.8), the translational parts of $g_0\Gamma g_0^{-1}$ can not contain a basis of C^n. This is a contradiction. Hence, we get $C_{ii}^1=0$ for $i=1, \cdots, n$. By the same way we get $C_{ii}^k=0$ for $k, i=1, \cdots, n$ from (4.7) inductively. Therefore each element $\gamma\in\Gamma$ can be written in the form

$$\gamma = \left(\begin{array}{ccc|c} 1 & & \sum_k C_{ij}^k\alpha_k(\gamma) & \alpha_1(\gamma) \\ & \ddots & & \vdots \\ \mathbf{0} & & 1 & \alpha_n(\gamma) \\ \hline 0 & & 0 & 1 \end{array}\right).$$

We define a subset G of $A(n, C)$ by

$$G = \left\{ \left(\begin{array}{ccc|c} 1 & & \sum_k C_{ij}^k z_k & z_1 \\ & \ddots & & \vdots \\ \mathbf{0} & & 1 & z_n \\ \hline 0 & \cdots\cdots & 0 & 1 \end{array}\right) \middle|\ z_k\in C,\ k=1, \cdots, n \right\}.$$

Then we can see that G is a simply connected complex abelian Lie group which contains Γ in the same way as for Lemma 2.5. Moreover the map $\phi: C^n\to G$ defined by

$$\phi(z) = \left(\begin{array}{ccc|c} 1 & & \sum_k C_{ij}^k z_k & z_1 \\ & \ddots & & \vdots \\ \mathbf{0} & & 1 & z_n \\ \hline 0 & \cdots\cdots & 0 & 1 \end{array}\right) \quad \text{for} \quad z\in C^n$$

is biholomorphic and $\phi(g(z))=g\cdot\phi(z)$ for any $g\in G$ and any $z\in C^n$. Since Γ acts on C^n freely and properly discontinuously, Γ is a discrete subgroup of G and ϕ induces a biholomorphic map $\bar{\phi}: C^n/\Gamma\to\Gamma\backslash G$. Since C^n/Γ is compact, C^n/Γ is biholomorphic to the complex torus $\Gamma\backslash G$. q. e. d.

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