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On real hypersurfaces of a complex projective space

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§ 0. Introduction.

Let $P^m(C)$ denote a complex projective space equipped with the Fubini-Study metric normalized so that the maximum sectional curvature is 4. We consider a real hypersurface M of $P^m(C)$. It is well-known that there does not exist a totally umbilical real hypersurface of $P^m(C)$ (See Tashiro-Tachibana [7].) More generally, there does not exist a real hypersurface of $P^m(C)$ with the parallel second fundamental tensor. This is immediately seen from the Codazzi equation of the immersion of M. From this point of view, in this paper, we will estimate the norm of the derivative of the second fundamental tensor, and we get,

THEOREM A. *Let M be a complete real hypersurface of $P^m(C)$. Then $\|\nabla H\|^2 \geqq 4(m-1)$, the equality holds if and only if M is congruent to $M^c_{p,q}$ for some p, q.*

The model space $M^c_{p,q}$ in the above theorem is discribed in the following.

Let S^{2m+1} be a Euclidean $(2m+1)$-sphere of curvature 1. We consider the Hopf fibration $\tilde{\pi}$:

$$S^1 \longrightarrow S^{2m+1} \xrightarrow{\tilde{\pi}} P^m(C)\,,$$

which is the Riemannian submersion with totally geodesic fibres.

Let $\bar{M}$ and M be Riemannian manifolds of dimension $2m$, $2m-1$ respectively and $\pi: \bar{M} \to M$ be a differentiable map. $(\bar{M}, M, \pi)$ is called a *Riemannian submersion compatible with the Hopf fibration* $\tilde{\pi}$ if the following conditions are satisfied.

(S1) $\bar{M}$ and M are (real) hypersurfaces of S^{2m+1} and $P^m(C)$ respectively.

(S2) $\pi: \bar{M} \to M$ is a Riemannian submersion with totally geodesic fibres such that the following diagram commutes:

$$\begin{array}{ccc} \bar{M} & \xrightarrow{\quad \bar{i} \quad} & S^{2m+1} \\ \downarrow \pi & & \downarrow \tilde{\pi} \\ M & \xrightarrow{\quad i \quad} & P^m(C) \end{array}$$

where $\bar{i}$ and i denote the immersions in (S1).

To consider a model space $M^c_{p,q}$ in this situation, we take a family of the products of spheres $M_{n,n'}=S^n\times S^{n'}$, where $n+n'=2m$. Choosing n and n' to be odd, namely $n=2p+1$, $n'=2q+1$, we put $\bar{M}=M_{2p+1,2q+1}$. Then we get a fibration π:

$$S^1 \longrightarrow M_{2p+1,2q+1} \overset{\pi}{\longrightarrow} M^c_{p,q}\,.$$

$(M_{2p+1,2q+1}, M^c_{p,q}, \pi)$ satisfies (S1) and (S2) (cf. [**2**], [**3**]).

$M^c_{p,q}$ thus obtained has a characteristic property, which can be used to prove M to be congruent to $M^c_{p,q}$ for some p, q. In general, a real hypersurface M of $P^m(C)$ has two structures, namely the contact structure induced from $P^m(C)$ and the submanifold structure represented by the second fundamental tensor of M on $P^m(C)$. It might be interesting to study the relations between the two structures. In particular, for the model space $M^c_{p,q}$, the relation is precisely obtained through the study of the submersion π. Okumura [**3**] proved the following theorem which is a characterization of $M^c_{p,q}$.

THEOREM 0. *Let M be a real hypersurface of $P^m(C)$ and $\pi:\bar{M}\to M$ the submersion which is compatible with the Hopf fibration $\tilde{\pi}: S^1\to S^{2m+1}\to P^m(C)$. Then the second fundamental tensor of $\bar{M}$ is parallel if and only if the contact structure of M induced from $P^m(C)$ commutes with the second fundamental tensor of M.*

Subsequently, a further observation on $M^c_{p,q}$ will be made. By use of the compatible submersion π, the hypersurface M of $P^m(C)$ related to $\bar{M}$ has been studied in [**2**], [**3**] and [**6**]. Namely, Lawson [**2**] studied the pinching problem of the second fundamental tensor when M is a minimal hypersurface of $P^m(C)$, and Okumura [**3**] also studied the pinching problem on the more general condition that the hypersurface M has the constant mean curvatute.

When $\bar{M}$ is 1) an Einstein space or 2) a locally symmetric space, it is well known that $\bar{M}$ has parallel second fundamental tensor. Projecting the quantities on $\bar{M}$ onto M in $P^m(C)$, we can consider the hypersurface with the conditions corresponding to 1) or 2). Using Theorem 0, we will study the above hypersurfaces in §4 and §5.

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§1. Preliminaries.

Let M be a real hypersurface of $P^m(C)$ and $i: M\to P^m(C)$ denote the isometric immersion. In a neighborhood of each point, we choose a unit normal vector field N in $P^m(C)$. The Riemannian connections D in $P^m(C)$ and ∇ in M are related by the following formulas for arbitrary vector fields X and Y

on M:

$$D_{i_*X}i_*Y=i_*(\nabla_X Y)+g(HX, Y)N, \tag{1.1}$$

$$D_{i_*X}N=-i_*(HX), \tag{1.2}$$

where g denotes the Riemannian metric induced from the Fubini-Study metric G on $P^m(C)$, i.e., $g(X, Y)=G(i_*X, i_*Y)$, and H is the second fundamental tensor of M in $P^m(C)$.

The mean curvature μ of M in $P^m(C)$ is defined by $\mu=\operatorname{trace} H$. If $\mu=0$, then M is called a *minimal hypersurface*.

An eigenvector X of the second fundamental tensor H is called a *principal curvature vector*, or simply a *P.C. vector*. Also an eigenvalue r of H is called a *principal curvature*. In what follows, we denote V_r the eigenspace of H with eigenvalue r.

It is known that M has an almost contact metric structure induced from the complex structure F on $P^m(C)$, (cf. [3]), i.e., we define a tensor f of type (1, 1), a vector field U and a 1-form u on M by the following:

$$g(fX, Y)=G(Fi_*X, i_*Y), \quad g(U, X)=u(X)=G(Fi_*X, N).$$

Then we have

$$f^2X=-X+u(X)U, \quad g(U, U)=1, \quad fU=0. \tag{1.3}$$

From the above remark and (1.1), we have easily

$$(\nabla_X f)Y=u(Y)HX-g(HY, X)U, \tag{1.4}$$

$$\nabla_Y U=fHY. \tag{1.5}$$

Let $\bar{R}$ and R be the curvature tensors of $P^m(C)$ and M respectively. Since the curvature tensor $\bar{R}$ has a nice form, we have the following Gauss and Codazzi equations.

$$\begin{aligned} g(R(X, Y)Z, W)=&\,g(Y, Z)g(X, W)-g(X, Z)g(Y, W)\\ &+g(fY, Z)g(fX, W)-g(fX, Z)g(fY, W)\\ &-2g(fX, Y)g(fZ, W)+g(HY, Z)g(HX, W)\\ &-g(HX, Z)g(HY, W) \end{aligned} \tag{1.6}$$

and

$$(\nabla_X H)Y-(\nabla_Y H)X=u(X)fY-u(Y)fX-2g(fX, Y)U. \tag{1.7}$$

Using (1.3), (1.6) and (1.7), we get

(1.8) $$g(R_0X, Y)=(2m+1)g(X, Y)-3u(X)u(Y)+\mu g(HX, Y)-g(H^2X, Y),$$

where μ=trace H and R_0 denotes the Ricci tensor on M.

(1.9) $$g((\nabla_X H)Y, U)-g((\nabla_Y H)X, U)=-2g(fX, Y).$$

§ 2. The fundamental lemmas on a real hypersurface of $P^m(C)$.

Let M be a real hypersurface of $P^m(C)$ and assume that the trajectories of the induced vector field U are geodesics, i. e.,

(2.1) $$\nabla_U U=0,$$

because U is a unit vector. Using (1.5), (2.1) becomes

(2.2) $$fHU=0.$$

Applying f to (2.2) and using (1.3), we get

(2.3) $$HU=\alpha U,$$

where $\alpha=g(HU, U)$. Thus we have

LEMMA 2.1. *In order that the trajectories of U be geodesics, it is necessary and sufficient that U be a P. C. vector.*

Differentiating (2.3) covariantly along X and making use of (1.4), we have

$$g((\nabla_X H)Y, U)+g(HfHX, Y)=(X\alpha)g(U, Y)+\alpha g(fHX, Y).$$

Making a similar equation by changing X and Y in the last equation and using (1.9), we get

(2.4) $$2g(HfHX-fX, Y)=(X\alpha)u(Y)-(Y\alpha)u(X)+g((fH+Hf)X, Y).$$

If we replace X by U in (2.4), we obtain

(2.5) $$Y\alpha=(X\alpha)u(Y).$$

Substituting (2.5) into (2.4), we have

(2.6) $$2HfH-2f=\alpha(Hf+fH).$$

LEMMA 2.2. *Assume that the trajectories of U are geodesics. If X belongs to V_r and is orthogonal to U, then fX belongs to $V_{(\alpha r+2)/(2r-\alpha)}$.*

PROOF. From (2.6), we get for a P.C. vector X which is orthogonal to U,

$$(2r-\alpha)HfX=(\alpha r+2)fX.$$

If $2r-\alpha=0$, then $\alpha r+2=0$. Hence we have the Lemma.

From Lemma 2.2, we easily obtain

PROPOSITION 2.3. *There exists no open set* O *of* M *such that at every point of* O, $fH+Hf=0$.

LEMMA 2.4. *If the trajectories of* U *are geodesics, then* α *is locally constant.*

PROOF. Since U is a P.C. vector of M, from Lemma 2.2 we get by (2.5), $\operatorname{grad}\alpha=\beta U$, where $\beta=U\alpha$. Differentiating this equation covariantly along X, we have

$$\nabla_X \operatorname{grad}\alpha=(X\beta)U+\beta fHX\,,$$

from which, together with the fact that

$$g(\nabla_X \operatorname{grad}\alpha,\, Y)=g(\nabla_Y \operatorname{grad}\alpha,\, X)\,,$$

we get

$$(Y\beta)u(Y)-(X\beta)u(X)=\beta g((fH+Hf)X,\, Y)\,. \tag{2.7}$$

Replacing X by U and making use of (2.5) and (2.6), we have

$$Y\beta=(U\beta)g(U,\, Y)\,. \tag{2.8}$$

Substituting (2.8) into (2.7), we obtain

$$\beta\cdot g((fH+Hf)X,\, Y)=0\,.$$

Thus we have the lemma by Proposition 2.3.

At each point, we can take orthonormal vectors U, X_a, fX_a $(a=1, \cdots, m-1)$ which are P.C. vectors. Then any tangent vector can be expressed in the following form:

$$X=xU+\sum_{a=1}^{m-1}x^a X_a+\sum_{a=1}^{m-1}y^a fX_a\,.$$

Using the above expression of X, we get

PROPOSITION 2.5. *Let* M *be a real hypersurface of* $P^m(C)$ *and assume that the trajectories of* U *are geodesics. Assume that* fX *belongs to* V_r *for any* $X\in V_r$. *Then* f *and* H *are commutative. Furthermore by Theorem* 0, *for the submersion* $(\bar{M}, M, \pi)$ *compatible with* $\tilde{\pi}$, $\bar{M}$ *has the parallel second fundamental tensor.*

§ 3. Proof of Theorem A.

For a compatible submersion $(\bar{M}, M, \pi)$ with the Hopf fibration $\tilde{\pi}$, it is well known (cf. Ishihara and Konishi [1]) that if $\bar{M}$ has the parallel second fundamental form, M satisfies

$$g((\nabla_Z H)X, Y) = -u(X)g(fZ, Y) - u(Y)g(fZ, X). \tag{3.1}$$

Now, we consider the converse problem, namely we determine the hypersurface M satisfying (3.1).

From (3.1) and the commutativity of the trace and the derivation, we have

LEMMA 3.1. *If M satisfies* (3.1), *then the mean curvature is constant.*

Using the Ricci identity, (3.1) and (1.9), we get

$$\begin{aligned} &g(HY, W)g(LX, Z) + g(HY, Z)g(LX, W) - g(HX, W)g(LY, Z) \\ &-g(HX, Z)g(LY, W) - g(fX, W)g(AY, Z) - g(fX, Z)g(AY, W) \\ &+g(fY, Z)g(AX, W) + g(fY, W)g(AX, Z) - 2g(fX, Y)g(AZ, W) \\ &= 0, \end{aligned} \tag{3.2}$$

where L and A are tensor fields of type (1, 1) which are respectively defined by the following:

$$LX = X - u(X)U - H^2X,$$

$$AX = (fH - Hf)X.$$

Then L and A are symmetric linear operators. If $A=0$, then f and H are commutative.

Contracting (3.2) with X and W, we have

$$\begin{aligned} &\mu g(LY, Z) - (2m+2-\text{trace } H^2)g(HY, Z) + 2g(HZ, U)u(Y) \\ &\qquad + 2g(HY, U)u(Z) - 4g(fHfY, Z) = 0. \end{aligned} \tag{3.3}$$

Replacing Y by U in (3.3) and using (1.3), we have

$$\mu g(H^2X, U) = 2\alpha u(X) - (2m - \text{trace } H^2)g(HX, U), \tag{3.4}$$

where $\alpha = g(HU, U)$.

On the other hand, replacing X and Z by U in (3.2) and exchanging Y and W, we get

$$g(HY, U)g(H^2W, U) = g(HW, U)g(H^2Y, U). \tag{3.5}$$

Considering (3.5), we get, for some scalar a,

$$g(H^2X, U) = ag(HX, U), \tag{3.6}$$

because of Schwarz's inequality.

Substituting (3.6) into (3.4), we have

(3.7) $$bg(HX, U)=2g(HU, U)u(X),$$

where $b=a\mu+2m-\text{trace}\, H^2$.

LEMMA 3.2. *For any point $p\in M$, U is a P.C. vector.*

PROOF. If $b\neq 0$, then U is a P.C. vector by (3.7). If $b=0$, then $g(HU, U)=0$, and we easily obtain $HU=0$ by (3.5).

We can put $HU=\alpha U$ for any point $p\in M$ because of Lemma 3.2. Then by Lemma 2.4, we see that α is constant.

Differentiating this equation and using (3.2), we get

(3.8) $$\alpha g(fHX, Y)=-g(fX, Y)+g(HfHX, Y).$$

Interchanging X and Y in (3.8), we have $\alpha g(AX, Y)=0$.

Now we prove

PROPOSITION 3.3. *Let M be a real hypersurface of $P^m(C)$ satisfying* (3.1). *Then f and H are commutative.*

PROOF. If $\alpha\neq 0$, it is clear from (3.8). In case $\alpha=0$, replacing W by fW in (3.2) and contracting X and W, we get

$$(2m+2)g(AX, Y)=0.$$

This means $A=0$. By Theorem 0, we have

THEOREM 3.4. *For a submersion $(\bar{M}, M, \pi)$ compatible with the Hopf fibration $\tilde{\pi}: S^1\to S^{2m+1}\to P^m(C)$, the second fundamental tensor of $\bar{M}$ is parallel if and only if M satisfies* (3.1).

From this fact and theorems in Ryan's paper [4], we have

THEOREM 3.5. *$M^c_{p,q}$ are only complete hypersurfaces of $P^m(C)$ satisfying* (3.1).

Define a tensor T by

$$T(X, Y)Z=g((\nabla_Z H)X, Y)+u(X)g(fZ, Y)+u(Y)g(fZ, X).$$

Calculating the norm of T and using (1.4) and (1.7), we get $\|\nabla H\|^2\geqq 4(m-1)$. Theorem A is thereby proved by Theorem 3.5.

§4. *C*-Einstein hypersurface of $P^m(C)$.

Let M be a real hypersurface of $P^m(C)$. If the Ricci tensor R_0 of M satisfies

(4.1) $$g(R_0X, Y)=ag(X, Y)+bu(x)u(Y),$$

where u is the induced 1-form defined in §1, we call M a *C-Einstein hypersurface*. When $b=0$, M is an Einstein space. Now we will consider a *C*-

Einstein hypersurface.

We define a symmetric tensor K of type (1, 1) by

$$K=H^2-\mu H\,, \tag{4.2}$$

where H is the second fundamental tensor of M.

LEMMA 4.1. *If M satisfies* (4.1) *and $b\neq -3$ at every point of M, then U is an eigenvector of K whose eigenvalue is equal to* $(2m-2-a-b)$. *Furthermore the other eigenvalues of K are equal to* $(2m+1-a)$.

PROOF. By the above assumption and (1.8), we get

$$KX=(2m+1-a)X-(b+3)u(X)U\,.$$

This equation implies the lemma.

On the other hand, at each point we can take $X_1, \cdots, X_{2m-1}$ which are P.C. vectors with principal curvature $r_1, \cdots, r_{2m-1}$ respectively and form as orthonormal bases. From (4.2), we get

$$KX_i=(r_i^2-\mu r_i)X_i\,. \tag{4.3}$$

LEMMA 4.2. *Under the assumptions of Lemma* 4.1, *U is a P.C. vector whose multiplicity is equal to* 1.

PROOF. (4.3) means that each X_i is the eigenvector of K. Then there exists a unique vector X with eigenvalue $(2m-2-a-b)$. It follows that the eigenspace of X coincides with the space of U. We get the lemma.

We can take an orthonormal basis $\{U, X_2, \cdots, X_{2m-1}\}$ each of which is a P.C. vector with principal curvature α, r_i $(i=2, \cdots, 2m-1)$ respectively. From Lemma 4.1 and (4.3), we have

$$r_i^2-\mu r_i-(2m+1-a)=0\,, \qquad (i=2, \cdots, 2m-1)\,, \tag{4.4}$$

$$\alpha^2-\mu\alpha-(2m-2-a-b)=0\,. \tag{4.5}$$

Thus we have proved

LEMMA 4.3. *Under the same assumptions as in Lemma* 4.1, *M has at most three distinct principal curvature at each point of M.*

On the other hand, by Lemma 2.2 we find that the only possibilities are the following cases at any point p of M.

Case 1) fX belongs to V_r for any P.C. vector $X\in V_r$.

Case 2) there exists a P.C. vector $X\in V_r$ such that fX dose not belong to V_r.

We assume that there exsists a point p of M in Case 2). Fix the above point p of M. From Lemma 2.2 and (4.4), we get

$$2(r_i^2+1)-\mu(2r_i-\alpha)=0\,, \tag{4.6}$$

where r_i denotes the principal curvature of X_i.

By the equation (4.6), we see easily that only Case 1) occurs when M is minimal. Using this fact and the Proposition 2.5, we have easily

THEOREM 4.4. *Let M be a complete minimal C-Einstein hypersurface of $P^m(C)$ such that $b \neq -3$. Then M is congruent to $M_{p,q}^c$ for some p, q.*

THEOREM 4.5. *Let M be a complete C-Einstein hypersurface of $P^m(C)$ with $m \geqq 3$. If $b \neq -3$ and $a+b \geqq 2(m-1)$ at each point of M, then M is congruent to $M_{p,q}^c$ for some p, q.*

PROOF. Let r, r' be the two real roots of (4.4). We only consider the following case by Lemma 4.3 and Lemma 4.4:

For any point p of M, the tangent space T_pM at p can be written as $T_pM = V_\alpha \oplus V_r \oplus V_{r'}$(direct sum), where $\dim V_\alpha = 1$, $r \neq r'$ and $\dim V_r = s$ $(0 \leqq s \leqq 2m-2)$.

From (4.5), the mean curvature μ and α have the same sign. If there exists a P.C. vector $X \in V_r$ such that $fX \notin V_r$, then by (4.6) we have $\mu r = 2(r^2+1)+\mu\alpha$. Similarly we get the same equation for r'. We see that μ, r, and r' are non-zero and have the same sign. By the definition of μ, we get

$$\mu = \operatorname{trace} H = \alpha + \mu + (s-1)r + (2m-3-s)r' ,$$

because $r+r' = \mu$.

We have $s=1$ and $2m-3=s$. This is a contradiction for $m \geqq 3$. Then V_r and $V_{r'}$ are invariant under f. This completes the proof by Proposition 2.5.

REMARK 1. We can consider the following special case of Case 2).

Case 2′)

$$fX \in V_r \qquad \text{for any} \quad X \in V_r .$$

Using the compatible submersion $(\bar{M}, M, \pi)$ in Case 2′), the second fundamental tensor of $\bar{M}$ has four principal curvatures whose multiplicities are 1, 1, $n-1$ and $n-1$. In this case if all the principal curvatures of M are constant, then so are the principal curvatures of $\bar{M}$. The hypersurfaces $\bar{M}$ of S^{m-1} with the above condition have been determined by R. Takagi [**5**].

REMARK 2. Through an Einstein space is a C-Einstein space with $b=0$, there exists no such hypersurface in the class of $M_{p,q}^c$ (cf. Proposition 5.5).

§ 5. The real hypersurfaces satisfying certain conditions.

We consider the compatible submersion $(\bar{M}, M, \pi)$. Using the Co-Gauss and the Co-Codazzi equations for this submersion (cf. [**1**], p. 31), we have easily the following:

LEMMA 5.1. *Let M be a real hypersurface of $P^m(C)$ and $(\bar{M}, M, \pi)$ a compatible submersion with the Hopf-fibration $\tilde{\pi}$. If $\bar{M}$ is a locally symmetric space,* *then M satisfies*

$$fHU=0, \tag{5.1}$$

$$f\cdot R=0, \tag{5.2}$$

where $\cdot$ means that f operates on R as a derivation, i.e., for any vector fields X, Y, Z and W on M

$$g((f\cdot R)(X, Y)Z, W)=g(R(fX, Y)Z, W)+g(R(X, fY)Z, W)$$
$$+g(R(X, Y)fZ, W)+g(R(X, Y)Z, fW).$$

In this section we want to discuss the converse problem. Namely the hypersurface M with the condition (5.1) and (5.2) will be determined.

The equation (5.1) implies that U is a P.C. vector with constant principal curvature by (2.3) and Lemma 2.1. So we can apply the results in § 2.

Contracting (5.2) we have

$$fR_0=R_0f. \tag{5.3}$$

By (1.6) we get for any vectors X, Y, Z and W on M

$$\begin{aligned}(f\cdot R)(X, Y, Z, W)=&g(HY, Z)g(HfX, W)-g(HfX, Z)g(HY, W)\\ &+g(HfY, Z)g(HX, W)-g(HX, Z)g(HfY, W)\\ &+g(HY, fZ)g(HX, W)-g(HX, fZ)g(HY, W)\\ &+g(HY, Z)g(HX, fW)-g(HX, Z)g(HY, fW).\end{aligned} \tag{5.4}$$

So we have by (5.2)

$$\begin{aligned}&g(HY, Z)g((Hf-fH)X, W)+g(HX, W)g((Hf-fH)Y, Z)\\ &-g(HY, W)g((Hf-fH)X, Z)-g(HX, Z)g((Hf-fH)Y, W)=0.\end{aligned} \tag{5.5}$$

Similarly the equation (5.3) is equivalent to

$$\mu(Hf-fH)X-(H^2f-fH^2)X=0. \tag{5.6}$$

LEMMA 5.2. *Let M be a real hypersurface of $P^m(C)$ with $m\geqq 3$ satisfying* (5.1) *and* (5.3). *If $\alpha=g(HU, U)=0$ at some point p of M, there exists a P.C. vector $X\in V_r$ such that $g(X, U)=0$ and $fX\in V_r$.*

PROOF. We remarked that fX is also a P.C.-vector if X is a P.C. vector (see § 2). Take the orthonormal basis $\{U, X_a, fX_a, (a=1, \cdots, m-1)\}$ consisting of P.C. vectors and denote their principal curvatures by α, r_a, $1/r_a$ respectively, because of Lemma 2.2. Suppose that $r_a\neq 1/r_a$, for all $a=1, \cdots, m-1$. In (5.6), replacing X by X_i, we get

$$(5.7) \qquad (r_a - 1/r_a)(r_a + 1/r_a - \mu) = 0 .$$

It follows $r_a + 1/r_a = \mu$. On the other hand, we have

$$\mu = g(HU, U) + \sum_{a=1}^{m-1} g(HX_a, X_a) + \sum_{a=1}^{m-1} g(HfX_a, fX_a)$$

$$= \sum_{a=1}^{m-1} (r_a + 1/r_a) = (m-1)\mu .$$

We have $\mu=0$, which is a contradiction.

LEMMA 5.3. *Under the assumptions of Lemma* 5.2, *the principal curvature of* fX_a *is equal to that of* X_a $(a=1, \cdots, m-1)$.

PROOF. There exists a P.C. vector X with principal curvature β such that $\beta^2=1$ because of Lemma 5.2. If we take any P.C. vector X_a with principal curvature r_a, then from (5.5), we have

$$\beta(1/r_a - r_a)(g(X, W)g(X_a, Z) - g(X, Z)g(X_a, W)) = 0 ,$$

where Z and W are any vectors on M. It follows that $r_a = 1/r_a$. When $\alpha \neq 0$, replacing Y and Z by U in (5.5), we see that f and H are commutative.

With the above fact and the above lemmas, we have

THEOREM 5.4. *Let M be a complete real hypersurface of* $P^m(C)$ $(m \geqq 3)$. *If M satisfies* (5.1) *and* (5.2), *then M is congruent to* $M^c_{p,q}$.

As a final remark, we will show that in $P^m(C)$ that there exists no real hypersurface with parallel Ricci tensor in the class of $M^c_{p,q}$. Assume that there exists a hypersurface $M^c_{p,q}$ with parallel Ricci tensor for some p, q. Since U is a P.C. vector with constant principal curvature, using Theorem 0 and (3.1), we have $2fH + (\mu - \alpha)f = 0$, where $\mu = \text{trace}\, H$. Multiplying this equation by f and contracting, we get $\mu = \alpha$. Consequently, $M^c_{p,q}$ has the parallel second fundamental tensor. It follows from (3.1) again that f vanishes identically. This is a contradiction.

Using Theorem 4.5 and the above fact, we have

PROPOSITION 5.5. *There exists no Einstein hypersurface of* $P^m(C)$ $(m \geqq 3)$ *with scalar curvature* $\geqq 2(m-1)(2m-1)$.

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