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Note on the positive definite integral quadratic lattice

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§ 0. Introduction.

By a quadratic lattice L we understand a finitely generated module over $\boldsymbol{Z}$, the ring of rational integers in which a metric is given in the sense of M. Eichler [2]. The bilinear form associated to the metric is denoted by (x, y) where x and y are elements of L. If for any $x \neq 0$ in L we have $(x, x) > 0$, we shall say L is positive definite, and if it holds that $(x, y) \in \boldsymbol{Z}$ for any pair x and y in L, we shall say L is integral. Since we shall confine ourselves to the positive definite integral quadratic lattice only, we shall call such a lattice merely a lattice. Since for any element x of a lattice L (x, x) is a positive rational integer, we shall say x is m-vector when (x, x) is equal to a positive rational integer m. It is known that the sublattice generated by 2-vectors in a lattice plays an important role in the classification theory of positive definite integral quadratic lattices (E. Witt [8], M. Kneser [4], H.-V. Niemeier [5]).

The first purpose of this paper is to show that when n is an integer not smaller than 17 among all lattices of fixed rank n a lattice has the largest number of 2-vectors if and only if L contains D_n or L is equal to B_n (D_n and B_n are defined in § 1). Roughly speaking, the set of 2-vectors and 1-vectors in a lattice exibits the order of the subgroup generated by reflections in the group of units of that lattice.

Our second purpose in this paper is to prove Theorem 2 which says that if the determinant of a lattice L exceeds 2^n, where n equals to the rank of L, then the rank of the sublattice of L generated by 2-vectors is smaller than n.

Though a fair part of the results in § 2 is not new, we think it is not worthless to expose it with abbreviated proofs because some of the standpoints is not found in previous literatures as far as we know.

§ 1. Some basic notations and definitions.

We shall use $e_1, e_2, \cdots, e_m, f_1, f_2, \cdots, f_k$ or $g_1, \cdots, g_p$ as orthonormal vectors in an Euclidean space $\boldsymbol{R}^n$ of sufficiently large dimension n $(n=1, 2, 3, \cdots)$. We shall order $\boldsymbol{R}^n$'s in a canonical manner;

$$\boldsymbol{R}^1 \subset \boldsymbol{R}^2 \subset \boldsymbol{R}^3 \subset \cdots .$$

Let L_1 and L_2 be two lattices, we understand by an isomorphism a bijective $\boldsymbol{Z}$-linear map σ from L_1 to L_2 satisfying the following condition;

$$(\sigma x, \sigma y) = (x, y)$$

holds for any pair x, y in L_1. Since $\boldsymbol{Z}$ is a principal ideal ring, any finitely generated torsion free $\boldsymbol{Z}$-module is a free module of finite rank, then a lattice has $\boldsymbol{Z}$-basis. Let L be a lattice of rank n and let $v_1, \cdots, v_n$ be its basis, then any element x in L can be written in the form;

$$x = \sum_{i=1}^{n} \xi_i v_i \qquad \xi_i \in \boldsymbol{Z} .$$

If we think ξ_i's as scalar variables, the form

$$(x, x) = \sum_{i,j=1}^{n} (v_i, v_j) \xi_i \xi_j$$

becomes a quadratic form. We shall denote this quadratic form by $Q(L)$, and this is uniquely determined from L up to integral equivalence. The determinant of the matrix $\|(v_i, v_j)\|$, $i, j=1, \cdots, n$ is called the determinant of $Q(L)$ or of L. We denote it by $d(L)$. We should remark that $Q(L)$ is always a positive definite integral quadratic form, because L is integral and has a positive metric. Conversely any positive definite integral quadratic form is expressed as $Q(L)$ for some lattice in $\boldsymbol{R}^n$, and to two integrally equivalent quadratic forms correspond isomorphic lattices. (As to the precise exposition see [2].) An isomorphism from L onto itself is called an automorphism or a unit of L. All the automorphisms of L form a group and we denote it by $\mathrm{Aut}(L)$. Let L be a lattice, then the dual $L^{\#}$ of L is defined by;

$$L^{\#} = \{ y \in L \otimes_{\boldsymbol{Z}} Q \mid (x, y) \in \boldsymbol{Z},\ \forall x \in L \} ,$$

where $L \otimes_{\boldsymbol{Z}} Q$ is the tensor product of L and Q, the field of rational numbers, over $\boldsymbol{Z}$. This lattice contains L and is not necessarily integral but this is useful for our later consideration.

We shall make a list of basic lattices namely;

$$A_n = [e_1 - e_2, e_2 - e_3, \cdots, e_n - e_{n+1}]_{\boldsymbol{Z}} .$$

This is a lattice in $\boldsymbol{R}^{n+1}$ of rank n with its generater $e_1 - e_2, \cdots, e_n - e_{n+1}$ over $\boldsymbol{Z}$. In the following by $[\ ]_{\boldsymbol{Z}}$ we shall express similar meaning as above.

$$B_n = [e_1, \cdots, e_n]_{\boldsymbol{Z}} .$$

In the usual expression in the theory of Lie algebra, B_n may be differently expressed, but both expressions are equivalent in the sense of quadratic lattice.

$$D_n=[e_1-e_2, \cdots, e_{n-1}-e_n, e_{n-1}+e_n]_{\boldsymbol{Z}}, \qquad n\geqq 4.$$

$$E_6=[e_1-e_2, \cdots, e_4-e_5, e_4+e_5, \frac{1}{2}(\sum_{i=1}^{5} e_i+\sqrt{3}\, e_6)]_{\boldsymbol{Z}}.$$

$$E_7=[e_1-e_2, \cdots, e_5-e_6, e_5+e_6, \frac{1}{2}(\sum_{i=1}^{6} e_i+\sqrt{2}\, e_7)]_{\boldsymbol{Z}}.$$

$$E_8=[e_1-e_2, \cdots, e_6-e_7, e_6+e_7, \frac{1}{2}\sum_{i=1}^{8} e_i]_{\boldsymbol{Z}}.$$

We shall call these lattices basic lattices, these are all integral lattices but have different determinants. In the basic lattices the generators used in the above are also the basis of those lattices and we shall call them canonical basis. For each one of these basic lattices the structure of its automorphism group is known and [1] is a standard reference. A lattice L is an orthogonal sum of sublattices L_1 and L_2 of L and we write as $L=L_1\oplus L_2$ if any x in L is expressed as $x=x_1+x_2$ with $x_1\in L_1$ and $x_2\in L_2$ such that the equation $(x_1, x_2)=0$ holds for any $x_1\in L_1$ and for any $x_2\in L_2$. A lattice L is called irreducible if there is no non-trivial orthogonal decomposition. Otherwise L is called reducible.

§2. Some preliminary results.

LEMMA 2-1. *If a lattice L has a 1-vector x, then L is reducible and is expressed as $L=\boldsymbol{Z}x\oplus L_1$, where $\boldsymbol{Z}x$ is a rank one sublattice of L generated by x over $\boldsymbol{Z}$.*

PROOF. Let $\alpha_1, \cdots, \alpha_n$ be the basis of L. Then L is also generated by $\alpha_1-(\alpha_1, x)x, \cdots, \alpha_n-(\alpha_n, x)x$ and x. Now we see that;

$$(\alpha_i-(\alpha_i, x)x, x)=(\alpha_i, x)-(\alpha_i, x)(x, x)=0, \qquad i=1, \cdots, n.$$

Set $L'=[\alpha_1-(\alpha_1, x)x, \cdots, \alpha_n-(\alpha_n, x)x]_{\boldsymbol{Z}}$. Then we have $L=L'\oplus \boldsymbol{Z}x$. Q.E.D.

For the given lattice L by repeating use of Lemma 2-1 we get the following decomposition;

$$L=\boldsymbol{Z}x_1\oplus\cdots\oplus\boldsymbol{Z}x_r\oplus L',$$

where x_i's are mutally orthogonal 1-vectors and $\boldsymbol{Z}x_i$ is a rank one sublattice of L generated by x_i over $\boldsymbol{Z}$ and L' does not contain any 1-vector. Apparently $\boldsymbol{Z}x_1\oplus\cdots\oplus\boldsymbol{Z}x_r$ is isomorphic to B_r, so we can write without loss of generality as follows;

$$L=B_r\oplus L', \quad \cdots\cdots\cdots\cdots (1)$$

where L' is 1-vector free. As to 2-vectors, we must take precise care. After the chain of lemmas we shall establish the following;

PROPOSITION 2-2. *If a lattice L is generated by 2-vectors over $\boldsymbol{Z}$, then L has basis consisting of 2-vectors.*

To establish the above proposition we pose the following problem;

(P) When L is a basic lattice or an orthogonal sum of basic lattices and x is a 2-vector with the condition that $(x, y)\in \boldsymbol{Z}$ holds for any $y\in L$ (henceforth we shall write this condition symbolically as $(x, L)\subseteqq \boldsymbol{Z}$), how becomes the lattice $L+\boldsymbol{Z}x$, generated by L and x over $\boldsymbol{Z}$?

The answer to this problem is that $L+\boldsymbol{Z}x$ is also isomorphic to a basic lattice or an orthogonal sum of basic lattices as the following consideration shows. First we can set $x=u+v$ with $(u, L)\subseteqq \boldsymbol{Z}$, $(u, v)=0$ and v is orthogonal to L (we shall write this condition as $(v, L)=0$). For the basic lattice we have the following;

LEMMA 2-3. (i) *Let L be a basic lattice other than A_n type and u be an element of $\boldsymbol{R}^m$ $(\supseteqq L)$ such that $(u, L)\subseteqq \boldsymbol{Z}$, then u can be taken from $L^{\#}$*, (ii) *in the case of A_n $(n\geqq 1)$ an element u of $\boldsymbol{R}^m$ $(\supseteqq A_n)$ such that $(u, A_n)\subseteqq \boldsymbol{Z}$ can be taken from $\boldsymbol{R}^{n+1}$ satisfying certain condition specified in the proof.*

PROOF. If L be a basic lattice other than A_n type, then we can say that;

$$L\otimes_{\boldsymbol{Z}}\boldsymbol{R}=\boldsymbol{R}^k ,$$

where k is the rank of L. In this case u can be taken from $\boldsymbol{R}^k$ and we write $u=\sum_{i=1}^{k} a_i w_i$, where $a_i\in \boldsymbol{R}$ $(1\leqq i\leqq k)$ and w_i $(1\leqq i\leqq k)$ are the canonical basis of L. The condition $(u, L)\subseteqq \boldsymbol{Z}$ implies that;

$$(u, w_i)\in \boldsymbol{Z} \quad \text{for} \quad i=1, \cdots, k . \qquad \cdots\cdots\cdots\cdots\cdots\cdots (2)$$

In each case of basic lattice L other than A_n type we can easily verify that (2) implies $a_i\in Q$ $(1\leqq i\leqq k)$ and then $u=\sum_{i=1}^{k} a_i w_i$ belongs to $L^{\#}$. The part (i) of the lemma is thus proved. In case of A_n, A_n can be embedded into B_{n+1} and without loss of generality we can take u from $B_{n+1}\otimes_{\boldsymbol{Z}}\boldsymbol{R}=\boldsymbol{R}^{n+1}$ and we put $u=\sum_{i=1}^{n+1} a_i e_i$ with $a_i\in \boldsymbol{R}$ $(1\leqq i\leqq n+1)$, where $e_1, e_2, \cdots, e_{n+1}$ are the canonical basis of B_{n+1}. The given condition $(u, A_n)\subseteqq \boldsymbol{Z}$ is equivalent to the condition;

$$(u, e_i-e_{i+1})=a_i-a_{i+1}\in \boldsymbol{Z} \quad \text{for} \quad i=1, \cdots, n .$$

We rewrite this condition as;

$$a_1\equiv a_2\equiv \cdots \equiv a_{n+1} \quad (\mathrm{mod}\ 1) . \qquad \cdots\cdots\cdots\cdots\cdots\cdots (3)$$

The part (ii) of the lemma is proved. Q. E. D.

Let $L=L_1\oplus\cdots\oplus L_t$ be an orthogonal sum of basic lattices and $x=u+v$ be a 2-vector in the problem (P) such that $(u, L)\subseteqq \boldsymbol{Z}$, $(u, v)=0$ and $(v, L)=0$, then we can write u as $u=u_1+\cdots+u_t$ with the following conditions;

$$\left.\begin{array}{ll}(u_i, u_j)=0 & (i\neq j)\,, \\ (u_i, L_j)=0 & (i\neq j) \text{ and} \\ (u_i, L_i)\subseteqq \boldsymbol{Z} & i=1, \cdots, t\,.\end{array}\right\} \quad\cdots\cdots\cdots\cdots\cdots\cdots \quad (4)$$

We shall justify the above settings. If L_i is a basic other than A_n type, then we know $L_i\otimes_{\boldsymbol{Z}}\boldsymbol{R}=\boldsymbol{R}^{n_i}$, where n_i is the rank of L_i. If L_i is A_n type, then we can not say that $A_n\otimes_{\boldsymbol{Z}}\boldsymbol{R}=\boldsymbol{R}^n$ as far as we adopt $[e_1-e_2, \cdots, e_n-e_{n+1}]_{\boldsymbol{Z}}$ as the model of A_n. But there is another model of A_n, namely;

$$\tilde{A}_n=\Big[\sqrt{2}\,e_i,\ -\sqrt{\frac{1}{2}}e_1, +\sqrt{\frac{3}{2}}e_2, \cdots, -\sqrt{\frac{r-1}{r}}e_{r-1}+\sqrt{\frac{r+1}{r}}e_r, \cdots$$

$$\cdots, -\sqrt{\frac{n-1}{n}}e_{n-1}+\sqrt{\frac{n+1}{n}}e_n\Big]_{\boldsymbol{Z}}\,.$$

It is easily seen that;

$$[e_1-e_2, \cdots, e_n-e_{n+1}]_{\boldsymbol{Z}}\cong \tilde{A}_n\,.$$

This time we can say that;

$$\tilde{A}_n\otimes_{\boldsymbol{Z}}\boldsymbol{R}=\boldsymbol{R}^n \quad\text{and}\quad \operatorname{rank}\tilde{A}_n=n\,.$$

If we can use $\tilde{A}_n$ as the model of A_n type, then the settings (4) can be easily justified because L_i and L_j $(i\neq j)$ are separated by the ambient spaces $\boldsymbol{R}^{n_i}$ and $\boldsymbol{R}^{n_j}$, where n_i (resp. n_j) is the rank of L_i (resp. L_j). Though $\tilde{A}_n$ is theoretically simpler model than $A_n=[e_1-e_2, \cdots, e_n-e_{n+1}]_{\boldsymbol{Z}}$, the calculations attached to $\tilde{A}_n$ are more complicated than those of A_n and we shall not use $\tilde{A}_n$. Let L be a basic lattice other than A_n type, then it is known that each element σ of $\operatorname{Aut}(L)$ is naturally extended to an orthogonal transformation of $L\otimes_{\boldsymbol{Z}}\boldsymbol{R}=\boldsymbol{R}^k$, where k is the rank of L. In A_n case $\operatorname{Aut}(A_n)$ is generated by the reflections with respect to 2-vectors in A_n and $(-1)\times$identity, and each element of $\operatorname{Aut}(A_n)$ is extended to an orthogonal transformation of $B_{n+1}\otimes_{\boldsymbol{Z}}\boldsymbol{R}=\boldsymbol{R}^{n+1}$. This remark will be used later.

Let L be a lattice and $x=u+v$ be a 2-vector in the problem (P) (the decomposition of x is like as above). Since our metric is positive, $(x, x)=2$ implies;

$$(u, u)\leqq 2\,. \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (5)$$

Besides A_n type the vector u can be taken from $L^{\#}$ by Lemma 2-3. As to A_n type we must take precise care and we shall discuss the case of A_n later.

We shall call an element u of $L^{\#}$ (L is a lattice) is a minimal representative (we shall abbreviate as m. r.) of $L^{\#}$ modulo L if u satisfies the following condition;

$$(u, u) \leqq (u+y, u+y) \qquad \forall y \in L .$$

When L is a basic lattice, the structure of $L^{\#}/L$ is known and Niemeier [5] remarked at pages 148-150 the following (the notations are a little different from his);

LEMMA 2-4. (i) *$A_n^{\#}/A_n \cong \boldsymbol{Z}/(n+1)\boldsymbol{Z}$ and complete m. r. of $A_n^{\#}$ modulo A_n are given by* 0 *and* $u_r = \frac{r}{n+1} \sum_{i=1}^{n-r+1} e_i - \frac{n-r+1}{n+1} \sum_{i=n-r+2}^{n+1} e_i$ *for* $r=1, \cdots, n$ *and by calculation we have;*

$$(u_r, u_r) = \frac{r(n+1-r)}{n+1} \qquad \textit{for} \quad 1 \leqq r \leqq n ,$$

(ii) *$D_{2n+1}^{\#}/D_{2n+1} \cong \boldsymbol{Z}/4\boldsymbol{Z}$ ($n \geqq 2$) and complete m. r. of $D_{2n+1}^{\#}$ modulo D_{2n+1} are* u_0, $u_1 = \frac{1}{2} \sum_{i=1}^{2n+1} e_i$, $u_2 = e_{2n+1}$ *and* $u_3 = \frac{1}{2} \sum_{i=1}^{2n+1} e_i - e_{2n+1}$ *and we have;*

$$(u_1, u_1) = (u_3, u_3) = \frac{2n+1}{4} \quad \textit{and} \quad (u_2, u_2) = 1 ,$$

(iii) *$D_{2n}^{\#}/D_{2n} \cong \boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/2\boldsymbol{Z}$ ($n \geqq 2$) and complete m. r. of $D_{2n}^{\#}$ modulo D_{2n} are* $u_0 = 0$, $u_1 = \frac{1}{2} \sum_{i=1}^{2n} e_i$, $u_2 = e_{2n}$ *and* $u_3 = \frac{1}{2} \sum_{i=1}^{2n} e_i - e_{2n}$ *and we haue;*

$$(u_1, u_1) = (u_3, u_3) = \frac{2n}{4} \quad \textit{and} \quad (u_2, u_2) = 1 ,$$

(iv) $E_8^{\#} = E_8$, (v) $B_n^{\#} = B_n$,

(vi) *$E_7^{\#}/E_7 \cong \boldsymbol{Z}/2\boldsymbol{Z}$ and complete m. r. of $E_7^{\#}$ modulo E_7 are* 0 *and* $e_6 + \frac{e_7}{\sqrt{2}}$ *and we have;*

$$\left(e_6 + \frac{e_7}{\sqrt{2}}, e_6 + \frac{e_7}{\sqrt{2}}\right) = \frac{3}{2} ,$$

(vii) *$E_6^{\#}/E_6 \cong \boldsymbol{Z}/3\boldsymbol{Z}$ and complete m. r. of $E_6^{\#}$ modulo E_6 are* 0, $e_5 + \frac{e_6}{\sqrt{3}}$ *and* $e_5 - \frac{e_6}{\sqrt{3}}$ *and we have;*

$$\left(e_5 + \frac{e_6}{\sqrt{3}}, e_5 + \frac{e_6}{\sqrt{3}}\right) = \left(e_5 - \frac{e_6}{\sqrt{3}}, e_5 - \frac{e_6}{\sqrt{3}}\right) = \frac{4}{3} .$$

REMARK 1. The values (u_r, u_r) ($1 \leqq r \leqq n$) of u_r, m. r. of $A_n^{\#}$ modulo A_n are necessary for our later argument and we give some as the table.

$r \backslash n$	1	2	3	4	5	6	7	8	9	10	11	12	....
1	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	$\frac{7}{8}$	$\frac{8}{9}$	$\frac{9}{10}$	$\frac{10}{11}$	$\frac{11}{12}$	$\frac{12}{13}$	....
2		$\frac{2}{3}$	1	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{10}{7}$	$\frac{3}{2}$	$\frac{14}{9}$	$\frac{8}{5}$	$\frac{18}{11}$	$\frac{5}{3}$	$\frac{22}{13}$	....
3			$\frac{3}{4}$	$\frac{6}{5}$	$\frac{3}{2}$	$\frac{12}{7}$	$\frac{15}{8}$	2	..	..	..	..	....
4				$\frac{4}{5}$	$\frac{4}{3}$	$\frac{12}{7}$	2	..	..	..	..	..	....
5					$\frac{5}{6}$	$\frac{10}{7}$	$\frac{15}{8}$	..	..	..	..	..	....
6						$\frac{6}{7}$	$\frac{3}{2}$	2	..	..	..	..	....

The following lemma may simplify later arguments.

LEMMA 2-5. *Let $L=L_1 \oplus \cdots \oplus L_t$ $(t \geqq 1)$ be an orthogonal sum of basic lattices and $x=u+v$ and $x'=u'+v'$ be a 2-vectors in the problem* (P), *where $u=u_1+\cdots+u_t$ and $u'=u_1'+\cdots+u_t'$ be the decompositions of u and u' in the manner of* (4) *and $(v, L)=(v', L)=0$ and $(v, u')=(v', u)=0$.*

(i) *It is clear that* $\mathrm{Aut}(L)$ *contains the direct product* $\mathrm{Aut}(L_1) \times \mathrm{Aut}(L_2) \times \cdots \times \mathrm{Aut}(L_t)$ *as the subgroup. If there exist $\sigma_i \in \mathrm{Aut}(L_i)$ $(i=1, \cdots, t)$ such that $u_i'=\sigma_i u_i$ $(i=1, \cdots, t)$, then $L+\boldsymbol{Z}x$ is isomorphic to $L+\boldsymbol{Z}x'$.*

(ii) *If there exist $w_i \in L_i$ $(i=1, \cdots, t)$ such that $u_i'=u_i+w_i$ and $(u_i', u_i')=(u_i, u_i)$ for $i=1, \cdots, t$, then $L+\boldsymbol{Z}x$ is isomorphic to $L+\boldsymbol{Z}x'$.*

(iii) *If the equation $(u_i, y_i)=(u_i', y_i)$ holds for any element y_i of L_i and for $i=1, \cdots, t$, then $L+\boldsymbol{Z}x$ is isomorphic to $L+\boldsymbol{Z}x'$.*

PROOF OF (i). Let $y=y_1+\cdots+y_t$ be the general element of L with $y_i \in L_i$ $(1 \leqq i \leqq t)$ and φ be the mapping from $L+\boldsymbol{Z}x$ to $L+\boldsymbol{Z}x'$ defined by;

$$\varphi(x)=x',$$

$$\varphi(y)=\sum_{i=1}^{t} \sigma_i y_i$$

and

$$\varphi(y+kx)=\varphi(y)+k\varphi(x) \qquad \text{for} \quad k \in \boldsymbol{Z},$$

then we can easily verify that φ is an isomorphism from $L+\boldsymbol{Z}x$ to $L+\boldsymbol{Z}x'$.

PROOF OF (ii). By the given condition we can say $L+\boldsymbol{Z}x=L+\boldsymbol{Z}x''$, where $x''=\sum_{i=1}^{t} u_i'+v$. Let φ be the mapping from $L+\boldsymbol{Z}x''$ to $L+\boldsymbol{Z}x'$ defined by;

$$\varphi(y)=y \quad \text{for} \quad \forall y \in L,$$

$$\varphi(x'')=x'$$

and

$$\varphi(y+kx'')=\varphi(y)+k\varphi(x'') \quad \text{for} \quad \forall k \in \boldsymbol{Z},$$

then we can say that φ is an isomorphism from $L+\boldsymbol{Z}x''$ to $L+\boldsymbol{Z}x'$.

PROOF OF (iii). Let φ be the mapping from $L+\boldsymbol{Z}x$ to $L+\boldsymbol{Z}x'$ defined by;

$$\varphi(y)=y \quad \text{for} \quad \forall y \in L,$$

$$\varphi(x)=x'$$

and

$$\varphi(y+kx)=\varphi(y)+k\varphi(x) \quad \text{for} \quad \forall k \in \boldsymbol{Z},$$

then we can say that φ is an isomorphism. Q. E. D.

We shall treat u part of 2-vector x in A_n case (described in Lemma 2-3 (ii)) more precisely. By Lemma 2-3 we can set $u=\sum_{i=1}^{n+1} a_i e_i$ with the conditions $a_i \in \boldsymbol{R}$ $(1 \leqq i \leqq n+1)$ and (3). Moreover u satisfies the inequality (5), so we have;

$$(u, u)=\sum_{i=1}^{n+1} a_i^2 \leqq 2. \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (6)$$

(I) Suppose that one of a_i's is an integer, then by (3) each a_i is also integer for $i=1, \cdots, n+1$. By the inequality (6) at most two of a_i's are not zero.

(I)-(i) When exactly two of a_i's are not zero (say a_{i_1} and a_{i_2}, $i_1 \neq i_2$), then we can say that $|a_{i_1}|=|a_{i_2}|=1$ and $x=u=a_{i_1}e_{i_1}+a_{i_2}e_{i_2}$. Since we aim the solution of the problem (P) and the lattice $L+\boldsymbol{Z}x$ is identical to $L+\boldsymbol{Z}(-x)$, we can assume that $a_{i_1}=1$. u must be one of the forms $e_{i_1}-e_{i_2}$ and $e_{i_1}+e_{i_2}$. Since there exists an element w (resp. w') of A_n such that $w+e_{i_1}-e_{i_2}=e_n-e_{n+1}$ (resp. $w'+e_{i_1}+e_{i_2}=e_n+e_{n+1}$) and $(e_{i_1}-e_{i_2}, e_{i_1}-e_{i_2})=(e_n-e_{n+1}, e_n-e_{n+1})$ (resp. $(e_{i_1}+e_{i_2}, e_{i_1}+e_{i_2})=(e_n+e_{n+1}, e_n+e_{n+1})$), by Lemma 2-5, (ii) we can set $u=e_n-e_{n+1}$ (resp. e_n+e_{n+1}). When $x=u=e_n-e_{n+1}$, then $A_n+\boldsymbol{Z}x=A_n$ and we shall neglect this case henceforth. When $x=u=e_n+e_{n+1}$, then $A_n+\boldsymbol{Z}x=D_{n+1}$ $(n \geqq 3)$, $A_2+\boldsymbol{Z}(e_2+e_3) \cong A_3$ and $A_1+\boldsymbol{Z}(e_1+e_2) \cong A_1 \oplus A_1$. We shall call this vector e_n+e_{n+1} singular vector of first kind for A_n.

(I)-(ii) When only one a_{i_0} $(1 \leqq i_0 \leqq n+1)$ is not zero, then we have $|a_{i_0}|=1$ and $u=\pm e_{i_0}$. Since $-1\times$identity is an element of $\mathrm{Aut}(A_n)$, by Lemma 2-5, (i) we can set $u=e_{i_0}$. Since $e_{i_0}-e_{n+1} \in A_n$ and $(e_{i_0}, e_{i_0})=(e_{n+1}, e_{n+1})$, by Lemma 2-5, (ii) we can set $u=-e_{n+1}$. We shall call this vector singular vector of second kind for A_n.

(II) Suppose that one of a_i's is not integer, then by (3) each a_i is not integer for $i=1, \cdots, n+1$. In this case we can take a_1 as positive by Lemma 2-5 (ii). By (3) we can set a_i as;

$$a_i=a_1+k_i \qquad (2\leqq i\leqq n+1),$$

where $k_i\in \boldsymbol{Z}$ and $a_1\notin \boldsymbol{Z}$.

By the inequality (6) at most one of a_i's has the absolute value larger than one and less than two.

(II)-(i) When each $|a_i|$ $(1\leqq i\leqq n+1)$ takes the value between zero and one, we can set as;

$$0<a_1<1$$

and

$$a_i=a_1+k_i \text{ with } k_i=0 \text{ or } -1 \qquad \text{for } i=2, \cdots, n+1.$$

Let a_1 appear m times and a_1-1 appear l times among a_i's with $l+m=n+1$ and l, $m\geqq 0$, then taking Lemma 2-5, (i) into account we can set;

$$u=a_1(e_1+\cdots+e_m)+(a_1-1)(e_{m+1}+\cdots+e_{m+l}). \quad \cdots\cdots\cdots\cdots (7)$$

In the case that $m<l$, we take $-u$ instead of u (this is justified by Lemma 2-5, (i)), so that we can assume that $m\geqq l$. By (7) we have;

$$\begin{aligned}(u, u)&=ma_1^2+l(a_1-1)^2\\ &=(m+l)\Big(a_1-\frac{l}{m+l}\Big)^2+\frac{ml}{m+l}\\ &\geqq\frac{ml}{m+l}. \qquad \cdots\cdots\cdots\cdots\cdots (8)\end{aligned}$$

In (8) the equality holds if and only if $a_1=\dfrac{l}{m+l}$ and then u is a m.r. of $A_n^\#$ modulo A_n. We shall call the vector u of the form (7) satisfying the inequality (6) the singular vector of third kind for A_n. In this case a_1 may vary continuously in the interval which is determined from (6). The vector u of the form (7) for fixed m and l with $a_1=\dfrac{l}{m+l}$ will be called the bottom vector (as we remarked above this vector is a m.r. of $A_n^\#$ modulo A_n).

(II)-(ii) When only one $|a_i|$ lies between one and two and other $|a_i|$'s lie between zero and one, then by Lemma 2-5, (i) we can set as;

$$1<a_1<2$$

and

$$a_i=a_1+k_i \quad \text{with } k_i=-1 \text{ or } -2 \qquad \text{for } i=2, \cdots, n+1.$$

If there is a $k_{i_0}=-2$, then we have;

$$\begin{aligned}(u, u)&=a_1^2+(a_1-2)^2\\ &=2(a_1-1)^2+2>2.\end{aligned}$$

This contradicts the inequality (6), so u must be of the form;

$$u=a_1e_1+(a_1-1)(e_2+\cdots+e_{n+1}), \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (9)$$

and

$$\begin{aligned}(u, u)&=a_1^2+n(a_1-1)^2\\ &>1. \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (10)\end{aligned}$$

The vector u of the form (9) satisfying the inequality (6) the singular vector of fourth kind for A_n. In the above we have enumerated the singular vectors for the completeness of the discussion of type A_n, but the following lemma will show that we can manage the later discussions without singular vectors.

LEMMA 2-6. *Let $L=L_1\oplus\cdots\oplus L_t$ be an orthogonal sum of basic lattices. Let $x=u+v$ be a 2-vector in the problem* (P) *and $u=u_1+\cdots+u_t$ be the decomposition of u in the manner of* (4). *If one of L_i is of type A_n (say $L_1=A_n$) and u_1 is the singular vector of j-th kind ($2\leqq j\leqq 4$), then there exists a m.r. u_1' of $A_n^\#$ modulo A_n and a 2-vector $x'=u'+v'$ with the following conditions;*

$$(u_1', u_1')\leqq(u_1, u_1),$$

$$u'=u_1'+u_2+\cdots+u_t,$$

$$u_1', u_2, \cdots, u_t \text{ and } L_1, \cdots, L_t \text{ satisfies (4)}$$

and

$$L+\boldsymbol{Z}x\cong L+\boldsymbol{Z}x'.$$

PROOF. When u_1 is the singular vector of second kind for A_n, then we put $u_1'=\frac{1}{n+1}\sum_{i=1}^{n}e_i-\frac{n}{n+1}e_{n+1}=\frac{1}{n+1}\sum_{i=1}^{n+1}e_i-e_{n+1}$ and we have;

$$(u_1', u_1')=\frac{n}{n+1}<1=(u_1, u_1).$$

By Lemma 2-4, (i) u_1' is a m.r. of $A_n^\#$ modulo A_n and we put $u'=u_1'+u_2+\cdots+u_t$. We shall verify the third condition of the lemma. If $(u_1', L_j)\neq 0$ for some j ($2\leqq j\leqq t$), then there exists b in L_j such that $(u_1', b)\neq 0$. By the assumption of the lemma that $(u_1, L_i)=0$ for $2\leqq i\leqq t$ we have $(u_1, b)=-(e_{n+1}, b)=0$ and consequently $(u_1', b)=\frac{1}{n+1}(\sum_{i=1}^{n+1}e_i, b)\neq 0$. So we have;

$$\begin{aligned}(\sum_{i=1}^{n+1}e_i, b)&=(\sum_{i=1}^{n}e_i, b)+(e_{n+1}, b)\\ &=(\sum_{i=1}^{n}e_i, b)\neq 0.\end{aligned}$$

Hence we can write b as;

$$b=\sum_{i=1}^{n} a_i e_i + b' ,$$

where $a_i \in \boldsymbol{R}$ and $(b', e_i)=0$ for $i=1, \cdots, n$ and $a_{i_0}\neq 0$ for some i_0 $(1\leqq i_0 \leqq n)$, and we have $(e_{i_0}-e_{n+1}, b)=(e_{i_0}, b)=a_{i_0}\neq 0$. But $e_{i_0}-e_{n+1}$ belongs to A_{n+1}. This contradicts to the assumption of the lemma, that is, $(A_n, L_j)=0$, so we can conclude that $u_1', u_2, \cdots, u_t$ and $L_1, \cdots, L_t$ satisfy the second condition of (4). By the similar reasoning we can say that;

$$(u_1', u_j)=0 \qquad \text{for} \quad 2\leqq j \leqq t ,$$

and consequently the first condition in (4) holds for $u_1', u_2, \cdots, u_t$. We know;

$$(u_1, e_r-e_{r+1})=(u_1', e_r-e_{r+1}) \qquad \text{for} \quad 1\leqq r \leqq n . \qquad \cdots\cdots\cdots\cdots (11)$$

The above equations (11) implies $(u_1', L_1)\subseteqq \boldsymbol{Z}$, and we can verify the last condition in (4). It remains to prove the last condition of the lemma. Since we know $(u_1', u_1')<(u_1, u_1)$, we get $(u', u')<(u, u)$. Let v' be determined so that the conditions, $(u_1', v')=(u_2, v')= \cdots =(u_t, v')=0$, $(L_1, v')=(L_2, v')= \cdots =(L_t, v')=0$ and $x=u'+v'$ is a 2-vector, are all satisfied. The choice of such v' is always possible in a sufficiently large Euclidean space. Since the assumptions of Lemma 2-5, (iii) is satisfied, we can conclude that;

$$L+\boldsymbol{Z}x \cong L+\boldsymbol{Z}x' .$$

Thus we have proved the lemma in the case of the singular vector of second kind.

When u_1 is the singular vector of third kind for A_n, then we take the bottom vector as u_1' and the process of the proof is similar to that in the case of the singular vector of second kind and we omit it. The case where u_1 is the singular vector of fourth kind is proved in the similar manner and we also omit it for the sake of brevity. Q. E. D.

When we consider the problem (P) in the situation that $L=L_1\oplus \cdots \oplus L_t$ is an orthogonal sum of basic lattices and $x=u+v$ is a 2-vector such that $u=u_1+\cdots +u_t$ is the decomposition of u in the manner of (4). If some of L_i's are A_n type and u_i's are the singular vectors of j-th kind $(2\leqq j\leqq 4)$, then by repeating use of Lemma 2-6 we can replace such u_i's by m. r. of $A_n^{\#}$ modulo A_n. If in the decomposition of $u=u_1+\cdots +u_t$ the singular vector of the first kind for A_n appears (say u_1), then by the fact $(u_1, u_1)=2$ we can say $u_2= \cdots =u_t=0$, $x=u$ and $L+\boldsymbol{Z}x=(L_1+\boldsymbol{Z}x)\oplus L_2\oplus \cdots \oplus L_t$. $A_n+\boldsymbol{Z}(e_n+e_{n+1})$ is already determined. In this case $L_2, \cdots, L_t$ are irrelevant to the problem (P). In general we shall call a component L_i of $L=L_1\oplus \cdots \oplus L_t$, an orthogonal sum of basic lattices, irrelevant to the problem (P) with respect to a 2-vector x if $L+\boldsymbol{Z}x=L_i\oplus(L_1\oplus \cdots \oplus L_{i-1}\oplus L_{i+1}\oplus \cdots \oplus L_t+\boldsymbol{Z}x)$ holds. We call a 2-vector x trivial for L if

$L+\boldsymbol{Z}x=L$ holds. This happens when and only when x belongs to L.

LEMMA 2-7. *Let $L=L_1\oplus\cdots\oplus L_t$ be an orthogonal sum of basic lattices and $x=u+v$ be a 2-vector in the problem* (P) *and $u=u_1+\cdots+u_t$ be the decomposition of u in the manner of* (4). *If one of E_8 and B_n appears among L_i's, then it is irrelevant to the problem* (P) *or x is trivial for L.*

PROOF. If say L_1 is E_8, then by Lemma 2-4, (iv) u_1 belongs to E_8 and by the inequality (5) u_1 satisfies $(u_1, u_1)\leqq 2$. Since any element y of E_8 satisfies $(y, y)\equiv 0 \pmod 2$, (u_1, u_1) is either 0 or 2. When (u_1, u_1) is 0, u_1 is zero and $L+\boldsymbol{Z}x=E_8\oplus(L_2\oplus\cdots\oplus L_t+\boldsymbol{Z}x)$. This implies that E_8 is irrelevant. When $(u_1, u_1)=2$, u_1 is a 2-vector in E_8 and $u_2=\cdots=u_t=v=0$. This implies that $x=u_1$ is trivial for L. If L_1 is B_n $(n\geqq 1)$ (by (1) and by the fact $B_{n_1}\oplus B_{n_2}=B_{n_1+n_2}$ we can assume without loss of generality that each L_i $(2\leqq i\leqq t)$ is not of type B_n), then by Lemma 2-4, (v) u_1 belongs to B_n and by the inequality (5) u_1 satisfies $(u_1, u_1)\leqq 2$. Since B_n contains both 1-vector and 2-vector (the latter occurs only when $n\geqq 2$), (u_1, u_1) is 0 or 1 or 2. When (u_1, u_1) is 0, u_1 is zero and in this case the lattice $L+\boldsymbol{Z}x$ in (P) has the form $B_n\oplus(L_2\oplus\cdots\oplus L_t+\boldsymbol{Z}x)$. This implies that B_n is irrelevant to the problem. When (u_1, u_1) is 1, $u_2=\cdots=u_t=0$ and v is such that $(v, v)=1$ bceause we assumed that each L_i $(2\leqq i\leqq t)$ is not of type B_n and hence L_i $(2\leqq i\leqq t)$ does not contain any 1-vector. In this case the lattice $L+\boldsymbol{Z}x$ in (P) has the form $B_n\oplus L_2\oplus\cdots\oplus L_t\oplus B_1$ and B_n is irrelevant to the problem (P). When (u_1, u_1) is 2, $u_2=\cdots=u_t=v=0$ and $x=u_1$ belongs to B_n (so we know $n\geqq 2$). Since $L+\boldsymbol{Z}x=L$, x is trivial for L.

Q. E. D.

From now on we can assume that neither E_8 nor B_n appears as an orthogonal component of the lattice L for the problem (P).

LEMMA 2-8. *Let L be a basic lattice other than B_n type and $L^{\#}$ be its dual. If an element u of $L^{\#}$ is not m. r., then we have;*

$$(u, u)\geqq 2\,,$$

where the equality in the above estimate holds only if u belongs to L.

PROOF. Let u_0 be a m. r. of $L^{\#}$ modulo L equivalent to u modulo L, then there exists y in L such that $u_0=u+y$ and we have;

$$\begin{aligned}(u, u)&=(u_0-y, u_0-y)\\&=(u_0, u_0)-2(u_0, y)+(y, y)\,.\end{aligned}$$

Since L is a basic lattice other than B_n, L has basis consisting of 2-vectors and (y, y) is an integer divisible by 2 for each y in L. The value (u_0, y) is also an integer, so (u, u) differs from (u_0, u_0) by a multiple of 2. Hence we have;

$$(u, u) \geqq (u_0, u_0) + 2 \geqq 2 .$$

When (u, u) is 2, then (u_0, u_0) is 0 and $u_0=0$. Such u is equal to y and $u \in L$. Q. E. D.

LEMMA 2-9. *Let $L=L_1 \oplus \cdots \oplus L_t$ be the orthogonal decomposition of a lattice L. The element $u=u_1+\cdots+u_t$, where u_i is in $L_i^\#$ for $1 \leqq i \leqq t$, of $L^\#$ is a m.r. of $L^\#$ modulo L if and only if each u_i is a m.r. of $L_i^\#$ modulo L_i $(1 \leqq i \leqq t)$.*

The proof of this lemma is easy and we omit it. The m. r. of each class of $L^\#$ modulo L is not necessarily unique, but we can prove the;

LEMMA 2-10. *Let $L=L_1 \oplus \cdots \oplus L_t$ be an orthogonal sum of basic lattices, where each L_i is not of type B_n for $i=1, \cdots, t$. Let $x=u+v$ and $x'=u'+v'$ be 2-vectors for L in the problem* (P), *where $u=u_1+u_2+\cdots+u_t$ and $u'=u_1'+u_2+\cdots+u_t$ are elements of $L^\#$ with $u_i \in L_i^\#$ $(2 \leqq i \leqq t)$ ond u_1 and $u_1' \in L_1^\#$. If u_1 and u_1' are two m.r. of the same class of $L_1^\#$ modulo L_1, then $L+\boldsymbol{Z}x$ is isomorphic to $L+\boldsymbol{Z}x'$.*

PROOF. By the condition there is y in L_1 such that $u_1=u_1'+y$. Since u_1 and L_1' are m. r. of the same class of $L_1^\#$ modulo L_1, we have $(u_1, u_1)=(u_1', u_1')$. Put $x''=u_1+u_2+\cdots+u_t+v'$, then it clearly holds that $L+\boldsymbol{Z}x'=L+\boldsymbol{Z}x''$ and $(x'', x'')=(x', x')=(x, x)=2$. It is easy to see that $L+\boldsymbol{Z}x$ is isomorphic to $L+\boldsymbol{Z}x''$. Q. E. D.

Let L be a basic lattice other than B_n type. Two m. r. u and u' of $L^\#$ modulo L are called complementary to each other if they satisfy the condition $u+u' \in L$. The m. r. u_r and u_{n+1-r} in Lemma 2-4 of $A_n^\#$ modulo A_n are complementary to each other. In the D_{2n+1} case m. r. u_1 and u_3 of $D_{2n+1}^\#$ modulo D_{2n+1} are complementary to each other. The m. r. $e_5+\frac{e_6}{\sqrt{3}}$ and $e_5-\frac{e_6}{\sqrt{3}}$ of $E_6^\#$ modulo E_6 are complementary to each other.

LEMMA 2-11. *Let $L=L_1 \oplus \cdots \oplus L_t$ be an orthogonal sum of basic lattices, where each L_i is not of type B_n for $i=1, \cdots, t$. Let $x=u+v$ and $x'=u'+v'$ be 2-vectors for L in the problem* (P), *where $u=u_1+u_2+\cdots+u_t$ and $u'=u_1'+u_2+\cdots+u_t$ are elements of $L^\#$ with $u_i \in L_i^\#$ $(2 \leqq i \leqq t)$ and u_1 and $u_1' \in L_1^\#$ and decompositions of u and u' satisfy the conditions of* (4). *If u_1 and u_1' are complementary m.r. of $L_1^\#$ modulo L_1, then $L+\boldsymbol{Z}x$ is isomorphic to $L+\boldsymbol{Z}x'$.*

PROOF. By the assumption there is y_1 in L_1 such that $u_1'=-u_1+y_1$, and by noting the fact $(u_1', u_1')=(u_1, u_1)$ we have;

$$L+\boldsymbol{Z}x' = L_1 \oplus \cdots \oplus L_t + \boldsymbol{Z}(-u_1+y_1+u_2+\cdots+u_t+v')$$
$$= L_1 \oplus \cdots \oplus L_t + \boldsymbol{Z}(-u_1+u_2+\cdots+u_t+v') .$$

It is easy to see that a mapping φ defined by;

$$\varphi(y_1+y_2+\cdots+y_t) = -y_1+y_2+\cdots+y_t$$

and

$$\varphi(-u_1+u_2+\cdots+u_t+v')=u_1+u_2+\cdots+u_t+v\,,$$

where $y_i\in L_i$ for $1\leqq i\leqq t$, can be extended to an isomorphism from $L+\boldsymbol{Z}x'$ onto $L+\boldsymbol{Z}x$. Q. E. D.

LEMMA 2-12. *Let $L=L_1\oplus\cdots\oplus L_t$ be an orthogonal sum of basic lattices, where each L_i is not of type B_n or of E_8 for $i=1,\cdots,t$. Let $u=u_1+\cdots+u_t$ be a m.r. of $L^{\#}$ modulo L in the manner of* (4). *The forms of L and 2-vector x in the problem* (P) *which satisfy none of two conditions;*

(a) *each L_i is irrelevant to the problem with respect to 2-vector $x=u+v$ and*

(b) *x is trivial for L,*

have the following possibilities, namely;

(i) $L=D_n$ $(n\geqq 4)$, $u=e_n$ *and* $x=u+v$,

(*here and henceforth v is an arbitrarily chosen vector with $(v, L)=0$ and $(x, x)=(u, u)+(v, v)=2$*),

(ii) $L=D_n$, $u=\frac{1}{2}\sum_{i=1}^{n}e_i$ $(4\leqq n\leqq 8)$ *and* $x=u+v$,

(iii) $L=E_6$, $u=e_5+\frac{e_6}{\sqrt{3}}$ *and* $x=u+v$,

(iv) $L=E_7$, $u=e_6+\frac{e_7}{\sqrt{2}}$ *and* $x=u+v$,

(v) $L=A_n$, $u=\frac{1}{n+1}\sum_{i=1}^{n}e_i-\frac{n}{n+1}e_{n+1}$ $(n\geqq 1)$ *and* $x=u+v$,

(vi) $L=A_n$, $u=\frac{2}{n+1}\sum_{i=1}^{n-1}e_i-\frac{n-1}{n+1}(e_n+e_{n+1})$ $(n\geqq 2)$ *and* $x=u+v$,

(vii) $L=A_n$, $u=\frac{3}{n+1}\sum_{i=1}^{n-2}e_i-\frac{n-2}{n+1}(e_{n-1}+e_n+e_{n+1})$ $(3\leqq n\leqq 8)$

and $x=u+v$,

(vii)′ $L=A_7$, $u=\frac{1}{2}(e_1+e_2+e_3+e_4)-\frac{1}{2}(e_5+e_6+e_7+e_8)$, $x=u$,

(viii) $L=E_7\oplus A_1$, $u=e_6+\frac{e_7}{\sqrt{2}}+\frac{1}{2}(f_1-f_2)$ *and* $x=u+v$,

where $A_1=[f_1-f_2]_{\boldsymbol{Z}}$,

(ix) $L=E_6\oplus A_1$, $u=e_5+\frac{e_6}{\sqrt{3}}+\frac{1}{2}(f_1-f_2)$ *and* $x=u+v$,

where $A_1=[f_1-f_2]_{\boldsymbol{Z}}$,

(x) $L=E_6\oplus A_2,\quad u=e_5+\frac{e_6}{\sqrt{3}}+\frac{1}{3}(f_1+f_2-2f_3)=x,$

where $A_2=[f_1-f_2,\ f_2-f_3]_{\boldsymbol{Z}},$

(xi) $L=D_n\oplus D_m\ (n,\ m\geqq 4),\quad u=e_n+f_m=x,$

where $D_m=[f_1-f_2, f_2-f_3, \cdots, f_{m-1}-f_m, f_{m-1}+f_m]_{\boldsymbol{Z}},$

(xii) $L=D_n\oplus D_4\ (n\geqq 4)$ *and* $u=e_n+\frac{1}{2}(f_1+f_2+f_3+f_4)=x,$

where $D_4=[f_1-f_2, f_2-f_3, f_3-f_4, f_3+f_4]_{\boldsymbol{Z}},$

(xiii) $L=D_4\oplus D_4=[e_1-e_2, e_2-e_3, e_3-e_4, e_3+e_4]_{\boldsymbol{Z}}\oplus[f_1-f_2, f_2-f_3,$

$f_3-f_4, f_3+f_4]_{\boldsymbol{Z}}$ *and* $u=\frac{1}{2}(e_1+\cdots+e_4+f_1+\cdots+f_4)=x,$

(xiv) $L=D_n\oplus A_m\ (n\geqq 4,\ m\geqq 1),\quad u=e_n+\frac{1}{m+1}\sum_{i=1}^{m}f_i-\frac{m}{m+1}f_{m+1}$

and $x=u+v,$ *where* $A_m=[f_1-f_2, \cdots, f_m-f_{m+1}]_{\boldsymbol{Z}},$

(xv) $L=D_4\oplus A_m\ (m\geqq 1),\quad u=\frac{1}{2}(e_1+\cdots+e_4)+\frac{1}{m+1}\sum_{i=1}^{m}f_i-\frac{m}{m+1}f_{m+1}$

and $x=u+v,$

(xvi) $L=D_5\oplus A_m\ (1\leqq m\leqq 3),\quad u=\frac{1}{2}(e_1+\cdots+e_5)+\frac{1}{m+1}\sum_{i=1}^{m}f_i$

$-\frac{m}{m+1}f_{m+1}$ *and* $x=u+v,$

(xvii) $L=D_6\oplus A_1=[e_1-e_2, \cdots, e_5-e_6, e_5+e_6]_{\boldsymbol{Z}}\oplus[f_1-f_2]_{\boldsymbol{Z}}$

and $u=\frac{1}{2}(e_1+\cdots+e_6)+\frac{1}{2}(f_1-f_2)=x,$

(xviii) $L=D_n\oplus A_3$ *and* $u=e_n+\frac{1}{2}(f_1+f_2-f_3-f_4)=x,$

(xix) $L=A_n\oplus A_m=[e_1-e_2, \cdots, e_n-e_{n+1}]_{\boldsymbol{Z}}+[f_1-f_2, \cdots, f_m-f_{m+1}]_{\boldsymbol{Z}},$

$u=\frac{1}{n+1}\sum_{i=1}^{n}e_i-\frac{n}{n+1}e_{n+1}+\frac{1}{m+1}\sum_{j=1}^{m}f_j-\frac{m}{m+1}f_{m+1}$ *and* $x=u+v,$

(xx) $L=A_3\oplus A_m\ (m\geqq 1),\quad u=\frac{1}{2}(e_1+e_2-e_3-e_4)+\frac{1}{m+1}\sum_{j=1}^{m}f_j-\frac{m}{m+1}f_{m+1}$

and $x=u+v,$

(xxi) $L=A_4\oplus A_m\ (1\leqq m\leqq 4),\quad u=\frac{2}{5}(e_1+e_2+e_3)-\frac{3}{5}(e_4+e_5)$

$+\frac{1}{m+1}\sum_{j=1}^{m}f_j-\frac{m}{m+1}f_{m+1}$ *and* $x=u+v,$

(xxii) $L=A_5\oplus A_m$ $(1\leqq m\leqq 2)$, $u=\frac{1}{3}(e_1+\cdots+e_4)-\frac{2}{3}(e_5+e_6)$

$+\frac{1}{m+1}\sum_{j=1}^{m}f_j-\frac{m}{m+1}f_{m+1}$ *and* $x=u+v$,

(xxiii) $L=A_1\oplus A_m=[e_1-e_2]_{\boldsymbol{Z}}\oplus[f_1-f_2,\cdots,f_m-f_{m+1}]_{\boldsymbol{Z}}$,

$u=\frac{1}{2}(e_1-e_2)+\frac{2}{m+1}\sum_{j=1}^{m-1}f_j-\frac{m-1}{m+1}(f_m+f_{m+1})$ $(6\leqq m\leqq 7)$

and $x=u+v$,

(xxiv) $L=A_1\oplus A_5$ *and* $u=\frac{1}{2}(e_1-e_2)+\frac{1}{2}(f_1+f_2+f_3-f_4-f_5-f_6)=x$,

(xxv) $L=A_1\oplus A_1\oplus A_m=[e_1-e_2]_{\boldsymbol{Z}}\oplus[f_1-f_2]_{\boldsymbol{Z}}\oplus[g_1-g_2,\cdots,g_m-g_{m+1}]_{\boldsymbol{Z}}$,

$u=\frac{1}{2}(e_1-e_2)+\frac{1}{2}(f_1-f_2)+\frac{1}{m+1}\sum_{i=1}^{m}g_i-\frac{m}{m+1}g_{m+1}$ *and* $x=u+v$,

(xxvi) $L=A_1\oplus A_1\oplus A_3$ *and* $u=\frac{1}{2}(e_1-e_2+f_1-f_2+g_1+g_2-g_3-g_4)=x$,

(xxvii) $L=A_1\oplus A_1\oplus D_4=[e_1-e_2]_{\boldsymbol{Z}}+[f_1-f_2]_{\boldsymbol{Z}}+[g_1-g_2,g_2-g_3,$

$g_3-g_4,g_3+g_4]_{\boldsymbol{Z}}$ *and* $u=\frac{1}{2}(e_1-e_2)+\frac{1}{2}(f_1-f_2)+\frac{1}{2}(g_1+\cdots+g_4)=x$,

(xxviii) $L=A_1\oplus A_1\oplus D_n$ *and* $u=\frac{1}{2}(e_1-e_2)+\frac{1}{2}(f_1-f_2)+g_n=x$,

(xxix) $L=A_1\oplus A_2\oplus A_m=[e_1-e_2]_{\boldsymbol{Z}}\oplus[f_1-f_2,f_2-f_3]_{\boldsymbol{Z}}\oplus[g_1-g_2,\cdots,$

$g_m-g_{m+1}]_{\boldsymbol{Z}}$ $(1\leqq m\leqq 5)$, $u=\frac{1}{2}(e_1-e_2)+\frac{1}{3}(f_1+f_2-2f_3)$

$+\frac{1}{m+1}\sum_{i=1}^{m}g_i-\frac{m}{m+1}g_{m+1}$ *and* $x=u+v$,

(xxx) $L=A_1\oplus A_3\oplus A_3=[e_1-e_2]_{\boldsymbol{Z}}\oplus[f_1-f_2,\cdots,f_3-f_4]_{\boldsymbol{Z}}\oplus[g_1-g_2,\cdots,$

$g_3-g_4]_{\boldsymbol{Z}}$ *and* $u=\frac{1}{2}(e_1-e_2)+\frac{1}{4}(f_1+f_2+f_3-3f_4)$

$+\frac{1}{4}(g_1+g_2+g_3-3g_4)=x$,

(xxxi) $L=A_2\oplus A_2\oplus A_2$ *and* $u=\frac{1}{3}(e_1+e_2-2e_3)+\frac{1}{3}(f_1+f_2-2f_3)$

$+\frac{1}{3}(g_1+g_2-2g_3)=x$,

(xxxii) $L=A_1\oplus A_1\oplus A_1\oplus A_1=[e_1-e_2]_{\boldsymbol{Z}}+[f_1-f_2]_{\boldsymbol{Z}}+[g_1-g_2]_{\boldsymbol{Z}}+[h_1-h_2]_{\boldsymbol{Z}}$

and $u=\frac{1}{2}(e_1-e_2+f_1-f_2+g_1-g_2+h_1-h_2)=x$.

This lemma is easily proved by using of Lemma 2-4, Remark 1, Lemma 2-7, 2-8 and Lemma 2-9 and we omit it. In the above lemma some of possibilities are omitted but this omission is justified by Lemma 2-5, Lemma 2-10 and Lemma 2-11.

LEMMA 2-13. *Let $\mathcal{L}=A_n+\boldsymbol{Z}x$, where x is a 2-vector such that $(x, A_n)\subseteqq \boldsymbol{Z}$, then the structure of $\mathcal{L}$ is determined according to the value of n in the following way;*

(i) *when $1\leqq n\leqq 4$ or $n\geqq 9$, then $\mathcal{L}=A_n$ or $\mathcal{L}=A_n\oplus A_1$ or $\mathcal{L}\cong A_{n+1}$ or $\mathcal{L}\cong$ $\cong D_{n+1}$,* (ii) *when $5\leqq n\leqq 7$, then $\mathcal{L}=A_n$ or $\mathcal{L}=A_n\oplus A_1$ or $\mathcal{L}\cong A_{n+1}$ or $\mathcal{L}\cong D_{n+1}$ or $\mathcal{L}\cong E_{n+1}$,* (iii) *when $n=8$, $\mathcal{L}=A_8$ or $\mathcal{L}=A_8\oplus A_1$ or $\mathcal{L}\cong A_9$ or $\mathcal{L}\cong D_9$ or $\mathcal{L}\cong E_8$.*

PROOF. Let u be the $A_n^\#$ part of 2-vector x and $x=u+v$ with $(v, A_n^\#)=0$. If u is zero, then we have $(v, v)=2$ and $A_n+\boldsymbol{Z}x=A_n\oplus A_1$. If $(u, u)=2$ and $u\in A_n$, then we have $v=0$ and $A_n+\boldsymbol{Z}x=A_n$. So we have only to consider the remaining possibilities, namely, $(u, u)\leqq 2$ and $u\in A_n^\#-A_n$. By Lemma 2-8 we can assume that u is a m. r. of $A_n^\#$ modulo A_n. M. r. $u\neq 0$ of $A_n^\#$ modulo A_n with $(u, u)\leqq 2$ are listed at Lemma 2-12, (v), (vi) and (vii). In case of (v) v is not zero and x is linearly independent over Q from $e_1-e_2, \cdots, e_n-e_{n+1}$, the basis of A_n. Hence $\mathcal{L}$ has the basis $e_1-e_2, \cdots, e_n-e_{n+1}$ and $-x$ over $\boldsymbol{Z}$, and $\mathcal{L}$ is isomorphic to A_{n+1} by Lemma 2-5, (iii). In case of (vi) v is not zero and x is linearly independent over Q from $e_1-e_2, \cdots, e_n-e_{n+1}$, the basis of A_n. Hence $\mathcal{L}$ has the basis $e_1-e_2, \cdots, e_n-e_{n+1}$ and $-x$ over $\boldsymbol{Z}$, so $\mathcal{L}$ is isomorphic to D_{n+1} by Lemma 2-5, (iii). (Note that $D_3\cong A_3$). In case of (vii) v is not zero for $3\leqq n\leqq 7$ and then x is linearly independent from $e_1-e_2, \cdots, e_n-e_{n+1}$ over Q. When $n=3$, with the basis $-x$, e_1-e_2, e_2-e_3 and e_3-e_4 $\mathcal{L}$ is isomorphic to A_4 by Lemma 2-5, (iii). (For the reason one should recall of the canonical basis of A_4.) When $n=4$, with the basis e_4-e_5, e_3-e_4, e_2-e_3, e_1-e_2 and $-x$ $\mathcal{L}$ is isomorphic to D_5 by Lemma 2-5, (iii). When $5\leqq n\leqq 7$, with the basis e_1-e_2, $e_2-e_3, \cdots, e_{n-2}-e_{n-1}, -x, e_{n-1}-e_n, e_{n+1}-e_n$ $\mathcal{L}$ is isomorphic to E_{n+1}. When $n=8$, v is zero and x is linearly dependent on $e_1-e_2, \cdots, e_8-e_9$ over Q. This time e_1-e_2 is linearly expressed over $\boldsymbol{Z}$ as;

$$e_1-e_2=\sum_{i=2}^{8} a_i(e_i-e_{i+1})+a_9x\,,$$

where $a_i=-i$ $(2\leqq i\leqq 6)$, $a_7=-4$, $a_8=-2$ and $a_9=3$. Hence 2-vectors $e_2-e_3, \cdots,$ e_6-e_7, $-x$, e_7-e_8, e_9-e_8 the basis of $\mathcal{L}$ and $\mathcal{L}$ is isomorphic to E_8. In case of (vii)′, by the same argument, $\mathcal{L}$ is isomorphic to E_7. By rearranging the above arguments we have the form of lemma. Q. E. D.

With similar arguments to the proof of Lemma 2-13 we can prove the following Lemmas 2-14, 2-15 and 2-16 and we shall omit those proofs. (The reader can prove these lemmas by using Lemmas 2-5, 2-9, 2-10, 2-11 and 2-12.)

LEMMA 2-14. *Let $\mathcal{L}=D_n+\boldsymbol{Z}x$, where x is a 2-vector such that $(x, D_n)\subseteqq \boldsymbol{Z}$*

and $n\geqq 4$, *then the structure of* $\mathcal{L}$ *is determined according to the value of* n *in the following way;*

(i) *when* $n=4$ *or* $n\geqq 9$, *then* $\mathcal{L}=D_n\oplus A_1$ *or* $\mathcal{L}=D_n$ *or* $\mathcal{L}\cong D_{n+1}$, (ii) *when* $5\leqq n\leqq 7$, *then* $\mathcal{L}=D_n\oplus A_1$ *or* $\mathcal{L}=D_n$ *or* $\mathcal{L}\cong D_{n+1}$ *or* $\mathcal{L}\cong E_{n+1}$, (iii) *when* $n=8$, *then* $\mathcal{L}=D_8\oplus A_1$ *or* $\mathcal{L}=D_8$ *or* $\mathcal{L}\cong D_9$ *or* $\mathcal{L}\cong E_8$.

LEMMA 2-15. *Let* $\mathcal{L}=E_j+\boldsymbol{Z}x$ $(6\leqq j\leqq 8)$, *where* x *is a 2-vector such that* $(x, E_j)\subseteqq \boldsymbol{Z}$, *then we have;*

(i) *if* $j=6$ *or* 7, *then* $\mathcal{L}=E_j$ *or* $\mathcal{L}=E_j\oplus A_1$ *or* $\mathcal{L}\cong E_{j+1}$, (ii) *if* $j=8$, *then* $\mathcal{L}=E_8$ *or* $\mathcal{L}=E_8\oplus A_1$.

LEMMA 2-16. *Let* $L=L_1\oplus\cdots\oplus L_t$ *be an orthogonal sum of basic lattices, where each component is other than* B_n *type and* $t\geqq 2$. $\mathcal{L}=L+\boldsymbol{Z}x$, *where* x *is a 2-vector such that* $(x, L)\subseteqq\boldsymbol{Z}$, *is the solution of the problem* (P) *which satisfies none of two conditions* (a) *and* (b) *stated in Lemma* 2-12 *if and only if* L *has one of the forms given in Lemma* 2-12, (viii)-(xxxii) *and* x *is a 2-vectors uniquely attached to such* L. *According to the numbering of such* L *we have the following isomorphisms;*

(viii) $\mathcal{L}\cong E_8$, (ix) $\mathcal{L}\cong E_7$, (x) $\mathcal{L}\cong E_8$, (xi) $\mathcal{L}\cong D_{m+n}$, (xii) $\mathcal{L}\cong D_{n+4}$, (xiii) $\mathcal{L}\cong D_8$, (xiv) $\mathcal{L}\cong D_{m+n+1}$, (xv) $\mathcal{L}\cong D_{m+5}$, (xvi) $\mathcal{L}\cong E_{5+m}$ $1\leqq m\leqq 3$, (xvii) $\mathcal{L}\cong E_7$, (xviii) $\mathcal{L}\cong D_{n+3}$, (xix) $\mathcal{L}\cong A_{m+n+1}$, (xx) $\mathcal{L}\cong D_{m+4}$, (xxi) $\mathcal{L}\cong E_{m+5}$ *for* $1\leqq m\leqq 3$ *and* $\mathcal{L}\cong E_8$ *for* $m=4$, (xxii) $\mathcal{L}\cong E_{m+6}$ $(1\leqq m\leqq 2)$, (xxiii) $\mathcal{L}\cong E_8$, (xxiv) $\mathcal{L}\cong E_6$, (xxv) $\mathcal{L}\cong D_{m+3}$, (xxvi) $\mathcal{L}\cong D_5$, (xxvii) $\mathcal{L}\cong E_6$, (xxviii) $\mathcal{L}\cong D_{n+2}$, (xxix) $\mathcal{L}\cong E_{n+4}$ *for* $2\leqq n\leqq 4$ *and* $\mathcal{L}\cong E_8$ *for* $n=5$, (xxx) $\mathcal{L}\cong E_7$, (xxxi) $\mathcal{L}\cong E_6$, (xxxii) $\mathcal{L}\cong D_4$.

Now it is an easy matter to prove Proposition 2-2. Suppose a lattice L is generated by 2-vectors $\alpha_1,\cdots,\alpha_l$ in L. $\boldsymbol{Z}\alpha_1$ is a sublattice of L isomorphic to A_1. $\boldsymbol{Z}\alpha_1+\boldsymbol{Z}\alpha_2$ is then equal to $\boldsymbol{Z}\alpha_1$ or $\boldsymbol{Z}\alpha_1\oplus\boldsymbol{Z}\alpha_2\cong A_1\oplus A_1$ or isomorphic to A_2 by Lemma 2-13, (i). We continue this argument inductively. If $\boldsymbol{Z}\alpha_1+\cdots+\boldsymbol{Z}\alpha_j$ $(j\leqq l)$ is isomorphic to an orthogonal sum of basic lattices whose components are other than B_n type, then $\boldsymbol{Z}\alpha_1+\cdots+\boldsymbol{Z}\alpha_j+\boldsymbol{Z}\alpha_{j+1}$ is also isomorphic to an orthogonal sum of basic lattices by Lemmas 2-13, 2-14, 2-15 and 2-16. In this way we can say that $\boldsymbol{Z}\alpha_1+\cdots+\boldsymbol{Z}\alpha_l$ is isomorphic to an orthogonal sum of basic lattices. Since $\alpha_1,\cdots,\alpha_l$ generate L, $\boldsymbol{Z}\alpha_1+\cdots+\boldsymbol{Z}\alpha_l=L$ and this implies that L has the basis consisting of 2-vectors. As an immediate consequence of the above considerations we have;

PROPOSITION 2-17. *Let* L *be an irreducible lattice generated by 2-vectors, then* L *is isomorphic to one of the lattices* A_n, D_n $(n\geqq 4)$, E_6, E_7 *and* E_8.

§ 3. Main results.

Let L be a lattice. For the first time we decompose L as (1). So that L' has no 1-vector. Since L has a positive definite metric, the number of 2-

vectors is finite. We consider the sublattice M of L' generated by 2-vectors of L'. In general it holds that rank $M \leqq$ rank L' and M is not necessarily irreducible. We can say that by the arguments in the proof of Proposition 2-2 M is written as;

$$M = M_1 \oplus \cdots \oplus M_k$$

where each M_j is irreducible and generated by 2-vectors. Then by Proposition 2-17 each M_j is isomorphic to one of A_n, D_n, E_6, E_7, or E_8. The sublattice $M \oplus B_r$ of L is called characteristic sublattice of L. We shall denote by $V_2(L)$ the number of 2-vectors in the lattice L.

LEMMA 3-1. *We have the following formulas;*

(i) $V_2(A_n) = n(n+1)$,

(ii) $V_2(B_n) = V_2(D_n) = 2n(n-1)$ if $n \geqq 4$,

(iii) $V_2(E_8) = 240$,

(iv) $V_2(E_7) = 126$,

(v) $V_2(E_6) = 72$.

These are well-known results and we neglect its proofs. Though A_0, D_0, B_0, D_1 and D_2 are meaningless, for the simplicity of the later description we shall write $V_2(A_0) = V_2(D_0) = V_2(B_0) = 0$, $V_2(D_1) = V_2(B_1) = 0$, $V_2(D_2) = V_2(B_2) = 4$ and $V_2(D_3) = V_2(B_3) = 12$. D_3 is isomorphic to A_3.

LEMMA 3-2. *Assume that the lattices $L_1, \cdots, L_k$ do not contain any 1-vector, then we have;*

$$L_2(L_1 \oplus \cdots \oplus L_k) = \sum_{i=1}^{k} V_2(L_i).$$

PROOF. It is sufficient to show that any 2-vector in $L_1 \oplus \cdots \oplus L_k$ belongs to some L_i. If it is not true, then there is a 2-vector u which can be written in the form $u = u_{i_1} + u_{i_2}$ with $i_1 \neq i_2$, $u_{i_1} \in L_{i_1}$ and $u_{i_2} \in L_{i_2}$. Then we have;

$$\begin{aligned}(u, u) &= (u_{i_1}, u_{i_1}) + (u_{i_2}, u_{i_2}) \\ &= 2.\end{aligned}$$

The case $(u_{i_1}, u_{i_1}) = (u_{i_2}, u_{i_2}) = 1$ does not appear by the assumption. So we can say that $(u_{i_1}, u_{i_1}) = 2$ or $(u_{i_2}, u_{i_2}) = 2$ and the rest vector equals to zero. In either case $u = u_{i_1}$ or $u = u_{i_2}$. Q. E. D.

LEMMA 3-3. *Assume that the lattice L_1 does not contain any 1-vector, then we have;*

$$\begin{aligned}V_2(B_r + L_1) &= V_2(B_r) + V_2(L_1) \\ &= 2r(r-1) + V_2(L_1)\end{aligned}$$

where r is a positive rational integer.

This lemma is proved in a similar way to the proof of Lemma 3-2, and we omit the proof.

LEMMA 3-4. *Let n_1, n_2 be integers not smaller than one, then we have the following inequalities;*

(i) $V_2(A_{n1}\oplus A_{n2}) < V_2(A_{n1+n2})$,

(ii) $V_2(D_{n1}\oplus D_{n2}) < V_2(D_{n1+n2})$,

where both n_1 and n_2 are not smaller than 3,

(iii) $V_2(A_{n1}) < V_2(D_{n1})$ *if* $n_1 \geqq 4$,

(iv) $V_2(A_{n1}\oplus D_{n2}) \leqq V_2(D_{n1+n2})$ *if* $n_1+n_2 \geqq 4$,

(v) $V_2(A_{n1}\oplus B_{n2}) \leqq V_2(D_{n1+n2})$ *if* $n_1+n_2 \geqq 4$.

PROOF. Proof of (i).

By Lemmas 3-1 and 3-2 we know that;

$$\begin{aligned} V_2(A_{n1}\oplus A_{n2}) &= V_2(A_{n1}) + V_2(A_{n2}) \\ &= n_1(n_1+1) + n_2(n_2+1) \\ &< (n_1+n_2)(n_1+n_2+1) \\ &= V_2(A_{n1+n2}) . \end{aligned}$$

In the same way (ii)~(v) can be proved. Q. E. D.

LEMMA 3-5.

(i) $V_2(E_8\oplus E_8) = V_2(D_{16})$,

(ii) $V_2(E_8\oplus E_7) < V_2(D_{15})$,

(iii) $V_2(E_7\oplus E_7) < V_2(D_{14})$,

(iv) $V_2(E_6\oplus E_7) < V_2(D_{13})$,

(v) $V_2(E_6\oplus E_7) < V_2(D_{13})$,

(vi) $V_2(E_6\oplus E_6) < V_2(D_{12})$.

PROOF. Proof of (i).

By Lemmas 3-1 and 3-2 we know that;

$$\begin{aligned} V_2(E_8\oplus E_8) &= V_2(E_8) + V_2(E_8) \\ &= 480 \\ &= V_2(D_{16}) . \end{aligned}$$

In the same way (ii)~(iv) can be proved. Q. E. D.

LEMMA 3-6. *The following inequalities hold;*

(i) $V_2(E_6\oplus A_n)<V_2(D_{n+6})$ *if* $n\geqq 1$,

(ii) $V_2(E_7\oplus A_n)<V_2(D_{n+7})$ *if* $n\geqq 2$,

(iii) $V_2(E_8\oplus A_n)<V_2(D_{n+8})$ *if* $n\geqq 4$,

(iv) $V_2(E_6\oplus D_n)<V_2(D_{n+6})$ *if* $n\geqq 4$,

(v) $V_2(E_7\oplus D_n)<V_2(D_{n+7})$ *if* $n\geqq 4$,

(vi) $V_2(E_8\oplus D_n)\leqq V_2(D_{n+8})$ *if* $n\geqq 4$.

PROOF. Proof of (i).

By Lemmas 3-1 and 3-2 we know that;

$$\begin{aligned}V_2(D_{n+6})-V_2(E_6\oplus A_n)&=2(n+6)(n+5)-72-n(n+1)\\&=n^2+21n-12>0 \quad \text{if } n\geqq 1.\end{aligned}$$

In the same way (ii)~(vi) can be proved. Q.E.D.

LEMMA 3-7. *Let s, t, p be non-negative real numbers and* $s+t+p=\sigma>0$, *then we have the following inequalities;*

$$72\sigma\leqq 72s+126t+240p\leqq 240\sigma,$$

$$6\sigma\leqq 6s+7t+8p\leqq 8\sigma.$$

Since the proof of this lemma is easy, we omit it.

LEMMA 3-8. *Let N be a lattice of the form;*

$$N=(\underset{s}{\oplus}E_6)\oplus(\underset{t}{\oplus}E_7)\oplus(\underset{p}{\oplus}E_8)$$

where s, t, p are non-negative rational integers and the symbol $\underset{s}{\oplus}E_6$ *means orthogonal sum of s times of* E_6*'s and so on. If* $s+t+p\geqq 2$, *then we get the following inequality;*

$$V_2(N)\leqq V_2(D_{6s+7t+8p}). \qquad (12)$$

PROOF. By Lemmas 3-1 and 3-2 we know that;

$$V_2(N)=72s+126t+240p$$

and

$$V_2(D_{6s+7t+8p})=2(6s+7t+8p)(6s+7t+8p-1).$$

If $s+t+p=2$, then the inequality (12) is nothing else one of the inequalities in Lemma 3-4. If $s+t+p=3$, then the inequality (12) is also proved by using Lemmas 3-4 and 3-5. If $s+t+p=\sigma\geqq 4$ then we know from Lemmas 3-1, 3-2 and 3-7 that;

$$V_2(D_{6s+7t+8p})-V_2(N)$$
$$=2(6s+7t+8p)(6s+7t+8p-1)-(72s+126t+240p)$$
$$\geqq 2\times 6\sigma(6\sigma-1)-240\sigma$$
$$=\sigma(72\sigma-252)>0 \qquad (\sigma\geqq 4)\,.$$

Q. E. D.

LEMMA 3-9. *Let l, m, q be non-negative rational integers such that $l+m+q \geqq 4$, then we have the inequality;*

$$V_2(A_l\oplus D_m\oplus B_q)\leqq V_2(D_{l+m+q})\,. \qquad \text{.............. (13)}$$

PROOF. When $l+q\leqq 3$, by Lemmas 3-1 and 3-2 we can verify that;

$$V_2(A_l\oplus B_q)\leqq V_2(A_{l+q})\,.$$

By the assumptions that $l+m+q\geqq 4$ and by (iv) of Lemma 3-3 we can say that;

$$V_2(A_{l+q}\oplus D_m)\leqq V_2(D_{l+q+m})\,.$$

When $l+q\geqq 4$, by (v) of Lemma 7 we know;

$$V_2(A_l\oplus B_q)\leqq V_2(D_{l+q})\,.$$

By the same assumption and by (ii) of Lemma 3-3 we can also say that;

$$V_2(D_{l+q}\oplus D_m)\leqq V_2(D_{l+m+q})\,.$$

In either case we have established the inequality (13) using Lemma 3-2.

Q. E. D.

THEOREM 1. *Let $\mathcal{L}_n$ be the set of all integral lattices with fixed* rank n. *If n is not less than* 17, *then we have that;*

$$\operatorname*{Max}_{L\in\mathcal{L}_n} V_2(L)=2n(n-1) \qquad \text{.................... (14)}$$

and $V_2(L)=2n(n-1)$ is attained by $L=B_n$ or $L=D_n$.

PROOF. Let $L\in\mathcal{L}_n$ and we decompose L into the form (1);

$$L=B_r\oplus L' \qquad r\geqq 0\,.$$

Let M be the characteristic sublattice of L', then M can be written in the following form by Propositions 2-2, 2-15;

$$M=(\bigoplus_s E_6)\oplus(\bigoplus_t E_7)\oplus(\bigoplus_p E_8)\oplus(\bigoplus_{i=1}^{l} A_{n_i})\oplus(\bigoplus_{j=1}^{h} D_{m_j})$$

where l and h are non-negative rational integers and n_i and m_j are positive rational integers. It clearly holds that;

$$\text{rank } L' \geqq 6s+7t+8p+\sum_{i=1}^{l} n_i+\sum_{j=1}^{h} m_j .$$

Put

$$M_1=(\bigoplus_s E_6)\oplus(\bigoplus_t E_7)\oplus(\bigoplus_p E_8)\oplus(\bigoplus_{i=1}^{l} A_{n_i})\oplus(\bigoplus_{j=1}^{h} D_{m_j})\oplus B_r\oplus B_q \quad \ldots . \ (15)$$

where $q=\text{rank } L'-\text{rank } M$.

Then we can see that $M_1\in\mathcal{L}_n$ and that

$$V_2(M_1)\geqq V_2(M) \qquad \text{by Lemmas 3-2 and 3-3}.$$

Put

$$M_2=(\bigoplus_s E_6)\oplus(\bigoplus_t E_7)\oplus(\bigoplus_p E_8)\oplus A_\lambda\oplus D_\mu\oplus B_{q_1}$$

where λ is the number $\sum_{i=1}^{l} n_i$ and μ is the number $\sum_{j=1}^{h} m_j$ and $q_1=r+q$. If $l=0$, then we shall understand that the part A_λ does not appear. If $h=0$, then we shall understand that the part D_μ does not appear. At any rate, we can see that $M_2\in\mathcal{L}_n$ and that $V_2(M_2)\geqq V_2(M_1)$ by using Lemmas 3-2, 3-3 and 3-4. When $s+t+p\geqq 2$, we put

$$M_3=D_{6s+7t+8p}\oplus A_\lambda\oplus D_\mu\oplus B_{q_1}.$$

Then by Lemmas 3-5 and 3-8 we can say that $M_3\in\mathcal{L}_n$ and that $V_2(M_3)\geqq V_2(M_2)$. Moreover we put

$$M_4=A_\lambda\oplus D_{6s+7t+8p+\mu}\oplus B_{q_1}.$$

Then by Lemmas 3-3 and 3-5 we can see that $M_4\in\mathcal{L}_n$ and that

$$V_2(M_4)\geqq V_2(M_2).$$

Thus we started from any element L of $\mathcal{L}_n$ and arrived a suitable element M of $\mathcal{L}_n$ such that $V_2(M)\geqq V_2(L)$, where M is one of the following forms;

(i) $E_\tau\oplus A_\lambda\oplus D_\mu\oplus B_{q_1}$

where $\tau=6$ or 7 or 8 and $\tau+\lambda+\mu+q_1=n=\text{rank } L$,

(ii) $A_\lambda\oplus D_\mu\oplus B_{q_1}$

where $\lambda+\mu+q_1=n=\text{rank } L$.

For the first case, by the assumption that $n\geqq 17$ we can see that

$$\lambda+\mu+q_1\geqq 9.$$

Then by Lemma 3-9 we can say that;

$$V_2(A_\lambda\oplus D_\mu\oplus B_{q_1})\leqq V_2(D_{\lambda+\mu+q_1}).$$

Now we can see that $E_\tau\oplus D_{\lambda+\mu+q_1}\in\mathcal{L}_n$ and that

$$V_2(L) \leqq V_2(E_\tau \oplus D_{\lambda+\mu+q_1}) .$$

But we know that $D_{\tau+\lambda+\mu+q_1} \in \mathcal{L}_n$ and that

$$V_2(E_\tau \oplus D_{\lambda+\mu+q_1}) \leqq V_2(D_{\tau+\lambda+\mu+q_1}) \qquad \text{by Lemma 3-6}.$$

For the second case, we can see that

$$D_{\lambda+\mu+q_1} \in \mathcal{L}_n$$

and that

$$V_2(A_\lambda \oplus D_\mu \oplus B_{q_1}) \leqq V_2(D_{\lambda+\mu+q_1}) \qquad \text{by Lemma 3-9}.$$

Theorem 1 is thus established if we take (ii) of Lemma 3-1 into account.

Q. E. D.

We call the rank of characteristic sublattice M of L as the characteristic number and we denote it by $c(L)$. In the following we shall prove;

THEOREM 2. *Let $\mathcal{L}_n$ be the same set as Theorem* 1, *and let $d(L)$ be the determinant of a lattice $L \in \mathcal{L}_n$. If $d(L) > 2^n$, then we have the inequality;*

$$c(L) < \operatorname{rank} L .$$

To prove this theorem, we need some lemmas.

LEMMA 3-10. *Let L_1 and L_2 be lattices, then we have;*

$$d(L_1 \oplus L_2) = d(L_1) d(L_2) .$$

Since this lemma is easy to prove, we omit the proof. For the basic lattices, we know the following:

LEMMA 3-11.

(i) $d(D_n) = 4 \qquad n \geqq 4$,

(ii) $d(E_8) = 1$,

(iii) $d(E_7) = 2$,

(iv) $E(E_6) = 3$,

(v) $d(A_n) = n+1$,

(vi) $d(B_n) = 1$.

We also omit the proof of this lemma. We shall prove Theorem 2. Let L be a lattice in $\mathcal{L}_n$, and assume that $c(L) = \operatorname{rank} L$, then the characteristic sublattice M of L is written in the form;

$$M = (\bigoplus_s E_6) \oplus (\bigoplus_t E_7) \oplus (\bigoplus_p E_8) \oplus (\bigoplus_{i=1}^{l} A_{ni}) \oplus (\bigoplus_{j=1}^{h} D_{mj}) \oplus B_q$$

where $6s+7t+8p+\sum_{i=1}^{l} n_i+\sum_{j=1}^{h} m_j+q=\text{rank}\,L$. It is known that the determinant of a sublattice of a lattice is divisible by the determinant of the latter if both rank have the same value, so we can say that;

$$d(M)\geqq d(L).$$

It is clearly that M also belongs to $\mathcal{L}_n$.

We put; $M'=\bigoplus_n A_1$.

We can see that M' belongs to $\mathcal{L}_n$. If we can show that $d(M')$ is not less than $d(M)$ and that $d(M')$ is equal to 2^n, then we complete the proof. By Lemmas 3-10 and 3-11 we have that;

$$d(M)=3^s\times 2^t\times(\prod_{i=1}^{l}(n_i+1))\times 2^{2h}.$$

Also by Lemmas 3-10 and 3-11 we have that;

$$\begin{aligned} d(M') &= 2^{6s}\times 2^{7t}\times 2^{8p}\times 2^{\sum_{i=1}^{l} n_i}\times 2^{\sum_{j=1}^{h} m_j}\times 2^q \\ &= 2^n. \end{aligned}$$

It is clear that the following inequalities hold;

$$\begin{aligned} &3^s\leqq 2^{6s}, \\ &2^t\leqq 2^{7t}, \\ &1\leqq 2^{8p}, \\ &2^{2h}\leqq 2^{\sum_{j=1}^{h} m_j}, \\ &1\leqq 2^q. \end{aligned}$$

If we can show that the inequality;

$$k+1\leqq 2^k \quad \text{.............................} \quad (16)$$

holds for each positive rational integer k, then we can say that;

$$\prod_{i=1}^{l}(n_i+1)\leqq 2^{\sum_{i=1}^{l} n_i}$$

and consequently that $d(M)\leqq d(M')$.

The inequality (16) is easily proved by the induction on k, and we omit its proof. Q. E. D.

It should be remarked that there always exists a lattice L whose characteristic number is equal to rank L minus one and its determinant is arbitrary positive rational integer d. Such a lattice L is given as follows;

$$L=[e_1, \cdots, e_{n-1}, \sqrt{d}\,e_n]_Z \qquad \text{where } n \text{ is the rank } L.$$

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