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Classification of metrisable regular s-manifolds with integrable symmetry tensor field

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Introduction.

A Riemannian regular s-manifold (M, g, s) is defined in essentially the same way as a Riemannian symmetric space but without the condition that the symmetry at each point should have order 2. In addition a regularity condition (trivially satisfied for symmetric spaces) is imposed on the composition of symmetries. A tensor field S of type (1, 1) is determined by the structure of (M, g, s) and, in turn, characterises it locally (cf. [**2**]). If there exists a Riemannian regular s-manifold structure (M, g, s), then M is a homogeneous space; thus, such structures provide one of the few known examples of a geometric condition on a manifold which implies homogeneity.

A regular s-manifold is called *quadratic* if its (orthogonal) symmetry tensor field S has a quadratic minimal polynomial; thus,

$$S^2-2(\cos\theta)S+I=0 \qquad \text{for} \quad 0<\theta<\pi\,,$$

where θ is called the *angular parameter*. In [**5**] we have given a classification for the compact case.

The purpose of this paper is to investigate those (M, g, s) for which the symmetry tensor field S is integrable in the sense that its Nijenhuis tensor vanishes; thus, for all $X,\ Y\in\mathfrak{X}(M)$

$$S^2[X, Y]-S[SX, Y]-S[X, SY]+[SX, SY]=0\,.$$

In the next section we give the definitions and basic properties for metrisable regular s-manifolds. (A more detailed account of the theory can be found in [**5**], but for completeness we include a summary in the present paper.) The subsequent section gives a statement of our results. Briefly Theorem A shows that integrability of S is equivalent to S being parallel; then the full

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classification of metrisable regular (M, s) with integrable S (Theorem C) is obtained via the simply connected case treated in Theorem B. Moreover, for integrable S it turns out that the Riemannian manifold (M, g) underlying (M, g, s) is Riemannian symmetric (Corollary to Theorem C). This last property is not shared by all Riemannian regular s-manifolds, as we show in an example concluding the final section of proofs.

Definitions and basic properties.

A *regular s-manifold* (M, s) is a smooth, connected manifold M together with a map s from M into the group Diff M of all diffeomorphisms of M, with the following properties:

(i) for each $p \in M$, the point p is an isolated fixed point of the diffeomorphism $s(p)$ (written s_p),

(ii) $s_p \circ s_q = s_{s_p(q)} \circ s_p$ for all $p, q \in M$,

(iii) the tensor field $S: M \to T^1_1(M)$ defined by $p \to S_p = (s_{p*})_p$ is smooth.

The diffeomorphism s_p is referred to as the *symmetry* at p, and S as the *symmetry tensor field*. Any smooth map $\phi: M \to M$ is called *s-preserving* (resp. *S-preserving*) if $\phi \circ s_p = s_{\phi(p)} \circ \phi$ for all $p \in M$ (resp. $\phi_*(SX) = S(\phi_* X)$ for all smooth vector fields X on M). Similarly any tensor field on M is called *s-invariant* if it is invariant under the action of s_p for each $p \in M$. In particular, a *Reimannian regular s-manifold* (M, g, s) is a regular s-manifold together with an s-invariant Riemannian metric g. For many purposes only the existence of such a metric is important, and we say (M, s) is a *metrisable regular s-manifold* if it admits an s-invariant metric.

An important special case arises when there is a smallest integer $k \geqq 2$ such that for each $p \in (M, s)$, s_p has order k, i. e. $(s_p)^k = $identity. We then call (M, s) a *k-symmetric space*. Thus for the Riemannian case, we may regard the symmetric spaces of E. Cartan as metrisable 2-symmetric spaces.

We say (M, s) and (M', s') are *equivalent* if there exists a diffeomorphism $\phi: M \to M'$ such that for all $p \in M$, $\phi \circ s_p = s'_{\phi(p)} \circ \phi$. Similarly, (M, g, s) and (M', g', s') are *equivalent* if there exists an isometry $\phi: (M, g) \to (M', g')$ with the above property. Classification of such manifolds is usually considered within this equivalence.

By analogy with symmetric spaces, it is often useful to consider the simply connected covering space of (M, g, s). We say (M, g, s) is *covered* by $(\tilde{M}, \tilde{g}, \tilde{s})$ if the projection $\pi: \tilde{M} \to M$ satisfies $\tilde{g} = \pi^* g$ and $\pi \circ \tilde{s}_p = s_{\pi(p)} \circ \pi$ for all $p \in \tilde{M}$. For each (M, g, s) there exists $(\tilde{M}, \tilde{g}, \tilde{s})$, unique up to equivalence, such that $\tilde{M}$ is simply connected and (M, g, s) is covered by $(\tilde{M}, \tilde{g}, \tilde{s})$ (cf. Remark 1.8(b) of [5]). Conversely, consider any simply connected $(\tilde{M}, \tilde{g}, \tilde{s})$ and any covering

$\pi:\tilde{M}\to M$ with group Γ of deck transformations. Then M admits a structure (M, g, s) covered by $(\tilde{M}, \tilde{g}, \tilde{s})$ if and only if Γ is a group of $\tilde{s}$-preserving isometries of $(\tilde{M}, \tilde{g}, \tilde{s})$ and is normalised by each $\tilde{s}_p$, $p\in\tilde{M}$ (cf. Proposition 1.6 of [5]).

Consider now (M, g, s) covered by $(\tilde{M}, \tilde{g}, \tilde{s})$, where $(\tilde{M}, \tilde{g})$ is a real Euclidean vector space and the symmetry tensor field $\tilde{S}$ is parallel. Then the group Γ of deck transformations is just a discrete translation group on $\tilde{M}$ such that the orbit $\Gamma(0)$ is invariant by $\tilde{s}_0$, where 0 is the origin of $\tilde{M}$. Let $\tilde{V}_1$ be the subspace of $\tilde{M}$ generated by $\Gamma(0)$ and $\tilde{V}_2$ its orthogonal complement. Now $\tilde{V}_1$ and $\tilde{V}_2$ are invariant under $\tilde{s}_0$; consequently $(\tilde{M}, \tilde{g}, \tilde{s})$ and $(\tilde{V}_1, \tilde{g}_1, \tilde{s}_1)\times(\tilde{V}_2, \tilde{g}_2, \tilde{s}_2)$ are equivalent, where for $i=1, 2$, $\tilde{g}_i$ and $\tilde{s}_i$ are the restrictions to $\tilde{V}_i$ of $\tilde{g}$ and $\tilde{s}$ respectively. Since the restriction of Γ to $\tilde{V}_2$ is trivial, it follows that (M, g, s) is equivalent to $(V_1, g_1, s_1)\times(\tilde{V}_2, \tilde{g}_2, \tilde{s}_2)$, where (V_1, g_1, s_1) is compact and covered by $(\tilde{V}_1, \tilde{g}_1, \tilde{s}_1)$.

Statement of results.

THEOREM A. *Let (M, s) be a metrisable regular s-manifold with symmetry tensor field S. If S is integrable, then $\nabla(S)=0$ for the Riemannian connection ∇ of any s-invariant metric g on M. Conversely, if $\nabla(S)=0$ for some s-invariant metric, then S is integrable.*

THEOREM B. (a) *Let $(\tilde{M}_0, \tilde{g}_0, \tilde{s}_0)$ be a simply connected Riemannian symmetric space with symmetry tensor field $\tilde{S}_0=-I$. For $\lambda=1, 2, \cdots, r$ let $(\tilde{M}_\lambda, \tilde{s}_\lambda, \theta_\lambda)$ be a simply connected quadratic $\tilde{s}_\lambda$-manifold associated with a Hermitian symmetric space $(\tilde{M}_\lambda, \tilde{g}_\lambda)$*† *with complex structure $\tilde{J}_\lambda$ (i.e. $\tilde{g}_\lambda$ is $\tilde{s}_\lambda$-invariant and the symmetry tensor field is given by $\tilde{S}_\lambda=(\cos\theta_\lambda)I+(\sin\theta_\lambda)\tilde{J}_\lambda$), where $0<\theta_1<\theta_2<\cdots<\theta_r<\pi$ are the respective angular parameters. Then the direct product*

$$(\tilde{M}, \tilde{s})=(\tilde{M}_0\times\tilde{M}_1\times\cdots\times\tilde{M}_r,\ \tilde{s}_0\times\tilde{s}_1\times\cdots\times\tilde{s}_r)$$

is a simply connected metrisable regular $\tilde{s}$-manifold with integrable symmetry tensor field $\tilde{S}=\tilde{S}_0\oplus\tilde{S}_1\oplus\cdots\oplus\tilde{S}_r$. particular, In $\tilde{g}=\tilde{g}_0\times\tilde{g}_1\times\cdots\times\tilde{g}_r$ is an $\tilde{s}$-invariant metric and $(\tilde{M}, \tilde{g}, \tilde{s})$ is a Riemannian regular $\tilde{s}$-manifold.

(b) *Conversely, any simply connected Riemannian regular s-manifold with integrable symmetry tensor field is equivalent to an $(\tilde{M}, \tilde{g}, \tilde{s})$ constructed as in* (a).

For the statement of Theorem C we require the following notation:

$(\boldsymbol{R}^n, \tilde{g}_0, \tilde{s}_0)$ is Riemannian symmetric of Euclidean type and $\boldsymbol{Z}^n$ denotes the integer lattice in $\boldsymbol{R}^n$.

$(\boldsymbol{C}^{m_1}, \tilde{g}_1, \tilde{s}_1)$, $(\boldsymbol{C}^{m_2}, \tilde{g}_2, \tilde{s}_2)$, and $(\boldsymbol{C}^{m_3}, \tilde{g}_3, \tilde{s}_3)$ are Riemannian 3-, 4-, 6-symmetric

† We sometimes denote a Riemannian symmetric space simply by (M, g), since its symmetries (of order 2) are uniquely determined.

spaces respectively (cf. Definition 1.2 of [5]), associated with Euclidean Hermitian symmetric spaces $(\boldsymbol{C}^{m_i}, \tilde{g}_i)$ with complex structures $\tilde{J}_i$ for $i=1, 2, 3$. Again for $i=1, 2, 3$, $\{\varepsilon_{i\alpha}, \tilde{J}_i(\varepsilon_{i\alpha})\}_{\alpha=1,2,\cdots,m_i}$ denotes the natural basis of the real vector space $\boldsymbol{R}^{2m_i}$ underlying $\boldsymbol{C}^{m_i}$; then

$$\Delta^{m_1} = \text{the lattice generated by } \{\varepsilon_{1\alpha}, \exp(\pi/3\tilde{J}_1)\varepsilon_{1\alpha}\}_{\alpha=1,2,\cdots,m_1},$$

$$\Sigma^{m_2} = \text{the lattice generated by } \{\varepsilon_{2\alpha}, \tilde{J}_2(\varepsilon_{2\alpha})\}_{\alpha=1,2,\cdots,m_2},$$

$$\Delta^{m_3} = \text{the lattice generated by } \{\varepsilon_{3\alpha}, \exp(\pi/3\tilde{J}_3)\varepsilon_{3\alpha}\}_{\alpha=1,2,\cdots,m_3}.$$

Define

$$(\tilde{M}', \tilde{g}', \tilde{s}') = (\boldsymbol{R}^n, \tilde{g}_0, \tilde{s}_0) \times (\boldsymbol{C}^{m_1}, \tilde{g}_1, \tilde{s}_1) \times (\boldsymbol{C}^{m_2}, \tilde{g}_2, \tilde{s}_2) \times (\boldsymbol{C}^{m_3}, \tilde{g}_3, \tilde{s}_3),$$

$$D' = \boldsymbol{Z}^n \times \Delta^{m_1} \times \Sigma^{m_2} \times \Delta^{m_3}.$$

$(\tilde{M}^+, \tilde{g}^+, \tilde{s}^+)$ denotes a simply connected Riemannian symmetric space of compact type, and D^+ is the centraliser of $I_0(\tilde{M}^+, \tilde{g}^+)$ in $I(\tilde{M}^+, \tilde{g}^+)$.

$$D = D^+ \times D' \text{ and } \pi' \text{ is the projection of } D \text{ onto } D'.$$

THEOREM C. (a) *With the above notation, let Γ be a subgroup of D such that $\pi'(\Gamma)=D'$. Then the manifold $\bar{M}=\tilde{M}^+\times\tilde{M}'/\Gamma$ admits a unique metrisable regular $\bar{s}$-manifold structure $(\bar{M}, \bar{s})$ covered by $(\tilde{M}^+\times\tilde{M}', \tilde{s}^+\times\tilde{s}')$ and $(\bar{M}, \bar{s})$ has integrable symmetry tensor field $\bar{S}$.*

(b) *Conversely, any metrisable regular (M, s) with integrable S is equivalent to the product of an $(\bar{M}, \bar{s})$ constructed as in* (a) *and a simply connected $(\tilde{M}, \tilde{s})$ described as in Theorem B.*†

By Theorem 8.3.12 of [8], the following corollary is an immediate consequence of Theorem C.

COROLLARY. *Let (M, g, s) be any Riemannian regular s-manifold with integrable symmetry tensor field. Then (M, g) is Riemannian symmetric.*

In general, if (M, g, s) is a Riemannian regular s-manifold, then (M, g) need not be Riemannian symmetric; in fact, we conclude the following section with an example of a Riemannian 3-symmetric space not even homeomorphic with any Riemannian symmetric space.

Proofs.

PROOF OF THEOREM A. Let ∇ be the Riemannian connection of an s-invariant metric g on the metrisable regular s-manifold (M, s). Let $G=I_0(M, g, s)$ be the connected group of s-preserving isometries of (M, g, s) as in Definition

† We allow the possibility of $\tilde{M}$, $\tilde{M}^+$ or any factor of $\tilde{M}'$ being a point.

1.10 of [5], and let H be the isotropy group of G at some point in M; then the reductive homogeneous space G/H is diffeomorphic to M (cf. Proposition 1.11 of [5]). Consider the canonical connection of the second kind $\bar{\nabla}$ on G/H [6]. Since S and g are G-invariant tensor fields we have $\bar{\nabla}(S)=0$ and $\bar{\nabla}(g)=0$. The torsion T of $\bar{\nabla}$ is given by $T(X, Y)=\bar{\nabla}_X Y-\bar{\nabla}_Y X-[X, Y]$ for $X, Y\in\mathfrak{X}(M)$, and a short calculation now shows that S is integrable if and only if $T=0$. Consequently, if S is integrable, $\bar{\nabla}$ is a torsion-free metric connection for g, whence $\bar{\nabla}=\nabla$ and so $\nabla(S)=\bar{\nabla}(S)=0$.

The converse is obvious.

PROOF OF THEOREM B. (a) Since each $\tilde{J}_\lambda$ is parallel on $(\tilde{M}_\lambda, \tilde{g}_\lambda)$, so $\tilde{S}_\lambda$ is parallel, $\tilde{S}$ is parallel on $(\tilde{M}, \tilde{g})$ and the conclusion of (a) follows from Theorem A.

(b) Consider any simply connected Riemannian regular $\tilde{s}$-manifold $(\tilde{M}, \tilde{g}, \tilde{s})$ with integrable $\tilde{S}$. For $\lambda=1, 2, \cdots, r$ let $e^{\pm i\theta_\lambda}$ (with $0<\theta_1<\theta_2<\cdots<\theta_r<\pi$) be the distinct eigenvalues of $\tilde{S}$ different from -1. Define the distributions T_0, T_λ as follows:

$$(T_0)_p=\{X\in M_p : \tilde{S}(X)=-X\}$$

$$(T_\lambda)_p=\{X\in M_p : (\tilde{S}^2-2(\cos\theta_\lambda)\tilde{S}+I)X=0\}$$

for $p\in M$, $\lambda=1, 2, \cdots, r$. We observe that the complexification of $(T_\lambda)_p$ is the sum of the eigenspaces of $\tilde{S}_p$ for the eigenvalues $e^{\pm i\theta_\lambda}$. By Theorem A, $\tilde{S}$ is parallel and so $T_0, T_1, \cdots, T_r$ are parallel distributions on $(\tilde{M}, \tilde{g})$.

Choose a point $p'\in\tilde{M}$ and for $\alpha=0, 1, \cdots, r$ let $\tilde{M}_\alpha$ be the maximal integral manifold of T_α through p'. Since $(\tilde{M}, \tilde{g})$ is complete (being Riemannian homogeneous [4]) and also simply connected, then by the de Rham decomposition theorem (cf. [3]) there exists an isometry $\phi:(\tilde{M}, \tilde{g})\to(\tilde{M}_0, \tilde{g}_0)\times(\tilde{M}_1, \tilde{g}_1)\times\cdots\times(\tilde{M}_r, \tilde{g}_r)$ where $\tilde{g}_\alpha$ is the restriction of $\tilde{g}$ to $\tilde{M}_\alpha$.

Consider any $\tilde{M}_\alpha$, $\alpha=0, 1, \cdots, r$, and any point $q\in\tilde{M}_\alpha$. Because $(\tilde{M}_\alpha, \tilde{g}_\alpha)$ is a complete, totally geodesic, topological submanifold of $(\tilde{M}, \tilde{g})$ and $(\tilde{s}_{q*})_q=\tilde{S}_q$ preserves $(T_\alpha)_q$, then $\tilde{s}_q|_{\tilde{M}_\alpha}$ is an isometry of $(\tilde{M}_\alpha, \tilde{g}_\alpha)$ with q as an isolated fixed point. Defining $(\tilde{s}_\alpha)_q=\tilde{s}_q|_{\tilde{M}_\alpha}$, we obtain the Riemannian regular $\tilde{s}_\alpha$-manifold $(\tilde{M}_\alpha, \tilde{g}_\alpha, \tilde{s}_\alpha)$ with parallel symmetry tensor field $\tilde{S}_\alpha$ (=the restriction of $\tilde{S}$ to $\tilde{M}_\alpha$). Observe that $\phi_*\tilde{S}\phi_*^{-1}$ is parallel on $(\tilde{M}_0, \tilde{g}_0)\times(\tilde{M}_1, \tilde{g}_1)\times\cdots\times(\tilde{M}_r, \tilde{g}_r)$ and agrees with the parallel tensor field $\tilde{S}_0\oplus\tilde{S}_1\oplus\cdots\oplus\tilde{S}_r$ at $\phi(p')$; consequently, $\phi_*\tilde{S}\phi_*^{-1}=\tilde{S}_0\oplus\tilde{S}_1\oplus\cdots\oplus\tilde{S}_r$, and so $\phi\circ\tilde{s}_p\circ\phi^{-1}=(\tilde{s}_0)_{p_0}\times(\tilde{s}_1)_{p_1}\times\cdots\times(\tilde{s}_r)_{p_r}$ for all $p\in\tilde{M}$ (where $\phi(p)=(p_0, p_1, \cdots, p_r)$). Thus $(\tilde{M}, \tilde{g}, \tilde{s})$ is equivalent to $(\tilde{M}_0, \tilde{g}_0, \tilde{s}_0)\times(\tilde{M}_1, \tilde{g}_1, \tilde{s}_1)\times\cdots\times(\tilde{M}_r, \tilde{g}_r, \tilde{s}_r)$.

Now $(\tilde{M}_0, \tilde{g}_0, \tilde{s}_0)$ is a Riemannian symmetric space, because T_0 is the (-1)-eigenspace distribution. For $\lambda=1, 2, \cdots, r$, $(\tilde{M}_\lambda, \theta_\lambda)$ is a quadratic s_λ-manifold with angular parameter θ_λ; moreover, the almost complex structure $\tilde{J}_\lambda=(\sin\theta_\lambda)^{-1}(\tilde{S}_\lambda-(\cos\theta_\lambda)I)$ is integrable (because $\tilde{S}_\lambda$ is integrable), hence $\tilde{J}_\lambda$ defines a Hermitian complex structure on $(\tilde{M}_\lambda, \tilde{g}_\lambda)$ (Hermitian because $\tilde{S}_\lambda$ is orthogonal for $\tilde{g}_\lambda$). Since $\tilde{S}_\lambda$ is parallel, $(\tilde{M}_\lambda, \tilde{g}_\lambda)$ is locally symmetric [4]; but being simply connected and complete, $(\tilde{M}_\lambda, \tilde{g}_\lambda)$ is therefore (globally) Riemannian symmetric. For each $q\in\tilde{M}_\lambda$ the 2-symmetry $(\sigma_\lambda)_q$ preserves $\tilde{J}_\lambda$, because $(\sigma_\lambda)_{q*}\circ\tilde{J}_\lambda\circ(\sigma_\lambda)_{q*}^{-1}$ is a parallel tensor field agreeing with $\tilde{J}_\lambda$ at q. Thus, for $\lambda=1, 2, \cdots, r$, $(\tilde{M}_\lambda, \tilde{g}_\lambda)$ is a Hermitian symmetric space with respect to $\tilde{J}_\lambda$. This completes the proof of (b) of Theorem B.

PROOF OF THEOREM C. (a) With the hypotheses of (a), Γ acts properly discontinuously on $(\tilde{M}^+\times\tilde{M}', \tilde{g}^+\times\tilde{g}')$ as a group of Clifford translations; moreover, Γ is $(\tilde{s}^+\times\tilde{s}')$-preserving and is normalised by the $(\tilde{s}^+\times\tilde{s}')$-symmetries. The result of (a) now follows from earlier remarks.

(b) Let $(\tilde{M}, \tilde{g}, \tilde{s})$ be the simply connected covering space of (M, g, s) where g is an s-invariant metric on M; let Γ be the group of deck transformations of the covering. Since S is integrable, the symmetry tensor field $\tilde{S}$ of $(\tilde{M}, \tilde{s})$ is integrable. Decompose $(\tilde{M}, \tilde{g}, \tilde{s})$ as in Theorem B. Now for $\alpha=0, 1, \cdots, r$, the Riemannian symmetric space $(\tilde{M}_\alpha, \tilde{g}_\alpha)$ decomposes into the direct produt $(\tilde{M}_\alpha^0, \tilde{g}_\alpha^0)\times(\tilde{M}_\alpha^+, \tilde{g}_\alpha^+)\times(\tilde{M}_\alpha^-, \tilde{g}_\alpha^-)$ where (0), (+) and (−) designate factors of Euclidean, compact, and non-compact type respectively; in the Hermitian case (i. e. $\alpha>0$) the complex structure $\tilde{J}_\alpha$ decomposes correspondingly as $\tilde{J}_\alpha^0\oplus\tilde{J}_\alpha^+\oplus\tilde{J}_\alpha^-$. It follows that:

$$\tilde{M}=\tilde{M}_0^0\times\tilde{M}_0^+\times\tilde{M}_0^-\times\tilde{M}_1^0\times\tilde{M}_1^+\times\tilde{M}_1^-\times\cdots\times\tilde{M}_r^0\times\tilde{M}_r^+\times\tilde{M}_r^-,$$

$$\tilde{g}=\tilde{g}_0^0\times\tilde{g}_0^+\times\tilde{g}_0^-\times\tilde{g}_1^0\times\tilde{g}_1^+\times\tilde{g}_1^-\times\cdots\times\tilde{g}_r^0\times\tilde{g}_r^+\times\tilde{g}_r^-,$$

$$\tilde{s}=\tilde{s}_0^0\times\tilde{s}_0^+\times\tilde{s}_0^-\times\tilde{s}_1^0\times\tilde{s}_1^+\times\tilde{s}_1^-\times\cdots\times\tilde{s}_r^0\times\tilde{s}_r^+\times\tilde{s}_r^-.$$

Since (M, g) is Riemannian homogeneous, Γ must be a group of Clifford translations of $(\tilde{M}, \tilde{g})$ (cf. [9]). Then each $\gamma\in\Gamma$ may be written

$$\gamma=\gamma_0^0\times\gamma_0^+\times\gamma_0^-\times\gamma_1^0\times\gamma_1^+\times\gamma_1^-\times\cdots\times\gamma_r^0\times\gamma_r^+\times\gamma_r^-,$$

where for $\alpha=0, 1, \cdots, r$, γ_α^0 and $\gamma_\alpha^\pm$ are Clifford translations of $(\tilde{M}_\alpha^0, \tilde{g}_\alpha^0)$ and $(\tilde{M}_\alpha^\pm, \tilde{g}_\alpha^\pm)$ respectively. Define the projections ρ_α^0, $\rho_\alpha^\pm$ of Γ onto its various factors by $\rho_\alpha^0(\gamma)=\gamma_\alpha^0$, $\rho_\alpha^\pm(\gamma)=\gamma_\alpha^\pm$ for each $\gamma\in\Gamma$.

Notice immediately that for $\alpha=0, 1, \cdots, r$, $\rho_\alpha^-(\Gamma)$ is trivial, because symmetric spaces of non-compact type have no non-trivial Clifford translations.

Since $(\tilde{M}, \tilde{s})$ covers (M, s), each $\gamma\in\Gamma$ is $\tilde{s}$-preserving (equivalently $\tilde{S}$-preserving) and Γ is normalised by each $\tilde{s}$-symmetry $\tilde{s}_p$. Consequently, $\rho_\alpha^0(\Gamma)$ (resp. $\rho_\alpha^+(\Gamma)$) is both $(\tilde{s}_\alpha^0)$-preserving (resp. $(\tilde{s}_\alpha^+)$-preserving) and also normalised by each $(\tilde{s}_\alpha^0)$-symmetry (resp. $(\tilde{s}_\alpha^+)$-symmetry). Because a Hermitian symmetric space of compact type admits no non-trivial Clifford translations which preserve the complex structure, it follows that for $\alpha>0$, $\rho_\alpha^+(\Gamma)$ is trivial. For

$\alpha=0$, $\tilde{M}_0^+/\rho_0^+(\Gamma)$ admits a Riemannian symmetric structure covered by $(\tilde{M}_0^+, \tilde{g}_0^+)$, whence $\rho_0^+(\Gamma)$ is a subgroup of the centraliser of $I_0(\tilde{M}_0^+, \tilde{g}_0^+)$ in $I(\tilde{M}_0^+, \tilde{g}_0^+)$ (cf. [8], Theorem 8.3.12).

We now turn to the Euclidean factors. For $\alpha=0, 1, \cdots, r$, $\rho_\alpha^0(\Gamma)$ is an $\tilde{S}_\alpha$-invariant lattice in the Euclidean vector space $\tilde{M}_\alpha^0$, so for $\alpha>0$, $\rho_\alpha^0(\Gamma)$ is trivial unless $\theta_\alpha=\pi/3$, $\pi/2$ or $2\pi/3$ (cf. Theorem A of [5]).

It follows that up to equivalence we have the following decomposition

$$\tilde{M}=\tilde{M}_0^+\times \boldsymbol{R}^n\times \boldsymbol{C}^{m_1}\times \boldsymbol{C}^{m_2}\times \boldsymbol{C}^{m_3}\times \tilde{M}_4$$

$$\tilde{g}=\tilde{g}_0^+\times\tilde{g}_0\times\tilde{g}_1\times\tilde{g}_2\times\tilde{g}_3\times\tilde{g}_4$$

$$\tilde{s}=\tilde{s}_0^+\times\tilde{s}_0\times\tilde{s}_1\times\tilde{s}_2\times\tilde{s}_3\times\tilde{s}_4\,,$$

where $(\tilde{M}_0^+, \tilde{g}_0^+, \tilde{s}_0^+)$ is (as above) Riemannian symmetric of compact type, $(\boldsymbol{R}^n, \tilde{g}_0, \tilde{s}_0)$ is a Euclidean symmetric space, $(\boldsymbol{C}^{m_1}, \tilde{g}_1, \tilde{s}_1)$, $(\boldsymbol{C}^{m_2}, \tilde{g}_2, \tilde{s}_2)$, and $(\boldsymbol{C}^{m_3}, \tilde{g}_3, \tilde{s}_3)$ are complex Euclidean 3-, 4- and 6-symmetric respectively as described prior to the statement of Theorem C, $(\tilde{M}_4, \tilde{g}_4, \tilde{s}_4)$ is a simply connected Riemannian regular $\tilde{s}_4$-manifold with integrable symmetry tensor field, and where we can suppose that the restriction of Γ to M_4 is trivial, that $\tilde{M}_0^+\times\boldsymbol{R}^n\times\boldsymbol{C}^{m_1}\times\boldsymbol{C}^{m_2}\times\boldsymbol{C}^{m_3}/\Gamma$ is compact, and that the projections of Γ onto $\boldsymbol{R}^n$, $\boldsymbol{C}^{m_1}$, $\boldsymbol{C}^{m_2}$, $\boldsymbol{C}^{m_3}$ are the lattices $\boldsymbol{Z}^n$, Δ^{m_1}, Σ^{m_2}, Δ^{m_3} respectively. Moreover, decomposing each $\gamma\in\Gamma$ as $\gamma=(\gamma_0^+, \gamma_0, \gamma_1, \gamma_2, \gamma_3, id)$ where $\gamma_0^+, \gamma_0, \gamma_1, \gamma_2, \gamma_3$ are Clifford translations of the respective factors, let ρ be the projection of Γ onto $\boldsymbol{R}^n\times\boldsymbol{C}^{m_1}\times\boldsymbol{C}^{m_2}\times\boldsymbol{C}^{m_3}$; thus, $\rho(\gamma)=(\gamma_0, \gamma_1, \gamma_2, \gamma_3)$. Since each symmetry (hence its square) normalises Γ and $(\tilde{S}_0)^2=I$, from $(\gamma_0, \gamma_1, \gamma_2, \gamma_3)\in\rho(\Gamma)$ we deduce that

$$(\gamma_0, (\tilde{S}_1)^2\gamma_1, (\tilde{S}_2)^2\gamma_2, (\tilde{S}_3)^2\gamma_3)\in\rho(\Gamma)\,,$$

and so $(0, (I-(\tilde{S}_1)^2)\gamma_1, (I-(\tilde{S}_2)^2)\gamma_2, (I-(\tilde{S}_3)^2)\gamma_3)\in\rho(\Gamma)$. Since $\tilde{S}_1$, $\tilde{S}_2$ and $\tilde{S}_3$ have no eigenvalue ±1, we conclude that $(0, \gamma_1, \gamma_2, \gamma_3)\in\rho(\Gamma)$ and hence $(\gamma_0, 0, 0, 0)\in\rho(\Gamma)$. Likewise using $(\tilde{S}_1)^3=(\tilde{S}_2)^4=(\tilde{S}_3)^6=I$, we conclude $(0, \gamma_1, 0, 0)$, $(0, 0, \gamma_2, 0)$ and $(0, 0, 0, \gamma_3)\in\rho(\Gamma)$; thus, $\rho(\Gamma)=\boldsymbol{Z}^n\times\Delta^{m_1}\times\Sigma^{m_2}\times\Delta^{m_3}$. This completes the proof of Theorem C.

EXAMPLE. From Theorem 6.1 of [10], $SU(3)$ admits an inner automorphism θ of order 3 such that a maximal torus T^2 of $SU(3)$ is the fixed point set of θ. So by Proposition 1.19 of [5], $M=SU(3)/T^2$ admits the structure (M, g, s) of a Riemannian regular s-manifold which is 3-symmetric. We claim that M is not homeomorphic with the underling manifold M' of any Riemannian symmetric space.

Since $M=SU(3)/T^2$ has dimension 6 and Euler characteristic 6, examination of the classification of Riemannian symmetric spaces (cf. [8]) shows that the only possibility for M' is $P^2(\boldsymbol{C})\times S^2$, the direct product of the complex projective space $P^2(\boldsymbol{C})$ (of (real) dimension 4, Euler characteristic 3, —cf. Theorem 8.10.10 of [8]) and the sphere S^2 (of dimension 2 and Euler characteristic 2). Consider the exact homotopy sequence (cf. [7])

$$\longrightarrow \pi_4(SU(3)) \longrightarrow \pi_4(SU(3)/T^2) \longrightarrow \pi_3(T^2) \longrightarrow .$$

Since $U(3)$ is homeomorphic to $S^1\times SU(3)$ (Proposition 7, § X, Chapter II of [1]) and $\pi_4(S^1)=0$ (§ 21.2 of [7]), we have by § 17.8 and § 25.4 of [7] that $\pi_4(SU(3))=0$. Moreover, $\pi_3(T^2)=0$, and so by the exactness of the above sequence we conclude that $\pi_4(SU(3)/T^2)=0$. But $\pi_4(P^2(\boldsymbol{C})\times S^2)=\pi_4(P^2(\boldsymbol{C}))+\pi_4(S^2)\neq 0$, because $\pi_4(S^2)=\boldsymbol{Z}_2$ (§ 21.7 of [7]). This establishes the above claim.

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