J. Math. Soc. Japan
Vol. 28, No. 4, 1976

On the finite element method for parabolic equations, I; approximation of holomorphic semi-groups[1)]

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(Received Jan. 27, 1976)

§ 1. Introduction.

In the present paper and a few papers to follow, we shall make an operator theoretical study of the finite element method applied to the initial boundary value problems for partial differential equations of parabolic type.

The particular objectives of the present paper are twofold. Firstly, we intend to develop a method of error analysis of a general nature which is in conformity with the theory of holomorphic semigroups (cf. Yosida [21] or Kato [13]). Specific descriptions of this method would be only possible after introduction of necessary notions and notation as in the following sections. However, it seems to be appropriate to give a few comments here of the motivation or the idea of our method. The initial boundary value problem which we are going to consider can be formulated as an evolution equation in a Hilbert space X of the following form with an m-sectorial operator A (Kato [13], Chap. V, VI):

$$\begin{cases} \dfrac{du}{dt}+Au=0, & (t>0), \\ u(0)=a, \end{cases}$$

where t is the time variable and a is the initial value. The solution $u:[0, \infty) \to X$ is given in terms of the semigroup e^{-tA} generated by $-A$ as

$$u(t)=e^{-tA}a.$$

Reflecting the parabolicity of the original equation, this semigroup e^{-tA} is a holomorphic semigroup and it admits of the integral representation (the Dunford integral)

$$e^{-tA}=\frac{1}{2\pi i}\int_{\Gamma} e^{-tz}(z-A)^{-1}dz \qquad (t>0).$$

1) A part of this paper was reported by the first author at the IRIA symposium in December, 1975, 2nd international symposium on computing methods in applied sciences and engineering (cf. Fujita [8]).

Here the path Γ of the complex integration is the positively oriented boundary of a certain sector in the complex z-plane. The above integral representation suggests to us that if we approximate A by some operator or other, say, by its finite element approximation A_h to be introduced in § 2, then investigation of the convergence of the resolvent $(z-A_h)^{-1}$ of A_h to the resolvent $(z-A)^{-1}$ of A will provide us with the corresponding information about the convergence of the approximate solution for the initial value problem. The rate of convergence of the so-called semi-discrete approximation, which approximates e^{-tA} essentially by e^{-tA_h}, will be derived along this line. Furthermore, if the time variable t is also discretized with time mesh $\tau>0$ and if we adopt, say, the forward difference finite element approximation, then e^{-tA_h} itself is approximated in turn by $G_n(A_h)=(1-\tau A_h)^n$ $(t=n\tau\,;\, n=1, 2, \cdots)$. This is nothing but approximation of the exponential function of A_h by the polynomial G_n of A_h, and can be treated again within the framework of the calculus of operator-valued holomorphic functions.

The second aim of the present paper is to deduce the optimum rate of convergence valid for $t>0$ of the semi-discrete as well as the difference finite element approximations. It is emphasized that in doing so we shall make neither the assumption of the self-adjointness of A nor the regularity assumption of a. Moreover, one of our typical results is the bound $Ch^2\|a\|/t$ of the error in the case of the semi-discrete approximation. Thus our results might be said to generalize or sharpen the existing estimates of the rate of convergence under consideration by various authors (e. g., Babuska and Aziz [**1**], Bramble and Thomée [**4**], Douglas and Dupont [**6**] and Zlámal [**24**]), although in this paper we deal only with basic but simple approximations mainly for the sake of simplicity of presentation.

Certain generalizations and modifications of the results obtained in this paper are immediate or straightforward. These will be discussed in forthcoming papers.

The present paper is composed of six sections. In § 2 we formulate the problem to be solved and the scheme of approximation to be analyzed more concretely. § 3 contains a theorem concerning the finite element approximation of the resolvent operators with complex argument. In § 4 we derive estimates of the rate of convergence of the semi-discrete approximation. § 5 is concerned with the semi-discrete approximation of the inhomogeneous equation $u_t+Au=f(t)$. § 6 is devoted to derivation of the rate of convergence of the difference finite element approximations.

§ 2. Notations and preliminaries.

Let Ω be a bounded domain in the plane R^2 and assume either that the boundary $\partial\Omega$ is smooth (of C^3-class) or that Ω is a convex polygon. In Ω we are given an elliptic operator L of the following form:

$$Lu=\sum_{i,j=1}^{2} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}+\sum_{j=1}^{2} b_j(x)\frac{\partial u}{\partial x_j}+c(x)u\,,$$

where the coefficients a_{ij}, b_j, c are possibly complex valued but are sufficiently smooth, e. g., $a_{ij}\in C^2$, $b_j\in C^1$ and $c\in C^0$ up to the boundary. L is assumed to be uniformly and strongly elliptic.

The initial boundary value problem which we are going to consider is composed of the following equations.

$$\frac{\partial u}{\partial t}=Lu \qquad (t>0,\ x\in\Omega)\,, \tag{2.1}$$

$$u=0 \qquad (t>0,\ x\in\partial\Omega)\,, \tag{2.2}$$

$$u|_{t=0}=a(x) \qquad (x\in\Omega)\,, \tag{2.3}$$

where the initial value $a=a(x)$ is a given function $\in L_2(\Omega)$. If we introduce the (complex) Hilbert space $X=L_2(\Omega)$ and define an operator $A:\mathcal{D}(A)\to X$ by

$$\mathcal{D}(A)=H^2(\Omega)\cap H_0^1(\Omega) \tag{2.4}$$

and

$$Au=-Lu \qquad (u\in\mathcal{D}(A))\,,$$

then the problem (2.1)~(2.3) is reduced to the following initial value problem of the evolution equation, for $u:[0,\infty)\to X$,

$$\frac{du}{dt}+Au=0\,, \tag{2.5}$$

with

$$u(0)=a\,. \tag{2.6}$$

As in (2.4) the symbol $H^j(\Omega)$ $(j=0, 1, \cdots)$ stands for the Sobolev space $W_2^j(\Omega)$ of order j, and the symbol $\|\ \|_j$ means the standard norm in $H^j(\Omega)$. For instance,

$$\|u\|_1^2=\|u\|_{H^1(\Omega)}^2=\|u\|_{L^2(\Omega)}^2+\|\nabla u\|_{L^2(\Omega)}^2\,. \tag{2.7}$$

We simply write $\|\ \|$ in place of $\|\ \|_0$. $H_0^1(\Omega)$ is the set $\{u\in H^1(\Omega)\,;\,u|_{\partial\Omega}=0\}$. $H^1(\Omega)$ and $H_0^1(\Omega)$ can be regarded as Hilbert spaces under their inner product

$(\ ,\ )_1$ consistent with the norm (2.7), while the inner product in $X=L_2(\Omega)$ will be written as $(\ ,\ )$. We put

$$V=H_0^1(\Omega).$$

As is well known, the operator A can be associated with a sesquilinear form $\sigma: V\times V\to \boldsymbol{C}$ such that

$$|\sigma(u, v)|\leqq c_1\|u\|_1\|v\|_1 \qquad (u, v\in V), \tag{2.8}$$

$$\operatorname{Re}\sigma(u, u)\geqq c_2\|u\|_1^2-\lambda_1\|u\|^2 \qquad (u\in V), \tag{2.9}$$

$$\sigma(u, v)=(Au, v) \qquad (u\in \mathcal{D}(A),\ v\in V), \tag{2.10}$$

where c_1, c_2 and λ_1 are positive constants. Since replacement of u by $e^{-\lambda_1 t}u$ in (2.5) implies replacement of A by $A+\lambda_1$, which in turn corresponds to changing $\sigma(\ ,\ )$ to $\sigma(\ ,\ )+\lambda_1(\ ,\ )$, we may assume $\lambda_1=0$ without loss of generality. Hence,

$$\operatorname{Re}\sigma(u, u)\geqq c_2\|u\|_1^2 \qquad (u\in V). \tag{2.9}$$

A consequence of (2.8)~(2.10) is that there exists an angle θ_1 $(0<\theta_1<\pi/2)$ with the following properties: if a subset of the complex plane G_1 is defined by

$$G_1=\{z\,;\ \theta_1\leqq|\arg z|\leqq\pi\}, \tag{2.11}$$

then $G_1\subset\rho(A)=$the resolvent set of A, and

$$\|(z-A)^{-1}\|\leqq\frac{M_1}{|z|} \qquad (z\in G_1) \tag{2.12}$$

for some positive constant M_1. Moreover, the semigroup $\{e^{-tA}\}_{t\geqq 0}$ which solves the initial value problem (2.5) and (2.6) by

$$u(t)=e^{-tA}a \qquad (t\geqq 0)$$

is a holomorphic semigroup (e.g., see Kato [**13**], Yosida [**21**], and Lions and Magenes [**17**]). It also admits of the representation

$$e^{-tA}=\frac{1}{2\pi i}\int_{\Gamma_1}e^{-tz}(z-A)^{-1}dz \tag{2.13}$$

where Γ_1 is the positively oriented boundary, running from $+\infty e^{i\theta_1}$ to $+\infty e^{-i\theta_1}$, of the sector

$$\Sigma_1=\{z\,;\ |\arg z|<\theta_1\}. \tag{2.14}$$

From (2.12) and (2.13) follow some useful inequalities, for instance,

$$\|A(z-A)^{-1}\|\leqq C,$$

and

$$\|A^{\alpha}e^{-tA}\| \leqq \frac{C_{\alpha}}{t^{\alpha}} \qquad (\alpha > 0). \tag{2.15}$$

Henceforth we may denote various positive constants (various positive constants depending on a parameter, say, α) indifferently by C (C_{α}) when the distinction seems unnecessary from the context. Finally we note that the graph norm $\|Au\|$ of A is equivalent to $H^2(\Omega)$-norm. This is the case also with A^*, the adjoint operator of A. Furthermore, $\mathcal{D}(A^{1/2})$ coincides with V (see Lions [**16**]).

We now turn to the finite element approximation. By V_h we mean the set of "trial functions", h being the size parameter of the subdivision of Ω. If Ω is a convex polygon, the subdivision should be a regular triangulation of Ω, and h represents the largest diameter of the element triangles. In this case, we set

V_h = "the set of all functions in V which are linear in each element".

If the boundary is curved, then we adopt such trial functions as were constructed by Zlámal [**22**], which are piecewise linear in interior elements, and are obtained by pulling back linear functions in elements adjacent to the boundary. In any case, we have

$$V_h \subset V$$

and thus our approximation is of a conforming type. If $\hat{u}^h$ represents the function $\in V_h$ which coincides with a given function u at every nodal point, then, we know that

$$\|u-\hat{u}^h\| \leqq Ch^2\|u\|_2 \qquad (u \in H^2(\Omega))$$

and

$$\|u-\hat{u}^h\|_1 \leqq Ch\|u\|_2 \qquad (u \in H^2(\Omega)) \tag{2.16}$$

(see Bramble and Zlámal [**3**], Zlámal [**22**]). For a later use, we introduce[1)]

$\tilde{u}^h = \tilde{P}_h u$ = the projection of $u \in V$ on V_h with respect to H^1-inner product,

for which we have

$$\|u-\tilde{u}^h\| \leqq Ch\|u\|_1 \qquad (u \in V), \tag{2.17}$$

$$\|u-\tilde{u}^h\|_1 \leqq Ch\|u\|_2 \qquad (u \in V \cap H^2(\Omega)) \tag{2.18}$$

1) Use of $\tilde{u}^h$ was kindly suggested to us by Dr. F. Kikuchi through an oral communication.

and

$$\|u-\tilde{u}^h\| \leqq Ch^2\|u\|_2 \qquad (u \in V \cap H^2(\Omega)) \tag{2.19}$$

as shown at the end of this section. Incidentally, we put

$$P_h u = \text{the projection of } u \in X \text{ on } V_h \\ \text{with respect to } L_2\text{-inner product}.$$

In this framework, our finite element approximation $A_h: V_h \to V_h$ of A is defined by

$$(A_h \varphi_h, \psi_h) = \sigma(\varphi_h, \psi_h) \qquad (\varphi_h, \psi_h \in V_h). \tag{2.20}$$

In other words, A_h is the operator associated with the sesquilinear form σ_h which is the restriction of σ on $V_h \times V_h$.

Let $u_h = u_h(t)$ be the approximate solution of the problem (2.5), (2.6) by the semi-discrete finite element approximation. Namely, we assume that $u_h: [0, \infty) \to V_h$ satisfies

$$\frac{d}{dt}(u_h, \varphi_h) + \sigma(u_h, \varphi_h) = 0 \qquad (\varphi_h \in V_h) \tag{2.21}$$

and

$$u_h(0) = a_h.$$

Then we can write instead of (2.21)

$$\frac{d}{dt} u_h + A_h u_h = 0,$$

and also have

$$u_h(t) = e^{-tA_h} a_h.$$

If $P_h a$ is taken as the approximation a_h of the initial value a, then we have

$$u_h(t) = e^{-tA_h} P_h a.$$

REMARK 2.1. (2.9) implies $\mathrm{Re}(Au, u) \geqq 0$ $(u \in \mathcal{D}(A))$ and, hence, A is accretive. Consequently, e^{-tA} is a contraction: $\|e^{-tA}\| \leqq 1$, $(t>0)$. This is also the case with e^{-tA_h}.

Moreover, if the time variable t is discretized with the time mesh $\tau > 0$, then the difference finite element approximation is given by

$$u_h(t+\tau) - u_h(t) + \tau A_h u_h(t) = 0 \qquad (t = n\tau,\ n \in \boldsymbol{N})$$

or

$$u_h(t+\tau) - u_h(t) + \tau A_h u_h(t+\tau) = 0 \qquad (t = n\tau,\ n \in \boldsymbol{N}),$$

according as the time difference is taken forward or backward. Therefore we have

(2.22) $$u_h(t)=(I-\tau A_h)^n a_h \qquad (t=n\tau,\ n\in \boldsymbol{N})$$

for the forward approximation, and

(2.23) $$u_h(t)=(I+\tau A_h)^{-n} a_h \qquad (t=n\tau,\ n\in \boldsymbol{N})$$

for the backward approximation. More sophisticated difference approximations, e. g., the Crank-Nicolson method, might be treated in this manner but will not be discussed in this paper.

Proof of (2.17), (2.18) and (2.19).

Putting $e=u-\tilde{u}^h$, we have

(2.24) $$(e, \varphi_h)_1=0 \qquad (\forall \varphi_h \in V_h)$$

and, hence, taking $\varphi_h=\tilde{u}^h-\hat{u}^h$, we have in view of (2.16)

$$\|e\|_1^2=(e, e+\varphi_h)_1=(e, u-\hat{u}^h)_1$$

$$\leqq \|e\|_1\|u-\hat{u}^h\|_1 \leqq Ch\|e\|_1\|u\|_2 ,$$

which implies (2.18). Making use of a modification of Nitsche's trick we can estimate $\|e\|$ as follows:

$$\|e\|=\sup_{\psi}\frac{|(e, \psi)|}{\|\psi\|}=\sup_{\psi}\frac{|(e, \eta)_1|}{\|\psi\|},$$

where $\eta\in V\cap H^2(\Omega)$ is related with ψ by

$$\eta=(-\Delta+I)^{-1}\psi$$

or, equivalently, $(\eta, \varphi)_1=(\psi, \varphi)$ $(\forall \varphi\in V)$. By (2.18) we have

$$\|\eta-\tilde{\eta}^h\|_1 \leqq Ch\|\eta\|_2 \leqq Ch\|\psi\| ,$$

whence follows

$$|(e, \eta)_1|=|(e, \eta-\tilde{\eta}^h)_1| \leqq \|e\|_1\|\eta-\tilde{\eta}^h\|_1$$

$$\leqq Ch\|e\|_1\cdot\|\psi\|$$

with the aid of (2.24). Consequently we have

(2.25) $$\|e\| \leqq Ch\|e\|_1 .$$

Since $\|e\|_1 \leqq \|u\|_1$ by the definition of $\tilde{u}^h$, (2.25) gives (2.17). Finally, substitution of (2.18) into (2.25) yields (2.19).

§ 3. Approximation of the resolvent.

The objective of this section is to prove the following

THEOREM 3.1. *There exist a positive constant C and an acute angle θ_1 such* *that for any* $f\in X$ *and for any* $z\in G_1=\{z\,;\,\theta_1\leqq|\arg z|\leqq\pi\}$ *we have*

$$\|e_h(z)\|_1\leqq Ch\|f\|\,, \tag{3.1}$$

$$\|e_h(z)\|\leqq Ch^2\|f\|\,, \tag{3.1}$$

$$\|e_h(z)\|\leqq Ch\|f\|/|z|^{1/2}\,, \tag{3.3}$$

where $e_h(z)=(z-A)^{-1}f-(z-A_h)^{-1}P_hf=w(z)-w_h(z)$.

We begin with consideration of the numerical range of the sesquilinear form σ.

LEMMA 3.2. *There exist positive constants* δ_0 *and* γ_0 *such that*

$$|\mathrm{Im}\,\sigma(\varphi,\varphi)|\leqq\delta_0(\mathrm{Re}\,\sigma(\varphi,\varphi)-\gamma_0\|\varphi\|_1^2) \tag{3.4}$$

for all $\varphi\in V$.

PROOF. From (2.8) and (2.9) it is easy to show

$$|\mathrm{Im}\,\sigma(\varphi,\varphi)|\leqq\frac{2c_1}{c_2}\Big(\mathrm{Re}\,\sigma(\varphi,\varphi)-\frac{c_2}{2}\|\varphi\|_1^2\Big)\,.$$

Thus (3.4) holds good with

$$\delta_0=\frac{2c_1}{c_2}\quad\text{and}\quad\gamma_0=\frac{c_2}{2}\,. \tag{3.5}$$

Q. E. D.

Henceforth δ_0 and γ_0 will denote those in (3.5), and we set

$$\theta_0=\tan^{-1}\delta_0\qquad\Big(0<\theta_0<\frac{\pi}{2}\Big)\,,$$

whereas we choose a θ_1 subject to $\theta_0<\theta_1<\pi/2$ once for all and define the sectors G_1, Σ_1 by (2.11) and (2.14) with this θ_1, accordingly. Γ_1 denotes the positively oriented boundary of Σ_1.

LEMMA 3.3. *There exists a positive constant* δ_1 *such that*

$$|z|\,\|\varphi\|^2+\|\varphi\|_1^2\leqq\delta_1|z\|\varphi\|^2-\sigma(\varphi,\varphi)| \tag{3.6}$$

for all $\varphi\in V$ *and* $z\in G_1$.

PROOF. We may assume $\varphi\neq0$, and put

$$\mu(\varphi)=\gamma_0\|\varphi\|_1^2/\|\varphi\|^2\,,$$

$$\zeta(\varphi)=\sigma(\varphi,\varphi)/\|\varphi\|^2\,.$$

Defining a sector Σ_φ with vertex $\mu(\varphi)$ by

$$\Sigma_\varphi=\{z\,;\,|\arg(z-\mu(\varphi))|\leqq\theta_0\}\,,$$

we notice that (3.4) implies $\zeta(\varphi)\in\Sigma_\varphi$. On the other hand, we see by an elementary consideration

$$\text{dist.}(z, \Sigma_\varphi) \geqq |z| \sin(\theta_1-\theta_0)+\mu(\varphi)\sin\theta_0$$

if $z\in G_1$. Thus putting

$$\delta_1=(\min\{\sin(\theta_1-\theta_0),\ \gamma_0\sin\theta_0\})^{-1},$$

we have

$$\begin{aligned}|z\|\varphi\|^2-\sigma(\varphi,\varphi)| &= \|\varphi\|^2|z-\zeta(\varphi)| \\ &\geqq \|\varphi\|^2\,\text{dist.}(z, \Sigma_\varphi) \\ &\geqq \frac{1}{\delta_1}(|z|\,\|\varphi\|^2+\|\varphi\|_1^2)\end{aligned}$$

which establishes the lemma. Q. E. D.

COROLLARY 3.4. *As for A_h we have*

$$|z|\,\|\varphi_h\|^2+\|\varphi_h\|_1^2\leqq\delta_1|((z-A_h)\varphi_h,\varphi_h)|\qquad(\varphi_h\in V_h). \tag{3.7}$$

Furthermore,

$$G_1\subset\rho(A_h)=\textit{the resolvent set of } A_h,$$

and the following inequalities hold good:

$$\begin{cases}\|(z-A_h)^{-1}f_h\|\leqq\delta_1\|f_h\|/|z| & (f_h\in V_h), \\ \|(z-A_h)^{-1}f_h\|_1\leqq\delta_1\|f_h\|/|z|^{1/2} & (f_h\in V_h), \\ \|A_h(z-A_h)^{-1}f_h\|\leqq(1+\delta_1)\|f_h\| & (f_h\in V_h),\end{cases}$$

for all $z\in G_1$.

PROOF. (3.7) is simply a restriction of (3.6) in view of (2.20). From (3.7) we can calculate with $w_h=(z-A_h)^{-1}f_h$ as

$$|z|\,\|w_h\|^2+\|w_h\|_1^2\leqq\delta_1|(f_h,w_h)|\leqq\delta_1\|f_h\|\,\|w_h\|,$$

and obtain

$$|z|\,\|w_h\|\leqq\delta_1\|f_h\|,$$

$$\|w_h\|_1^2\leqq\delta_1\|f_h\|\,\|w_h\|\leqq\delta_1^2\|f_h\|^2/|z|$$

$$\|A_hw_h\|=\|zw_h-f_h\|\leqq|z|\,\|w_h\|+\|f_h\|\leqq(\delta_1+1)\|f_h\|,$$

whence follows the corollary. Q. E. D.

REMARK 3.5. From the general theory we know that

$$\|A^\alpha(z-A)^{-1}\|\leqq C_\alpha|z|^{-1+\alpha}\qquad(z\in G_1,\ 0\leqq\alpha\leqq1). \tag{3.8}$$

On the other hand, by making use of (3.6) we can show

$$\|(z-A)^{-1}f\|_1\leqq\delta_1\|f\|/|z|^{1/2} \tag{3.9}$$

by an elementary argument as above, which is equivalent to the special case $\alpha=1/2$ of (3.8) in view of $\mathcal{D}(A^{1/2})=V$. See Lions [16]. Moreover, (3.8) remains valid with the same C_α even if A is replaced by A_h.

PROOF OF THEOREM 3.1. w and w_h satisfy

$$z(w, \varphi)-\sigma(w, \varphi)=(f, \varphi) \qquad (\varphi \in V)$$

and

$$z(w_h, \varphi_h)-\sigma(w_h, \varphi_h)=(f, \varphi_h) \qquad (\varphi_h \in V_h),$$

respectively. Therefore,

$$z(e_h, \varphi_h)-\sigma(e_h, \varphi_h)=0 \qquad (\varphi_h \in V_h). \tag{3.10}$$

Thus in virtue of (3.6) we have for any $\varphi_h \in V_h$

$$\begin{aligned}|z|\,\|e_h\|^2+\|e_h\|_1^2 &\leqq \delta_1 |z\|e_h\|^2-\sigma(e_h, e_h)| \\ &= \delta_1 |z(e_h, e_h+\varphi_h)-\sigma(e_h, e_h+\varphi_h)|.\end{aligned}$$

Setting $\varphi_h=w_h-\tilde{w}^h$, we have

$$\begin{aligned}|z|\,\|e_h\|^2+\|e_h\|_1^2 &\leqq \delta_1 |z(e_h, w-\tilde{w}^h)-\sigma(e_h, w-\tilde{w}^h)| \\ &\leqq \delta_1(|z|\,\|e_h\|\,\|w-\tilde{w}^h\|+c_1\|e_h\|_1\|w-\tilde{w}^h\|_1) \\ &\leqq Ch(|z|\,\|e_h\|\,\|w\|_1+\|e_h\|_1\|w\|_2)\end{aligned} \tag{3.11}$$

by (2.17) and (2.18). On the other hand,

$$\begin{aligned}&\|w\|_1=\|(z-A)^{-1}f\|_1 \leqq C\|f\|/|z|^{1/2}, \\ &\|w\|_2 \leqq C\|Aw\|_0 \leqq C\|f\|.\end{aligned}$$

Hence we are lead to

$$|z|\,\|e_h\|^2+\|e_h\|_1^2 \leqq Ch\|f\|(|z|^{1/2}\|e_h\|+\|e_h\|_1)$$

and moreover, to

$$|z|\,\|e_h\|^2+\|e_h\|_1^2 \leqq Ch^2\|f\|^2 \tag{3.12}$$

with a suitable constant C. (3.12) gives (3.1) and (3.3). In order to derive (3.2) we resort to Nitsche's trick. Let A^* be the adjoint operator of A. A^* is nothing but the accretive operator associated with the sesquilinear form $\sigma^*(u, v)=\overline{\sigma(v, u)}$. And we know $\mathcal{D}(A^*)=V\cap H^2(\Omega)=\mathcal{D}(A)$. For any $g\in X$ let us estimate $|(e_h, g)|$. Defining η by $\eta=(\bar{z}-A^*)^{-1}g$, we have

$$\begin{aligned}|(e_h, g)| &= |z(e_h, \eta)-\sigma(e_h, \eta)| \\ &= |z(e_h, \eta-\tilde{\eta}^h)-\sigma(e_h, \eta-\tilde{\eta}^h)| \\ &\leqq C(|z|\,\|e_h\|\,\|\eta-\tilde{\eta}^h\|+\|e_h\|_1\|\eta-\tilde{\eta}^h\|_1) \\ &\leqq C(h|z|^{1/2}\|f\|\cdot h\|\eta\|_1+h\|f\|\cdot h\|\eta\|_2)\end{aligned}$$

by means of (3.10), (3.3), (3.1), (2.17) and (2.18). Noting that the inequalities (3.8) and (3.9) are true with A replaced by A^*, namely, that

$$\|\eta\|_1 \leqq C\|g\|/|z|^{1/2}$$

and

$$\|\eta\|_2 \leqq C\|g\|,$$

we see from the inequalities above that

$$|(e_h, g)| \leqq Ch^2\|f\|\|g\| \qquad (g \in X).$$

This implies (3.2). Q. E. D.

REMARK 3.6. (3.1) and (3.2) hold true for $z=0$.

REMARK 3.7. (3.2) and (3.3) can be written, respectively, as

$$\|(z-A)^{-1}-(z-A_h)^{-1}P_h\| \leqq Ch^2 \tag{3.13}$$

and

$$\|(z-A)^{-1}-(z-A_h)^{-1}P_h\| \leqq Ch/|z|^{1/2}. \tag{3.14}$$

In other words, the estimates in Theorem 3.1 give the rate of convergence measured in the operator norm.

REMARK 3.8. When combined with the obvious inequality

$$\|e_h\| \leqq \|w\|+\|w_h\| \leqq C\|f\|/|z|+C\|P_h f\|/|z| \leqq C\|f\|/|z|, \tag{3.15}$$

(3.2) yields

$$\|e_h(z)\| \leqq C_\alpha \frac{(|z|h^2)^\alpha}{|z|}\|f\| \qquad (0 \leqq \alpha \leqq 1).$$

These slight generalizations of the error estimates are apparent with the estimates in the other theorems in the present paper and will not be stated explicitly (see Fujita [8]).

§ 4. Rate of convergence of the semi-discrete finite element approximation.

In this section we suppose that $u(t)=e^{-tA}a$ is the solution of the original problem, and $u_h=u_h(t)=e^{-tA_h}a_h$ is the semi-discrete approximating solution with the initial value $a_h \in V_h$. First of all we claim the following

THEOREM 4.1. *Suppose that $a_h=P_h a$. Then we have for the error $\varepsilon_h(t)=u(t)-u_h(t)$ the following estimates:*

$$\|\varepsilon_h\|_1 \leqq Cht^{-1}\|a\|, \qquad (t>0), \tag{4.1}$$

$$\|\varepsilon_h\| \leqq Ch^2t^{-1}\|a\|, \qquad (t>0). \tag{4.2}$$

PROOF. Let θ_1 and Γ_1 be as mentioned in § 3. We then have

$$\varepsilon_h(t)=\frac{1}{2\pi i}\int_{\Gamma_1} e^{-tz}((z-A)^{-1}-(z-A_h)^{-1}P_h)a\,dz\,. \tag{4.3}$$

Hence it follows by means of (3.1), f being replaced by a, that

$$\begin{aligned}\|\varepsilon_h(t)\|_1 &\leqq C'\int_0^\infty e^{-t\rho\cos\theta_1}h\|a\|d\rho \\ &=Cht^{-1}\|a\|\,.\end{aligned}$$

Similarly, (3.2) yields (4.2), because

$$\begin{aligned}\|\varepsilon_h(t)\| &\leqq C'h^2\|a\|\int_0^\infty e^{-t\rho\cos\theta_1}d\rho \\ &=Ch^2t^{-1}\|a\|\,.\end{aligned}$$

Q. E. D.

REMARK 4.2. (4.2) can be written as

$$\|e^{-tA}-e^{-tA_h}P_h\|\leqq Ch^2t^{-1}\,.$$

We proceed to the case of a more general choice of a_h. Namely, we suppose that a_h is in V_h but not necessarily equal to $P_h a$. The following theorem is concerned with such a case.

THEOREM 4.3. *Let $a_h\in V_h$. Then with the same notation as in Theorem 4.1, we have*

$$\|\varepsilon_h(t)\|_1\leqq C(ht^{-1}\|a_h\|+t^{-1/2}\|a-a_h\|)$$

and

$$\|\varepsilon_h(t)\|\leqq C(h^2t^{-1}\|a_h\|+\|a-a_h\|)\,.$$

PROOF. In view of $P_h a_h=a_h$, we can write

$$\begin{aligned}\varepsilon_h(t) &= e^{-tA}a-e^{-tA_h}a_h \\ &= e^{-tA}(a-a_h)+(e^{-tA}-e^{-tA_h}P_h)a_h \\ &\equiv \varepsilon_h^{(1)}(t)+\varepsilon_h^{(2)}(t)\,.\end{aligned}$$

$\varepsilon_h^{(1)}(t)$ can be estimated as

$$\|e^{-tA}(a-a_h)\|_1\leqq Ct^{-1/2}\|a-a_h\|$$

or

$$\|e^{-tA}(a-a_h)\|\leqq \|a-a_h\|\,,$$

while $\varepsilon_h^{(2)}(t)$ is dealt with by making use of Theorem 4.1. Then the theorem is obvious. Q. E. D.

THEOREM 4.4. *Suppose that $a\in V$ and $a_h=P_h a$. Then with the same nota-*

tion as in Theorem 4.1 *we have*

$$\|\varepsilon_h(t)\|_1 \leqq Cht^{-1/2}\|a\|_1,$$

$$\|\varepsilon_h(t)\| \leqq Ch\|a\|_1,$$

and

$$\|\varepsilon_h(t)\| \leqq Ch^2t^{-1/2}\|a\|_1.$$

PROOF. Going back to Theorem 3.1, we have to modify (3.1) and (3.2) as

$$\|e_h(z)\|_1 \leqq Ch|z|^{-1/2}\|f\|_1 \tag{4.4}$$

and

$$\|e_h(z)\| \leqq Ch^2|z|^{-1/2}\|f\|_1, \tag{4.5}$$

respectively. This can be done by noting that (3.11) yields the inequality

$$|z|\,\|e_h\|^2 + \|e_h\|_1^2 \leqq C(|z|\,\|w-\tilde{w}^h\|^2 + \|w-\tilde{w}^h\|_1^2). \tag{4.6}$$

By (2.17) and (2.18), (4.6) gives

$$\begin{aligned} |z|\,\|e_h\|^2 + \|e_h\|_1^2 &\leqq Ch^2(|z|\,\|w\|_1^2 + \|w\|_2^2) \\ &\leqq Ch^2|z|^{-1}\|f\|_1^2, \end{aligned} \tag{4.7}$$

since

$$\begin{aligned} \|w\|_1 \leqq C\|A^{1/2}w\| &= C\|(z-A)^{-1}A^{1/2}f\| \\ &\leqq C\|(z-A)^{-1}\|\,\|A^{1/2}f\| \\ &\leqq C|z|^{-1}\|f\|_1, \end{aligned}$$

and since

$$\begin{aligned} \|w\|_2 \leqq C\|Aw\| &= C\|A^{1/2}(z-A)^{-1}A^{1/2}f\| \\ &\leqq C\|A^{1/2}(z-A)^{-1}\|\,\|A^{1/2}f\| \\ &\leqq C|z|^{-1/2}\|f\|_1. \end{aligned}$$

(4.4) is obvious from (4.7), which gives also

$$\|e_h\| \leqq Ch|z|^{-1}\|f\|_1. \tag{4.8}$$

Again with resort to Nitsche's trick as in the proof of Theorem 3.1, we obtain (4.5) from (4.4) and (4.8). In order to complete the proof we have only to estimate the contour integral in (4.3) by means of (4.4), (4.5) and (4.8).

Q. E. D.

§5. A remark on the semi-discrete approximation for inhomogeneous equations.

In order to exemplify the convenience of the error estimates in terms of the operator norm which have been obtained in the preceding section, we deal with the semi-discrete approximation for the following initial value problem:

$$\text{(5.1)} \qquad \begin{cases} \dfrac{du}{dt}+Au=f(t) & (0\leqq t\leqq T), \\ u(0)=a, \end{cases}$$

where the given function $f:[0, T]\to X$ is assumed to be Hölder continuous with exponent θ $(0<\theta<1)$; namely, there exists a constant N_1 such that

$$\text{(5.2)} \qquad \|f(t)-f(s)\|\leqq N_1|t-s|^{\theta}, \qquad (t, s\in[0, T]).$$

The semi-discrete approximate $u_h:[0, T]\to V_h$ for u is determined by the differential equation

$$\text{(5.3)} \qquad \frac{du_h}{dt}+A_h u_h=P_h f(t) \qquad (0\leqq t\leqq T)$$

and the initial condition

$$\text{(5.4)} \qquad u_h(0)=P_h a.$$

Then we have the following

THEOREM 5.1. *Let u and u_h be as above. Then for the error $\varepsilon_h(t)=u(t)-u_h(t)$ we have*

$$\text{(5.5)} \qquad \|\varepsilon_h(t)\|\leqq Ch^2(t^{-1}\|a\|+\|f(t)\|+t^{\theta}N_1),$$

where the constant C is independent of t, h, a and f.

PROOF. In view of Duhamel's principle we have

$$u(t)=e^{-tA}a+\int_0^t e^{-(t-s)A}f(s)ds$$

and

$$u_h(t)=e^{-tA_h}P_h a+\int_0^t e^{-(t-s)A_h}P_h f(s)ds.$$

Therefore we can write as

$$\varepsilon_h(t)=\varepsilon_h^{(1)}+\varepsilon_h^{(2)}+\varepsilon_h^{(3)}$$

with

$$\varepsilon_h^{(1)}(t)=e^{-tA}a-e^{-tA_h}P_h a,$$

$$\varepsilon_h^{(2)}(t)=\int_0^t (e^{-(t-s)A}-e^{-(t-s)A_h}P_h)(f(s)-f(t))ds$$

and

$$\varepsilon_h^{(3)}(t)=\int_0^t (e^{-(t-s)A}-e^{-(t-s)A_h}P_h)f(t)ds\,.$$

From Theorem 4.1 the inequality $\|\varepsilon_h^{(1)}\|\leqq Ch^2t^{-1}\|a\|$ is obvious. Also, by virtue of Theorem 4.1 and Remark 4.2, we have

$$\begin{aligned}\|\varepsilon_h^{(2)}\| &\leqq \int_0^t \|e^{-(t-s)A}-e^{-(t-s)A_h}P_h\|\cdot\|f(s)-f(t)\|ds \\ &\leqq \int_0^t C'h^2(t-s)^{-1}\cdot N_1(t-s)^{\theta}ds \\ &\leqq C'N_1h^2t^{\theta}/\theta=Ch^2N_1t^{\theta}\,.\end{aligned}$$

In order to estimate $\varepsilon_h^{(3)}$, we first carry out the integration on the right hand side:

$$\varepsilon_h^{(3)}=-A^{-1}(e^{-tA}-I)f(t)+A_h^{-1}(e^{-tA_h}-I)P_hf(t)\,.$$

The operator $K_h(t)=A^{-1}(e^{-tA}-I)-A_h^{-1}(e^{-tA_h}-I)P_h$, $(t>0)$, is represented by the contour integral

$$K_h(t)=\frac{1}{2\pi i}\int_\Gamma \frac{e^{-tz}-1}{z}\{(z-A)^{-1}-(z-A_h)^{-1}P_h\}\,dz\,, \tag{5.6}$$

where Γ is the positively oriented infinite path $\{z=\rho e^{\pm i\theta_1}, 0\leqq\rho<\infty\}$. In fact, (5.6) can be easily verified in consideration that $0\in\rho(A)$, $0\in\rho(A_h)$ and the function $(e^{-tz}-1)/z$ is regular at $z=0$. Then in (5.6) we may deform Γ to the following $\Gamma^{(r)}$, r being any positive number:

$$\Gamma^{(r)}=\Gamma_0^{(r)}\cup\Gamma_1^{(r)} \qquad \text{(as sets)}\,,$$

$$\Gamma_0^{(r)}=\{re^{i\theta}:\theta_1\leqq|\theta|\leqq\pi\}\,,$$

$$\Gamma_1^{(r)}=\{\rho e^{\pm i\theta_1}:r\leqq\rho<\infty\}\,.$$

Moreover, we may write

$$K_h(t)=\frac{1}{2\pi i}\int_{\Gamma^{(r)}}\frac{e^{-tz}}{z}W_h(z)dz-\frac{1}{2\pi i}\int_{\Gamma^{(r)}}\frac{1}{z}W_h(z)dz \tag{5.7}$$

with $W_h(z)=(z-A)^{-1}-(z-A_h)^{-1}P_h$. The second integral on the right hand side of (5.7) turns out to be 0, since $(1/z)W_h(z)$ is holomorphic in the domain $G^{(r)}$ exterior to $\Gamma^{(r)}$, i.e., in $G^{(r)}=\{z\in G_1: |z|\geqq r\}$, and since $\|(1/z)W_h(z)\|=O(|z|^{-2})$ as $|z|\to\infty$ in $G^{(r)}$. On the other hand, choosing $r=1/t$, we can estimate the first integral on the right hand side of (5.7) as follows.

$$\begin{aligned}&\left\|\int_{\Gamma^{(1/t)}}\frac{e^{-tz}}{z}W_h(z)dz\right\| \\ &=\int_{\Gamma_0^{(1/t)}}\left|\frac{e^{-tz}}{z}\right|\|W_h(z)\|\,|dz|+\int_{\Gamma_1^{(1/t)}}\left|\frac{e^{-tz}}{z}\right|\|W_h(z)\|\,|dz|\end{aligned}$$

$$\leqq Ch^2\Big(\int_{\Gamma_0^{(1/t)}} te|dz| + \int_{\Gamma_1^{(1/t)}} \frac{e^{-t\,\mathrm{Re}\,z}}{|z|}|dz|\Big)$$

$$\leqq Ch^2\Big(te\cdot\frac{1}{t}2(\pi-\theta_1)+\int_1^\infty \frac{e^{-\rho\cos\theta_1}}{\rho}d\rho\Big)=Ch^2.$$

Here use has been made of the estimate $\|W_h(z)\|\leqq Ch^2$ due to Theorem 3.1. Thus we have $\|K_h(t)\|\leqq Ch^2$ and, therefore,

$$\|\varepsilon_h^{(3)}\|\leqq Ch^2\|f(t)\|.$$

Summing up the obtained bounds for $\varepsilon_h^{(j)}$ $(j=1, 2, 3)$, we establish the theorem.

Q. E. D.

§6. Rate of convergence of the difference finite element approximation.

In this section, we shall make a study of the case where the time variable is also discretized, i. e., the difference finite element approximation. Only basic but simple schemes are considered, while we leave schemes involving more sophisticated approximation of the time derivative to a forthcoming paper.

6.1. Backward difference approximation.

We first deal with the backward difference approximation (2.23). Thus $u_h(t)=(I+\tau A_n)^{-n}a_h$ $(t=n\tau,\ n=0, 1, \cdots)$ is taken as the approximation to $u(t)=e^{-tA}a$. For the sake of completeness, we state the following

THEOREM 6.1. *The backward difference approximation* (2.23) *is unconditionally stable.*[1] *Precisely, we have*

$$\|(I+\tau A_n)^{-n}\|\leqq 1 \tag{6.1}$$

for any n.

PROOF. By virtue of (2.9) we have $\mathrm{Re}(A_h\varphi_h, \varphi_h)\geqq 0$ for all $\varphi_h\in V_h$. In other words, A_h is accretive and hence, $(I+\tau A_h)^{-1}$ is a contraction for any $\tau>0$:

$$\|(I+\tau A_h)^{-1}\|\leqq 1.$$

This yields (6.1). Q. E. D.

REMARK 6.2. If $\mathrm{Re}(A_h\varphi_h, \varphi_h)\geqq -\lambda_1\|\varphi_h\|^2$ as follows from (2.9)′, then we have instead of (6.1)

$$\|(I+\tau A_h)^{-n}\|\leqq e^{\gamma n\tau} \qquad (n=0, 1, \cdots)$$

1) The stability condition $\sup_{\tau,h}\tau\|A_h\|<+\infty$ assumed in Fujita [8] is superfluous as long as L_2-stability of (2.23) is concerned.

with some $\gamma>\lambda_1$ for sufficiently small τ.

As for the rate of convergence we have

THEOREM 6.3. *Let $a_h=P_h a$ in the backward difference finite element approximation* (2.23) *and denote the error $u(t)-u_h(t)$ by $\varepsilon(t)$; namely*

$$\varepsilon(t)=e^{-tA}a-(1+\tau A_h)^{-n}a_h .$$

Then we have

$$\|\varepsilon(t)\| \leqq C(h^2+\tau)t^{-1}\|a\| \qquad (t=n\tau,\ n=1, 2, \cdots). \tag{6.2}$$

PROOF. Putting

$$\varepsilon^{(1)}=e^{-tA}a-e^{-tA_h}P_h a=e^{-tA}a-e^{-tA_h}a_h$$

and

$$\varepsilon^{(2)}=(e^{-tA_h}-(I+\tau A_h)^{-n})a_h \equiv K_h a_h ,$$

we notice $\varepsilon=\varepsilon^{(1)}+\varepsilon^{(2)}$. According to Theorem 4.1,

$$\|\varepsilon^{(1)}\| \leqq Ch^2 t^{-1}\|a\| .$$

Therefore, it is enough to show

$$\|K_h\| \leqq C\tau t^{-1} . \tag{6.3}$$

Now we can easily verify for $n\geqq 1$

$$\begin{aligned} -K_h &= \int_0^\tau \frac{d}{ds}((I+sA_h)^{-n}e^{-n(\tau-s)A_h})ds \\ &= n\int_0^\tau sA_h^2(I+sA_h)^{-n-1}e^{-n(\tau-s)A_h}ds \\ &= n\int_0^\tau sA_h^{3/2}(I+sA_h)^{-(n+1)}\cdot A_h^{1/2}e^{-n(\tau-s)A_h}ds . \end{aligned} \tag{6.4}$$

On the other hand, we have

$$\| A_h^{1/2}e^{-n(\tau-s)A_h}\| \leqq C(n(\tau-s))^{-1/2} \tag{6.5}$$

as is seen from (2.15) and Remark 3.5. Also by the inequality

$$\|A_h^\alpha(1+sA_h)^{-k}\| \leqq C_\alpha(ks)^{-\alpha} \qquad (k>\alpha,\ s>0), \tag{6.6}$$

which we shall prove below, we obtain

$$\|A_h^{3/2}(1+sA_h)^{-(n+1)}\| \leqq C((n+1)s)^{-3/2} \leqq C(ns)^{-3/2} \tag{6.7}$$

for $n=1, 2, \cdots$. By means of (6.5) and (6.7) we can estimate $\|K_h\|$ from (6.4), namely,

$$\|K_h\| \leqq C'n\int_0^\tau s(ns)^{-3/2}(n(\tau-s))^{-1/2}ds$$

$$=C'n^{-1}B\left(\frac{1}{2}, \frac{1}{2}\right)=C\tau/(n\tau)=C\tau t^{-1}.$$

Thus we have (6.3). In order to prove (6.6) we define $F_k=F_k(\lambda)$ by

$$F_k(\lambda)=(k\lambda)^\alpha(1+\lambda)^{-k}$$

and write

$$F_k(sA_h)=(ksA_h)^\alpha(I+sA_h)^{-k} \tag{6.8}$$

$$=\frac{1}{2\pi i}\int_{\Gamma_1}F_k(sz)(z-A_h)^{-1}dz,$$

where the path of integration Γ_1 is mentioned in § 3.

For $z\in\Gamma_1$ we note

$$|1+sz|\geqq 1+s\rho\cos\theta_1$$

and, denoting the smallest positive integer $>\alpha$ by k_0, we have for $k\geqq k_0$

$$|1+sz|^k\geqq(1+s\rho\cos\theta_1)^k\geqq 1+\frac{k(k-1)\cdots(k-k_0+1)}{k_0!}(s\rho\cos\theta_1)^{k_0}$$

by means of the binomial theorem. Therefore we can choose a positive constant γ_0 such that

$$|1+sz|^{-k}\leqq(1+\gamma_0(ks\rho)^{k_0})^{-1} \qquad (z\in\Gamma_1). \tag{6.9}$$

In view of $\|(z-A_h)^{-1}\|\leqq C|z|^{-1}$ $(z\in\Gamma_1)$ we obtain from (6.9)

$$\|F_k(sA_h)\|\leqq C\int_0^\infty(ks\rho)^\alpha(1+\gamma_0(ks\rho)^{k_0})^{-1}\frac{d\rho}{\rho}$$

$$=C\int_0^\infty\eta^\alpha(1+\gamma_0\eta^{k_0})^{-1}\frac{d\eta}{\eta}<+\infty,$$

noting $k_0>\alpha$.

In this way we get $\|F_k(sA_h)\|\leqq C_\alpha$ which implies (6.6) and completes the proof of the theorem. Q. E. D.

6.2. Forward difference approximation.

We turn to the forward difference approximation (2.22). Namely, $u_h(t)=(I-\tau A_h)^n a_h$ $(t=n\tau,\ n=1, 2, \cdots)$ is taken as the approximate to $u(t)=e^{-tA}a$. For this approximation, we need a stability condition to be imposed on h and τ. We claim the following

THEOREM 6.4. *Suppose that the condition*

$$\tau\|A_h\|\leqq 2\cos\theta_0 \tag{6.10}$$

is satisfied as τ, $h\to 0$, *where* θ_0 *is the angle defined by* $\theta_0=\tan^{-1}\delta_0$ *in* § 3. *Then the forward difference approximation* (2.22) *is stable. Precisely, we have*

$$\|(I-\tau A_h)^n\| \leqq 2 \qquad (n=1, 2, \cdots). \tag{6.11}$$

PROOF. Take any $\varphi_h \in V_h$ with $\|\varphi_h\|=1$. Then the complex number $\zeta=((I-\tau A_h)\varphi_h, \varphi_h)=1-\tau\sigma(\varphi_h, \varphi_h)$ satisfies $|\zeta|\leqq 1$ as is easily verified in account of $|\arg \tau\sigma(\varphi_h, \varphi_h)|=|\arg \sigma(\varphi_h, \varphi_h)|\leqq \theta_0$ and $|\tau\sigma(\varphi_h, \varphi_h)|\leqq \tau\|A_h\|\leqq 2\cos\theta_0$. Thus the numerical range of the operator $S_h=I-\tau A_h$ is included in the unit disk in the complex plane. Therefore by virtue of a mapping theorem for the numerical range (see Kato [**12**], for instance), the numerical range of S_h^n is also included in the unit disk. Obviously, this implies $\|S_h^n\|\leqq 2$. Q. E. D.

REMARK 6.5. If A_h is self-adjoint, then we can take $\theta_0=0$ and (6.10) is reduced to

$$\frac{\sigma(\varphi_h, \varphi_h)}{\|\varphi_h\|^2} \leqq \frac{2}{\tau}.$$

REMARK 6.6. If σ satisfies, for some $\lambda_1>0$,

$$|\arg(\sigma(\varphi_h, \varphi_h)+\lambda_1\|\varphi_h\|^2)| \leqq \theta_0$$

in place of $|\arg \sigma(\varphi_h, \varphi_h)|\leqq \theta_0$, which occurs when (2.9)′ is assumed, then we have under the condition $\tau(\|A_h\|+\lambda_1)\leqq 2\cos\theta_0$

$$\|(I-\tau(A_h+\lambda_1 I))^n\| \leqq 2 \qquad (n=1, 2, \cdots),$$

whence follows

$$\begin{aligned} \|(I-\tau A_h)^n\| &= \|\{(I-\tau(A_h+\lambda_1 I))+\tau\lambda_1 I\}^n\| \\ &\leqq 2\sum_{k=0}^{n}\binom{n}{k}(\tau\lambda_1)^k \leqq 2(1+\tau\lambda_1)^n \\ &\leqq 2e^{n\tau\lambda_1}=2e^{t\lambda_1} \end{aligned}$$

with the aid of the binomial theorem. Therefore the forward difference approximation is again stable under the above-mentioned condition in this case.

As to the rate of convergence we have

THEOREM 6.7. *Let* $a_h=P_h a$ *in the forward difference finite element approximation* (2.22). *Also suppose that the following stability condition is satisfied:*

$$\sup_{\tau, h} \tau\|A_h\| < 2\cos\theta_0. \tag{6.12}$$

Then for the error

$$\varepsilon(t)=u(t)-u_h(t)=e^{-tA}a-(I-\tau A_h)^n a_h \qquad (t=n\tau,\ n=1, 2, \cdots),$$

we have

$$\|\varepsilon(t)\| \leqq C(h^2+\tau)t^{-1}\|a\|. \tag{6.13}$$

PROOF. Since θ_1 chosen in §3 was arbitrary except for the condition $\theta_0<\theta_1<\pi/2$, we may assume

(6.14)
$$\sup_{\tau,h} \tau\|A_h\| < 2\cos\theta_1$$

without loss of generality.

As before, let us put

$$\varepsilon^{(1)} = e^{-tA}a - e^{-tA_h}a_h$$

and

$$\varepsilon^{(2)} = e^{-tA_h}a_h - (I-\tau A_h)^n a_h \equiv K_h a_h .$$

Again, it is enough to show $\|K_h\| \leqq C\tau t^{-1}$, for $\|\varepsilon^{(1)}\| \leqq Ch^2 t^{-1}\|a\|$ follows from Theorem 4.1. Similarly to (6.4), we have

(6.15)
$$\begin{aligned} K_h &= -\int_0^\tau \frac{d}{ds}((I-sA_h)^n e^{-n(\tau-s)A_h})ds \\ &= n\int_0^\tau sA_h^2(I-sA_h)^{n-1}e^{-n(\tau-s)A_h}ds \\ &= n\int_0^\tau sA_h^{3/2}(I-sA_h)^{n-1}\cdot A_h^{1/2}e^{-n(\tau-s)A_h}ds . \end{aligned}$$

Therefore, if we use (6.5) and the inequality

(6.16)
$$\|A_h^\alpha(I-sA_h)^n\| \leqq C_\alpha(ns)^{-\alpha} \qquad (\alpha>0,\ 0<s\leqq\tau)$$

to be proved below, we obtain for $n\geqq 2$

$$\begin{aligned} \|K_h\| &\leqq C'n\int_0^\tau s((n-1)s)^{-3/2}(n(\tau-s))^{-1/2}ds \\ &= C'n^{1/2}(n-1)^{-3/2}B\Big(\frac{1}{2}, \frac{1}{2}\Big) \leqq Cn^{-1} \\ &= C\tau/(n\tau) = C\tau t^{-1} . \end{aligned}$$

On the other hand, for $n=1$, $\|K_h\| = \|e^{-\tau A_h}-(1-\tau A_h)\| \leqq C = C\tau/\tau = C\tau/t$ is obvious from (6.14). Thus it remains to prove (6.16). To this end we choose positive constants κ and μ subject to

(6.17)
$$\tau\|A_h\|(1+\kappa) \leqq \mu < 2\cos\theta_1 .$$

This is possible in view of (6.14). Then we introduce a positively oriented contour Γ which is composed of the following two portions (as sets):

$$\Gamma^{(1)} = \{re^{\pm i\theta_1};\ 0\leqq r\leqq R\} ,$$

$$\Gamma^{(2)} = \{Re^{i\theta};\ -\theta_1\leqq\theta\leqq\theta_1\} ,$$

where $R=\mu/s$. We define a function $F_n=F_{n,\alpha}$ by

$$F_n(\lambda)=(n\lambda)^\alpha(1-\lambda)^n\,.$$

We want to show $\|F_n(sA_h)\|\leqq C$. Let us consider the Dunford integral

$$F_n(sA_h)=\frac{1}{2\pi i}\int_\Gamma F_n(sz)\frac{1}{z-A_h}\,dz=\frac{1}{2\pi i}(I^{(1)}+I^{(2)})\,,$$

where $I^{(j)}$ $(j=1, 2)$ means the contribution to the integral from $\Gamma^{(j)}$. Suppose that $z=\mathrm{re}^{\pm i\theta_1}\in\Gamma^{(1)}$. Then we have

$$|1-sz|^2=1+s^2r^2-2sr\cos\theta_1 \qquad (0\leqq sr\leqq\mu)\,.$$

Hence in view of $\mu<2\cos\theta_1$, we can choose a positive γ which depends only on θ_1 and μ such that

$$|1-sz|\leqq 1-\gamma sr \qquad (0\leqq sr\leqq\mu)\,.$$

Therefore, we have

$$\|I^{(1)}\|\leqq C\int_0^R(nsr)^\alpha(1-\gamma sr)^n\frac{dr}{r} \tag{6.18}$$

$$\leqq C\int_0^\infty(nsr)^\alpha e^{-\gamma nsr}\frac{dr}{r}=C\int_0^\infty\xi^{\alpha-1}e^{-\gamma\xi}d\xi=C_\alpha^{(1)}\,.$$

We proceed to $I^{(2)}$ and suppose that $z\in\Gamma^{(2)}$. Then

$$\|(z-A_h)^{-1}\|\leqq\frac{1}{|z|}\frac{1}{1-\dfrac{\|A_h\|}{|z|}}=\frac{1}{R}\frac{1}{1-\dfrac{\|A_h\|}{R}}$$

$$\leqq\frac{1}{R}\frac{1}{1-\dfrac{1}{1+\kappa}}=\frac{1}{R}\frac{1+\kappa}{\kappa}$$

by (6.17). Also we can verify that for $z=s^{-1}\mu e^{i\theta}\in\Gamma^{(2)}$

$$|1-sz|\leqq|1-\mu e^{i\theta_1}|=(1+\mu^2-2\mu\cos\theta_1)^{1/2}$$

$$=(1-\mu(2\cos\theta_1-\mu))^{1/2}\equiv\delta<1\,.$$

In this way, we have

$$\|I^{(2)}\|\leqq C\int_{-\theta_1}^{\theta_1}(nsR)^\alpha\cdot\delta^n\cdot\frac{(1+\kappa)}{R\kappa}R\,d\theta\leqq C(n\mu)^\alpha\delta^n\leqq C_\alpha^{(2)} \tag{6.19}$$

since $n^\alpha\delta^n\to 0$ as $n\to\infty$ by $0<\delta<1$. (6.18) and (6.19) yield $\|F_n(sA_h)\|\leqq C_\alpha$ and, hence, (6.16). Q. E. D.

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