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Comments on Satake compactification and the great Picard theorem

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§ 1. Introduction.

The purpose of this note is to clarify some points in [3], [4], [7] concerning Pyatezkii-Šapiro compactification and hyperbolic imbedding of an arithmetic quotient of a symmetric domain into its compactification. In order to explain our results, we need to consider the concept of hyperbolic imbedding in non-Hausdorff spaces.

DEFINITION. Let Z be a compact, second countable topological space (which is not necessarily Hausdorff) and let $Y\subset Z$ be an open set which is a complex (Hausdorff) space. We say that Y is *hyperbolically imbedded in* Z if the following two conditions are satisfied:

(1) Y is hyperbolic, i.e., if the intrinsic pseudo-distance d_Y is a true distance;

(2) For every $z\in\partial Y(=\bar{Y}-Y)$ and every open neighborhood U of z in Z, there exists a smaller open neighborhood $V\subset\bar{V}\subset U$ such that

$$d_Y(V\cap Y, Y-V)>0.$$

Note that if Z is Hausdorff, then condition (2) can be replaced by

(2′) For all sequences $\{p_n\}$ and $\{q_n\}$ in Y such that $p_n\to p\in\partial Y$ and $q_n\to q\in\partial Y$ and such that $d_Y(p_n, q_n)\to 0$, we have $p=q$.

If Z is not Hausdorff, (2′) is stronger than (2).

Let $\mathcal{D}$ be a symmetric bounded domain and Γ an arithmetically defined discontinuous group of automorphisms of $\mathcal{D}$. Let $Y=\Gamma\backslash\mathcal{D}$. Let Y^s denote the Satake compactification of Y defined in [9]. By Baily-Borel [2], Y^s is a normal complex projective variety. On the other hand, Pyatezkii-Šapiro [8] compactified Y by introducing a topology in the set Y^s by a different method. We denote this compactification by Y^p. By [1], the identity map $i\colon Y^s\to Y^p$ is continuous, i.e., the topology of Y^p is at least as coarse as that of Y^s. Until recently, it has been a haunting question whether the identity map i is

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a homeomorphism or, equivalently, if Y^p is Hausdorff. In the meantime the following theorems have been obtained:

THEOREM 1 [7]. *Y is hyperbolically imbedded in Y^p.*

THEOREM 2 [3]. *Y is hyperbolically imbedded in Y^s.*

In both theorems, the intrinsic distance d_Y has to be modified when the action of Γ on $\mathscr{D}$ is not free. For this technical point, see [7]. Clearly, Theorem 2 is stronger than Theorem 1, but its proof is more involved. Making use of Theorem 2 and the result of our earlier paper [5], one of us [4] showed that Y^p is also Hausdorff. According to a private communication from Borel, the fact that Y^p is Hausdorff can be established by means of Borel-Serre's theory of corners, but his proof is rather involved and has not been written up.

In the next section we shall show that Theorem 1 easily implies Theorem 2. As a consequence, the argument in [4] now yields a relatively simple proof that Y^p is Hausdorff.

§ 2. Proof of Theorem 2.

Let $B_1, \cdots, B_k$ be the boundary components of Y^s so that $\partial Y = Y^s - Y = \cup B_i$. The fact that there are only finitely many boundary components plays an essential rôle.

Let $p \in \partial Y$ and let A be the subset of ∂Y consisting of those points $q \in \partial Y$ satisfying the following condition:

"There exist sequences $\{p_n\}$ and $\{q_n\}$ in Y such that $p_n \to p$ and $q_n \to q$ in Y^s and such that $d_Y(p_n, q_n) \to 0$."

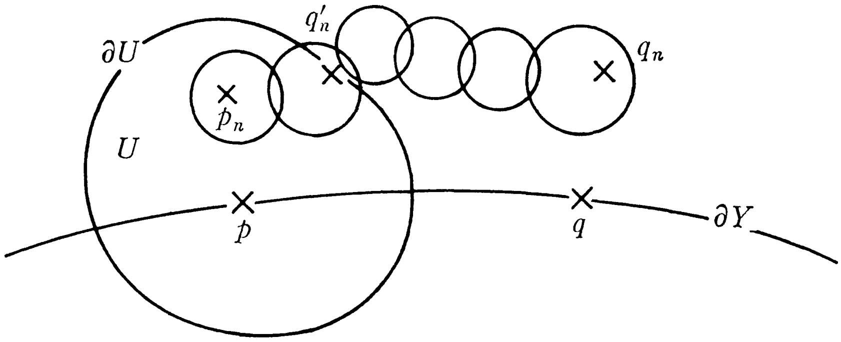
We must show that A contains only one point p. We first show that A is a finite set. It suffices to show that each boundary component B_i contains at most one point of A. Assume the contrary. Without loss of generality we assume that B_1 contains two points of A, say q and r. Then there exist disjoint open sets U_1 and U_2 in Y^p (not only in the topology of Y^s but also in the topology of Y^p!) such that $q \in U_1$ and $r \in U_2$. (This follows immediately from the way Pyatezkii-Šapiro defines his topology and from the condition that q and r are in the same boundary component). This contradicts Theorem 1, thus showing that A is a finite set.

Now we want to show that p is the only point in A. Assume that there is another point q in A. Then we have sequences $\{p_n\}$ and $\{q_n\}$ in Y such that $p_n \to p$ and $q_n \to q$ in Y^s and such that $d_Y(p_n, q_n) \to 0$.

Let U be an open neighborhood of p in Y^s such that

$$A \cap U = A \cap \bar{U} = \{p\}$$

where $\bar{U}$ is the closure of U in Y^s. Such an open set U exists because A is a finite set. In particular, q is not in $\bar{U}$. We may also assume that none of the q_n are in $\bar{U}$ and every p_n is in U. If we recall the definition of d_Y, we see that $d_Y(p_n, q_n)$ can be approximated by the "length" of a chain of analytic disks from p_n to q_n and this chain meets the boundary ∂U of U. Hence there exists a sequence $\{q'_n\} \subset \partial U \cap Y$ such that $d_Y(p_n, q'_n) \to 0$, (see the figure).



By taking a subsequence, we may assume that $q'_n \to q' \in \partial U \cap \partial Y$. Clearly, q' is in A. But this is a contradiction since $\partial U \cap A = \emptyset$. This completes the proof of Theorem 2.

§ 3. Proof that Y^p is Hausdorff.

We repeat the argument in [4] for the convenience of the reader. In [5] we proved the following

THEOREM 3. *Let M be a complex space hyperbolically imbedded in a (Hausdorff) complex space W. Then every holomorphic map $f: Y\ (=\Gamma \backslash \mathcal{D}) \to M$ extends to a continuous map $\bar{f}: Y^p \to W$.*

(We are referring to Theorem 1 on p. 245 of [5], which was stated for Y^s, but we used only the weaker topology Y^p in the proof).

We apply Theorem 3 to the following situation:

$$M = Y, \quad W = Y^s, \quad f = j: Y \to Y \qquad \text{(the identity map)}.$$

Since Y is hyperbolically imbedded in Y^s by Theorem 2, we can conclude that j extends to a continuous map $\bar{j}: Y^p \to Y^s$. Clearly, $\bar{j}$ is the inverse of $i: Y^s \to Y^p$. This completes the proof of the fact that $i: Y^s \to Y^p$ is a homeomorphism and hence Y^p is Hausdorff.

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