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Determination of homotopy spheres that admit free actions of finite cyclic groups

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Introduction.

In this paper, we shall determine the homotopy spheres that admit free actions of the finite cyclic group $\boldsymbol{Z}_m$ where m is an integer. In the case of free involutions, namely when $m=2$, Lopez de Medrano gave an answer in [**6**] using the results of Browder [**2**] on Kervaire invariants. Also, Orlik [**9**] showed that every homotopy sphere that bounds a parallelizable manifold admits a free $\boldsymbol{Z}_{p^r}$-action where p is an odd prime by constructing explicit examples on Brieskorn spheres.

If one tries to follow the line of Lopez de Medrano when m is an arbitrary integer, one faces with the difficulty when $m\equiv 0 \pmod 4$. So we shall adopt the philosophy of Brumfiel [**3**]. In this process, we must construct a surgery theory on manifolds with singularity which are called $\tilde{\boldsymbol{Z}}_m$-manifolds in this paper (§§ 4, 5). We shall give a brief view of our program:

§ 1: We state our main result (Theorem 6.1) together with notations which will be frequently used in this paper.

§ 2: We construct a free $\boldsymbol{Z}_m$-action on a Brieskorn sphere of dimension $=4k+1$. This example plays an important rôle in later sections.

§ 3: We discuss the surgery theory on odd-dimensional manifolds with $\pi_1=\boldsymbol{Z}_m$ improving the result of Wall [**13**] 14E.4.

§ 4: The definition and elementary properties of $\tilde{\boldsymbol{Z}}_m$-manifolds are stated.

§ 5: The results of § 3 and § 4 are combined to yield the surgery theory for "simply connected" $\tilde{\boldsymbol{Z}}_m$-manifolds.

§ 6: The results of § 3 and § 5 are applied to give a proof of our main theorem.

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§ 1. Statement of the main theorem.

We have a linear $\boldsymbol{Z}_m$-action on $S^{2n+1}\subset \boldsymbol{C}^{n+1}$ where the action is given by $(z_0, z_1, \cdots, z_n)\mapsto(\alpha z_0, \alpha^{p_1}z_1, \cdots, \alpha^{p_n}z_n)$ with $\alpha=\exp(2\pi i/m)$ and $(p_j, m)=1$. The quotient space of S^{2n+1} under this action is the lens space denoted by $L^{2n+1}(m;p_1,\cdots,p_n)$. It is well known that two lens spaces $L^{2n+1}(m;p_1,\cdots,p_n)$ and $L^{2n+1}(m;q_1,\cdots,q_n)$ are homotopy equivalent preserving the natural orientations if $p_1\cdots p_n\equiv q_1\cdots q_n \pmod m$. Also it is known that for any free $\boldsymbol{Z}_m$-action on a homotopy sphere Σ^{2n+1}, the quotient space is homotopy equivalent to $L^{2n+1}(m;p_1,\cdots,p_n)$ for some appropriate choice of $p_1,\cdots,p_n$. Hence $\Sigma^{2n+1}/\boldsymbol{Z}_m$ is homotopy equivalent to $L_q^{2n+1}=L^{2n+1}(m;q,1,\cdots,1)$ for some q. In this case, we shall call this action a free $\boldsymbol{Z}_m$-action of type q. Our main result is

MAIN THEOREM. *A homotopy sphere Σ^{2n+1} admits a free $\boldsymbol{Z}_m$-action of type q if and only if its normal invariant $\eta(\Sigma)$ belongs to the subgroup $\pi_q^*([L_q^{2n+1}, G/O])$ of $\pi_{2n+1}(G/O)$ where π_q^* is the natural map induced by the projection $\pi_q: S^{2n+1}\to L_q^{2n+1}$ and $n\geqq 3$.*

We shall fix some notations which will be frequently used in this paper. We have a standard CW-decomposition of the lens space $L^{2n+1}(m;p_1,\cdots,p_n)$ with cells $e^0, e^1, \cdots, e^{2n+1}$ where

$$e^{2r}=\{[z_0,\cdots,z_r,0,\cdots,0]\mid z_r\neq 0 \text{ and } \arg(z_r)=0\}$$

and

$$e^{2r+1}=\{[z_0,\cdots,z_r,0,\cdots,0]\mid z_r\neq 0 \text{ and } 0<\arg(z_r)<2\pi/m\}\,.$$

Let $\hat{L}^{2n}(m;p_1,\cdots,p_{n-1})$ be the mapping cone of the natural projection $S^{2n-1}\to L^{2n-1}(m;p_1,\cdots,p_{n-1})$. Then $\hat{L}^{2n}(m;p_1,\cdots,p_{n-1})$ is homeomorphic to the $2n$-skeleton of $L^{2n+1}(m;p_1,\cdots,p_{n-1},p_n)$ under the standard CW-decomposition above. The following notations are used when there is no fear of confusion:

$$L_q^{2n+1}=L^{2n+1}(m;q,1,\cdots,1)\,,$$

$$\hat{L}_q^{2n}=\hat{L}^{2n}(m;q,1,\cdots,1)\,,$$

$$L^{2n+1}=L^{2n+1}(m;p_1,\cdots,p_n)$$

and

$$\hat{L}^{2n}=\hat{L}^{2n}(m;p_1,\cdots,p_{n-1})\,.$$

§ 2. Free $\boldsymbol{Z}_m$-actions on Brieskorn spheres.

Let $f(z_0,z_1,\cdots,z_{2k+1})=z_0^s+z_1^2+\cdots+z_{2k+1}^2$ be a complex valued function on $\boldsymbol{C}^{2k+2}$ with $s\equiv\pm 3 \pmod 8$ and $(s,m)=1$. The existence of such an integer s is assured by the existence of infinitely many primes which are of the form $8j\pm 3$. Then it is well known that the manifold $\Sigma_s^{4k+1}=f^{-1}(0)\cap S^{4k+3}$ is a homotopy sphere bounding a parallelizable manifold and that Σ_s^{4k+1} is not diffeomorphic to the standard sphere in dimensions where "Kervaire invariant conjecture" holds. We define a $\boldsymbol{Z}_m$-action on $\boldsymbol{C}^{2k+2}$ by

$$(z_0,z_1,\cdots,z_{2k+1})\longmapsto(\alpha^{2t}z_0,\alpha z_1,\cdots,\alpha z_{2k+1})$$

where $\alpha=\exp(2\pi i/m)$ and $st\equiv 1 \pmod m$. Clearly, this Z_m-action keeps S^{4k+3} invariant. It also keeps $f^{-1}(0)$ invariant since $f(\alpha^{2t}z_0, \alpha z_1, \cdots, \alpha z_{2k+1})=\alpha^2 f(z_0, z_1, \cdots, z_{2k+1})$ holds. Hence this action induces a Z_m-action T_s on Σ_s^{4k+1}. We can easily verify that the Z_m-action (Σ_s^{4k+1}, T_s) is free.

Now let $\varphi_1: \Sigma_s^{4k+1}/T_s \to L_1^{4k+1}$ and $\varphi_2: L_1^{4k+1} \to L_t^{4k+1}$ be defined by

$$\varphi_1([z_0, z_1, \cdots, z_{2k+1}])=[z_1/c_1, \cdots, z_{2k+1}/c_1]$$

$$\varphi_2([u_0, u_1, \cdots, u_{2k}])=[u_0/c_2, u_1^t/c_2, u_2/c_2, \cdots, u_{2k}/c_2]$$

where $c_1=(\sum_{j=1}^{2k+1}|z_j|^2)^{1/2}$ and $c_2=(|u_0|^2+|u_1|^{2t}+\sum_{j=2}^{2k}|u_j|^2)^{1/2}$. Then φ_1 (resp. φ_2) is an s-fold (resp. t-fold) ramified covering map and we have $\deg(\varphi_2\varphi_1)\equiv 1 \pmod m$. Therefore by the theorem of Olum [8] the quotient manifold Σ_s^{4k+1}/T_s is homotopy equivalent to L_t^{4k+1} since both φ_1 and φ_2 induce isomorphisms of fundamental groups. Thus we obtain the following

PROPOSITION 2.1. *The quotient space of the free Z_m-action (Σ_s^{4k+1}, T_s) is homotopopy equivalent to $L^{4k+1}(m; p_1, \cdots, p_{2k})$ with $sp_1 \cdots p_{2k}\equiv 1 \pmod m$.*

PROPOSITION 2.2. *Every homotopy $(4k+1)$-sphere that bounds a parallelizable manifold admits a free Z_m-action for any integer m.*

Proposition 2.2 is an affirmative answer to the conjecture of Orlik [**9**] in dimensions $4k+1$.

When m is even, by restricting this action to the subgroup $Z_2 \subset Z_m$, one obtains the so-called Brieskorn-Hirzebruch involution $(\Sigma_s^{4k+1}, T_s|Z_2)$ (see [**6**] V.4).

LEMMA 2.3. *When m is even, (Σ_s^{4k+1}, T_s) does not admit codimension 2 characteristic spheres.*

PROOF. In dimension$=4k+1$, the obstruction to the existence of codim$=2$ Z_2-characteristic spheres, Browder-Livesay invariant and abstract codim$=1$ and 2 surgery obstructions are all equal ([**6**]). These obstructions do not vanish for $(\Sigma_s^{4k+1}, T_s|Z_2)$ ([**2**], [**5**]).

§ 3. Surgery on odd-dimensional manifolds with $\pi_1=Z_m$.

In this section we shall discuss the surgery obstructions for odd dimensional manifolds with $\pi_1=Z_m$. Surgery theories for $\pi_1=\{1\}$, Z_2 and Z are assumed to be known. The main reference here is Wall's book [**13**]. First we quote two lemmas due to Wall [**13**].

LEMMA 3.1 (Wall). *The transfer homomorphism $\tau: L_0^\varepsilon(Z_m)\to L_0(1)$ $(\varepsilon=h, s)$ is surjective.*

LEMMA 3.2 (Wall). *For $\varepsilon=h$, s,*

i) $L_{2n-1}(Z) \xrightarrow{\alpha} L_{2n-1}^\varepsilon(Z_m) \xrightarrow{p} L_{2n+1}^\varepsilon(Z\to Z_m)$ *is zero.*

ii) $L_{2n-1}(\boldsymbol{Z}) \xrightarrow{\alpha} L^{\epsilon}_{2n-1}(\boldsymbol{Z}_m)$ *is zero unless* n, m *are even.*

In the above, α is induced by the natural epimorphism $\boldsymbol{Z}\to\boldsymbol{Z}_m$. The map p is characterized as follows: Let $f: M^{2n-1}\to X^{2n-1}$ be a normal map with $\pi_1(X)\cong\boldsymbol{Z}_m$ and surgery obstruction $x=\theta(f)\in L^{\epsilon}_{2n-1}(\boldsymbol{Z}_m)$. Denote by $\tilde{X}\to X$ the universal covering of X. It induces an m-fold covering $\tilde{M}\to M$ and a map $\tilde{f}: \tilde{M}\to\tilde{X}$ covering f. Then we have a well-defined normal map

$$\bar{f}=\tilde{f}\times_{\boldsymbol{Z}_m} id: \tilde{M}\times_{\boldsymbol{Z}_m} D^2 \longrightarrow \tilde{X}\times_{\boldsymbol{Z}_m} D^2 .$$

We have $p(x)=\theta(\bar{f})$ in $L^{\epsilon}_{2n+1}(\boldsymbol{Z}\to\boldsymbol{Z}_m)$.

The surgery obstructions define a homomorphism

$$\theta: \Omega_n(K(\boldsymbol{Z}_m, 1)\times G/O) \longrightarrow L^{\epsilon}_n(\boldsymbol{Z}_m)$$

as stated in [**13**] 13B3.

LEMMA 3.3. *The composition of maps*

$$p\theta: \Omega_3(K(\boldsymbol{Z}_m, 1)\times G/O) \xrightarrow{\theta} L^{\epsilon}_3(\boldsymbol{Z}_m) \xrightarrow{p} L^{\epsilon}_5(\boldsymbol{Z}\to\boldsymbol{Z}_m)$$

is zero.

PROOF. The Conner-Floyd bordism spectral sequence [**4**] shows that the Hurewicz map

$$\mu: \Omega_3(K(\boldsymbol{Z}_m, 1)\times G/O) \longrightarrow H_3(K(\boldsymbol{Z}_m, 1)\times G/O\,;\boldsymbol{Z})$$

is an isomorphism.

Case I. m is odd:

$\Omega_3(K(\boldsymbol{Z}_m, 1)\times G/O)$ is isomorphic to $\boldsymbol{Z}_m$ generated by

$$\varphi: L^3_1 \xrightarrow{(\varphi_1, \varphi_2)} K(\boldsymbol{Z}_m, 1)\times G/O$$

where $\varphi_1: L^3_1\to K(\boldsymbol{Z}_m, 1)$ is the classifying map of the universal covering $S^3\to L^3_1$ and $\varphi_2: L^3_1\to G/O$ is the trivial map. Then we have $\theta(\varphi)=0$ since $\theta(\varphi_2)$ is already zero.

Case II. m is even:

The group $\Omega_3(K(\boldsymbol{Z}_m, 1)\times G/O)\cong\boldsymbol{Z}_m\oplus\boldsymbol{Z}_2$ has two generators:

$$\varphi: L^3_1 \longrightarrow K(\boldsymbol{Z}_m, 1)\times G/O$$

as above and

$$\psi: S^1\times S^2 \xrightarrow{j\times k} K(\boldsymbol{Z}_m, 1)\times G/O$$

where $[j]\in\pi_1(K(\boldsymbol{Z}_m, 1))$ and $[k]\in\pi_2(G/O)$ are generators of respective groups. We have $\theta(\varphi)=0$ as above. Denote by $\psi': S^1\times S^2\to G/O$ the map

$$S^1\times S^2 \xrightarrow{\psi} K(\boldsymbol{Z}_m, 1)\times G/O \xrightarrow{\text{proj}} G/O .$$

Then we have $\theta(\psi)=(j)_*\theta(\psi')$ where

$$(j)_*:\ L_3(\boldsymbol{Z}) \longrightarrow L_3^{\varepsilon}(\boldsymbol{Z}_m)$$

is equal to α. Therefore $\theta(\psi)=0$ holds by Lemma 3.2 (i).

LEMMA 3.4. *For any normal map* $\varphi: L^5(m; p_1, p_2)\rightarrow G/O$, *its surgery obstruction* $\theta(\varphi)$ *in* $L_5^{\varepsilon}(\boldsymbol{Z}_m)$ *vanishes* ($\varepsilon=h, s$).

PROOF. Let N be a closed tubular neighborhood of $L^3=L^3(m; p_1)$ in $L^5=L^5(m; p_1, p_2)$ and put $E=L^5-\text{int}\,N$. Then the surgery obstruction for $\varphi|N: N\rightarrow G/O$ is given by $\theta(\varphi|N)=p\theta(\varphi|L^3)$ in $L_5^{\varepsilon}(\boldsymbol{Z}\rightarrow\boldsymbol{Z}_m)$. This is zero by Lemma 3.3. Now consider the normal map

$$\bar{f}=f\circ p_1:\ L^5\times CP(2) \longrightarrow G/O .$$

Then $\theta(\bar{f}|N\times CP(2))=0$ by the periodicity of surgery obstructions, and we obtain an ε-equivalence ($\varepsilon=h, s$) at $N\times CP(2)$. The remaining surgery obstruction lies in $L_9(\pi_1(E\times CP(2))=L_9(\boldsymbol{Z})$ which is mapped to $\theta(\bar{f})\in L_9^{\varepsilon}(\boldsymbol{Z}_m)$ by the natural map

$$\alpha:\ L_9(\boldsymbol{Z}) \longrightarrow L_9^{\varepsilon}(\boldsymbol{Z}_m)$$

since surgery obstructions are natural for inclusions ([**13**], 3.2). We have $\theta(\bar{f})=0$ by Lemma 3.2, and by periodicity again we see that $\theta(f)=0$. This completes the proof.

The argument above can be taken as the first step of the induction used by Wall ([**13**], 14E4) to calculate the surgery obstructions for lens spaces. Hence Wall's theorem 14E4 holds for $\varepsilon=s$ as well as $\varepsilon=h$. Instead of giving a reproduction of his proof, we shall turn to the general situation with $\pi_1=\boldsymbol{Z}_m$ here.

LEMMA 3.5. *The surgery obstruction map*

$$\theta:\ \Omega_5(K(\boldsymbol{Z}_m, 1)\times G/O) \longrightarrow L_5^{\varepsilon}(\boldsymbol{Z}_m)$$

is zero ($\varepsilon=h, s$).

PROOF. Consider the Conner-Floyd spectral sequence for $\Omega_*(K(\boldsymbol{Z}_m, 1)\times G/O)$ with $E_{p,q}^2=H_p(K(\boldsymbol{Z}_m, 1)\times G/O; \Omega_q)$ ([**4**]). Then $E_{5,0}^2$ is a torsion group since $H_5(G/O; \boldsymbol{Z})$ is. Hence all differentials vanish on $E_{5,0}^r$. Therefore, we have $E_{5,0}^2=E_{5,0}^{\infty}$, namely the Hurewicz map

$$\mu:\ \Omega_5(K(\boldsymbol{Z}_m, 1)\times G/O) \longrightarrow H_5(K(\boldsymbol{Z}_m, 1)\times G/O; \boldsymbol{Z})$$

is surjective. Put

$$A_i=\text{Image}\ \{\Omega_i(K(\boldsymbol{Z}_m, 1))\otimes\Omega_{5-i}(G/O) \longrightarrow \Omega_5(K(\boldsymbol{Z}_m, 1)\times G/O)\} .$$

Then we can verify that A_0, A_1, A_3 and A_5 generate $\Omega_5(K(\boldsymbol{Z}_m, 1)\times G/O)$.

I. $\theta(A_0)=0$: An element of A_0 is represented by

$$\varphi: M^5 \xrightarrow{(\varphi_1, \varphi_2)} K(\boldsymbol{Z}_m, 1)\times G/O$$

where φ_1 is the trivial map. We can therefore assume that M^5 is simply connected. Then we have $\theta(\varphi)=(\varphi_1)_*\theta(\varphi_2)=0$ since $\theta(\varphi_2)\in L_5(1)=0$.

II. $\theta(A_1)=0$: Take a representative

$$\varphi: S^1\times M^4 \xrightarrow{\varphi'\times\varphi''} K(\boldsymbol{Z}_m, 1)\times G/O$$

of A_1. Then as before we may assume that M^4 is simply connected. We have $\theta(\varphi)=(\varphi')_*\theta(p_2\varphi'')$ by definition. If $[\varphi']=q[j]\in\pi_1(K(\boldsymbol{Z}_m, 1))$ where $j: S^1\to K(\boldsymbol{Z}_m, 1)$ represents the generator, $(\varphi')_*$ factors as

$$L_5(\boldsymbol{Z}) \xrightarrow{(q)_*} L_5(\boldsymbol{Z}) \xrightarrow{\alpha} L_5^\varepsilon(\boldsymbol{Z}_m)$$

which is zero by Lemma 3.2 (ii).

III. $\theta(A_5)=0$: Take a representative

$$\varphi: M^5 \xrightarrow{(\varphi_1, \varphi_2)} K(\boldsymbol{Z}_m, 1)\times G/O$$

of A_5 where φ_2 is trivial. Then $\theta(\varphi_2)$ is already zero in this case.

IV. Final case: When m is odd, we have $\theta(A_3)=0$ since $\Omega_3(K(\boldsymbol{Z}_m, 1))\otimes\Omega_2(G/O)\cong\boldsymbol{Z}_m\otimes\boldsymbol{Z}_2=0$. Let us assume that m is even. The free $\boldsymbol{Z}_m$-action (Σ_s^5, T_s) of § 2 defines a homotopy smoothing $\Sigma_s^5/T_s\to L^5=L^5(m; t, 1)$ whose normal invariant is denoted by $\varphi_2: L^5\to G/O$. We know that the k_2-class for this normal invariant does not vanish [2] or equally we have $\theta(\varphi_2|L^3)\neq 0$ in $L_3^\varepsilon(\boldsymbol{Z}_m)$ where $L^3=L^3(m; t)\subset L^5$. Let $\varphi_1: L^5\to K(\boldsymbol{Z}_m, 1)$ classify the universal cover and put

$$\varphi: L^5 \xrightarrow{(\varphi_1, \varphi_2)} K(\boldsymbol{Z}_m, 1)\times G/O\,.$$

Denote by $x\in H^3(K(\boldsymbol{Z}_m, 1); \boldsymbol{Z}_2)$ and $k_2\in H^2(G/O; \boldsymbol{Z}_2)$ the generators. Then $\varphi^*(xk_2)[L^5]$ does not vanish whereas xk_2 is annihilated by elements which belong to A_0, A_1 and A_5. This shows that A_0, A_1, A_5 and φ generate the whole group $\Omega_5(K(\boldsymbol{Z}_m, 1)\times G/O)$ since $A_3\cong\boldsymbol{Z}_2$. The surgery obstruction for φ vanishes by Lemma 3.4. This completes the proof.

LEMMA 3.6. *Let X^n be a compact n-manifold with $\pi_1(X)\cong\boldsymbol{Z}_m$ and $n\geqq 6$. Then there exists a submanifold Y^{n-2} of X^n satisfying the following conditions: Let N be a closed tubular neighborhood of Y in X and put $E=X-\mathrm{int}\,N$. The natural inclusions $Y\to X$ and $\partial E\to E$ induce isomorphisms $\pi_1(Y)\cong\pi_1(X)\cong\boldsymbol{Z}_m$ and*

$\pi_1(\partial E)\cong\pi_1(E)\cong \boldsymbol{Z}$.

PROOF. Consider the map $f: X^n\to L_1^{2\infty+1}$ which classifies the universal cover of X. Then we can apply the theorem of Quinn [10] to deduce our assertion since $L_1^{2\infty-1}\to L_1^{2\infty+1}$ is a homotopy equivalence and $(L_1^{2\infty+1}-L_1^{2\infty-1})\to L_1^{2\infty+1}$ is homotopically an S^1-bundle.

When n and m are even, we have a canonical map $d': L_{2n-1}^{\varepsilon}(\boldsymbol{Z}_m)\to L_{2n-1}(\boldsymbol{Z}_2)\cong \boldsymbol{Z}_2$ ([13]).

THEOREM 3.7. *Let M^{2n-1} be an oriented manifold with $\pi_1(M)\cong \boldsymbol{Z}_m$ $(n\geqq 3)$ and $f: M^{2n-1}\to G/O$ be a normal map. Then $\theta(f)=0$ in $L_{2n-1}^{\varepsilon}(\boldsymbol{Z}_m)$ $(\varepsilon=h, s)$ unless both n and m are even and in this case $\theta(f)=0$ if and only if $d'\theta(f)=0$.*

PROOF. We use the induction. Let (a_k) and (b_k) be the following statements:

(a_k): The assertion of the theorem holds for $n=k$.

(b_k): The image of $\theta: [M^{2k-1}, G/O]\to L_{2k-1}^{\varepsilon}(\boldsymbol{Z}_m)$ lies in the images of $\alpha: L_{2k-1}(\boldsymbol{Z})\to L_{2k-1}^{\varepsilon}(\boldsymbol{Z}_m)$ when $\pi_1(M)\cong \boldsymbol{Z}_m$.

We know that (a_3) and (b_3) hold by Lemma 3.5. Now we assume (a_n) and (b_n). Let $f: M^{2n+1}\to G/O$ be a normal map. By Lemma 3.6, there exists a submanifold M'^{2n-1} of M^{2n+1} satisfying the conditions of Lemma 3.6. Let N be a closed tubular neighborhood of M' in M and put $E=M-\operatorname{int} N$. The surgery obstruction for $f|N$ is given by $p\theta(f|M')\in L_{2n+1}^{\varepsilon}(\boldsymbol{Z}\to \boldsymbol{Z}_m)$. But since $\theta(f|M')$ is in the image of $\alpha: L_{2n-1}(\boldsymbol{Z})\to L_{2n-1}^{\varepsilon}(\boldsymbol{Z}_m)$ by (b_n), we have $\theta(f|N)=0$ from Lemma 3.2 (i). Therefore we obtain a homotopy equivalence (ε-equivalence) at N. The remaining surgery obstruction lies in $L_{2n+1}(\pi_1(E))=L_{2n+1}(\boldsymbol{Z})$. This obstruction is mapped to $\theta(f)\in L_{2n+1}^{\varepsilon}(\boldsymbol{Z}_m)$ by α. Thus we get (b_{n+1}). $(b_{n+1})\Rightarrow(a_{n+1})$ follows from Lemma 3.2 (ii) and the fact that the composition

$$L_{2n+1}(\boldsymbol{Z}) \xrightarrow{\alpha} L_{2n+1}^{\varepsilon}(\boldsymbol{Z}_m) \xrightarrow{d'} L_{2n+1}(\boldsymbol{Z}_2)=\boldsymbol{Z}_2$$

is an isomorphism when n is odd and m is even ([13]).

§4. $\tilde{\boldsymbol{Z}}_m$-manifolds.

Let X^n be an oriented smooth manifold with an orientation preserving free $\boldsymbol{Z}_m$-action T on the boundary ∂X. Then a closed $\tilde{\boldsymbol{Z}}_m$-manifold associated to (X^n, T) is the space $\hat{X}^n=X^n/\sim$ where $x\sim y$ if and only if $x, y\in\partial X$ and $T^k(x)=y$ for some integer k. The singular subset $\delta\hat{X}=\partial X/\sim$ and $\hat{X}^n-\delta\hat{X}$ have natural smooth structures induced by that of X^n. But $\hat{X}^n$ fails to be a manifold unless $m=2$, and in this case $\hat{X}$ is a non-orientable manifold if $\partial X\neq\emptyset$. A $\tilde{\boldsymbol{Z}}_m$-manifold with boundary is defined similary by an object (W^n, V^{n-1}, T) where W^n is an oriented manifold and T is an orientation preserving free $\boldsymbol{Z}_m$-action on a submanifold $V^{n-1}\subset\partial W$. We define $\hat{W}^n=W/\sim$ where $x\sim y$ if and only if $x, y\in V$ and $T^k(x)=y$ for some k. We write $\delta\hat{W}=V/\sim$ and the boundary $\partial\hat{W}$ of $\hat{W}$ is defined to be $(\partial W-\text{int } V)/\sim$.

EXAMPLE 4.1. Let $X^{2n}=D^{2n}$ and the $\boldsymbol{Z}_m$-action on $X=S^{2n-1}$ be given by

$$T(z_0, z_1, \cdots, z_{n-1})=(\alpha z_0, \alpha^{p_1}z_1, \cdots, \alpha^{p_{n-1}}z_{n-1})$$

where $\alpha=\exp(2\pi i/m)$ and $(p_j, m)=1$. Then $\hat{X}^{2n}=\hat{L}^{2n}(m; p_1, \cdots, p_{n-1})$ and $\delta\hat{X}=L^{2n-1}(m; p_1, \cdots, p_{n-1})$.

EXAMPLE 4.2. Let T_0 be an orientation preserving free $\boldsymbol{Z}_m$-action on an oriented manifold M^n. Define

$$(W^{n+1}, V^n, T)=(M^n\times I, M\times\{0\}, T_0\times id).$$

Then $\hat{W}^{n+1}$ is homeomorphic to the mapping cylinder of $M^n\to M^n/T_0$ with $\delta\hat{W}=M^n/T_0$ and $\partial\hat{W}=M^n$.

The notion of $\tilde{\boldsymbol{Z}}_m$-manifolds with boundary enables us to define cobordism relations among closed $\tilde{\boldsymbol{Z}}_m$-manifolds and thus we obtain cobordism groups of $\tilde{\boldsymbol{Z}}_m$-manifolds denoted by $\Omega_*(\tilde{\boldsymbol{Z}}_m)$ where addition is given by disjoint unions. Before giving an explicit description of these cobordism groups, we make some preparations which will be useful in later sections.

Let the objects $(X_i^{n_i}, T_i)$ $(i=0, 1)$ define $\tilde{\boldsymbol{Z}}_m$-manifolds $\hat{X}_i^{n_i}$. A map

$$f: (X_0, \partial X_0)\longrightarrow(X_1, \partial X_1)$$

which is $\boldsymbol{Z}_m$-equivariant on the boundary induces a map

$$\hat{f}: \hat{X}_0\longrightarrow\hat{X}_1$$

of $\tilde{\boldsymbol{Z}}_m$-manifolds. In this case, we call $\hat{f}$ a $\tilde{\boldsymbol{Z}}_m$-map (associated to f). When $n_0=n_1$, the degree of $\hat{f}$ is defined to be the degree of f.

Let $\hat{X}^n$ be a $\tilde{\boldsymbol{Z}}_m$-manifold associated to (X^n, T). We fix a $\boldsymbol{Z}_m$-action on a cone on m-points

$$C(m)=\{z\in\boldsymbol{C} \mid |z|\leq 1, \arg(z)=2\pi j/m \text{ or } z=0\}$$

given by $z\mapsto\alpha z$, $\alpha=\exp(2\pi i/m)$. Let J be defined by

$$J=\partial X\times_{\boldsymbol{Z}_m}D^2$$

where $(x, v)\sim(T^k(x), \alpha^k v)$ for $x\in\partial X$ and $v\in D^2$. Then J contains as subsets

$$K=\partial X\times_{\boldsymbol{Z}_m}C(m),$$

$$\dot{K}=\{[x, v]\in K \mid |v|=1\},$$

and boundary $\partial J=\partial X\times_{\boldsymbol{Z}_m}S^1$.

$\dot{K}$ can be be identified with ∂X by the map $[x, \alpha^k]\mapsto T^{-k}(x)$. Hence we have an embedding $\partial X=\dot{K}\to\partial J$ which has a product tubular neighborhood $\partial X\times I$. We obtain an $(n+1)$-dimensional manifold

$$\overline{\overline{X}}^{n+1}=X\times I\bigcup_{\partial X\times I} J$$

by glueing along $\partial X\times I$. We call $\overline{\overline{X}}^{n+1}$ the regularization of the $\tilde{\boldsymbol{Z}}_m$-manifold $\hat{X}^n$. $\overline{\overline{X}}$ contains $\hat{X}$ as a deformation retract since $\hat{X}$ is homeomorphic to $X\bigcup_{\partial X=\dot{K}} K$. It can also be seen that a $\tilde{\boldsymbol{Z}}_m$-map

$$\hat{f}:\hat{X}_0\longrightarrow\hat{X}_1$$

between $\tilde{\boldsymbol{Z}}_m$-manifolds extends to a map

$$\overline{\overline{f}}:(\overline{\overline{X}}_0,\partial\overline{\overline{X}}_0)\longrightarrow(\overline{\overline{X}}_1,\partial\overline{\overline{X}}_1)$$

which is called the regularization of $\hat{f}$.

Let M^q be a smooth manifold. An embedding $\hat{X}^n\to M^q$ is called regular if it factors through an embedding of $\overline{\overline{X}}^{n+1}$ in M^q as

$$\hat{X}^n\subset\overline{\overline{X}}^{n+1}\longrightarrow M^q.$$

The regularization $\overline{\overline{X}}$ of $\hat{X}$ has a stable normal bundle $\nu_{\overline{\overline{X}}}$. The stable normal bundle $\nu_{\hat{X}}$ is defined to be its restriction to $\hat{X}$, $\nu_{\overline{\overline{X}}}|\hat{X}$.

As a direct application of the notion of regularizations, we can describe the cobordism and bordism groups of $\tilde{\boldsymbol{Z}}_m$-manifolds in the following form.

THEOREM 4.3. *The cobordism groups and bordism groups of $\tilde{\boldsymbol{Z}}_m$-manifolds are represented as follows:*

$$\Omega_n(\tilde{\boldsymbol{Z}}_m)\cong\tilde{\Omega}_{n+1}(K(\boldsymbol{Z}_m,1)),$$

$$\Omega_n(A;\tilde{\boldsymbol{Z}}_m)\cong\tilde{\Omega}_{n+1}(A^+\wedge K(\boldsymbol{Z}_m,1)).$$

PROOF.

I. Definition of a map $\Omega_n(\tilde{\boldsymbol{Z}}_m)\to\tilde{\Omega}_{n+1}(K(\boldsymbol{Z}_m,1))$: Take a representative $\hat{X}^n$ of $\Omega_n(\tilde{\boldsymbol{Z}}_m)$. Let $\varphi:\delta\hat{X}\to L_1^{2r-1}$ (r large) classify the covering $\partial X\to\delta\hat{X}$. Then we get a $\boldsymbol{Z}_m$-equivariant map $\tilde{\varphi}:\partial X\to S^{2r-1}$, which extends to a map

$$f:(X,\partial X)\longrightarrow(D^{2r},S^{2r-1})$$

and f induces a $\tilde{\boldsymbol{Z}}_m$-map $\hat{f}:\hat{X}\to\hat{L}_1^{2r}$. $\hat{f}$ extends to a regularization

$$\overline{\overline{f}}:\overline{\overline{X}}^{n+1}\longrightarrow\overline{\overline{(\hat{L}_1^{2r})}}=L_1^{2r+1}-\text{int}\,D^{2r+1}.$$

$\overline{\overline{f}}$, continued by the collapsing map

$$\overline{\overline{(L_1^{2r})}} \longrightarrow \overline{\overline{(L_1^{2r})}}/\partial\overline{\overline{(L_1^{2r})}} = L_1^{2r+1}$$

yields a map $(\overline{\overline{X}}^{n+1}, \partial\overline{\overline{X}}) \to (L_1^{2r+1}, *)$ which determines an element of $\tilde{\Omega}_{n+1}(K(\boldsymbol{Z}_m, 1))$.

II. Definition of a map $\tilde{\Omega}_{n+1}(K(\boldsymbol{Z}_m, 1)) \to \Omega_n(\tilde{\boldsymbol{Z}}_m)$: Take a representative $F: (W^{n+1}, \partial W) \to (K(\boldsymbol{Z}_m, 1), *)$ of $\tilde{\Omega}_{n+1}(K(\boldsymbol{Z}_m, 1))$. By taking r large, F can be regarded as a map (also denoted by F) $F: (W^{n+1}, \partial W) \to (L_1^{2r+1}, *)$. We may assume that the base point is not included in $\hat{L}_1^{2r}(\subset L_1^{2r+1})$. First make F t-regular to the submanifold L_1^{2r-1} in L_1^{2r+1}. Since t-regularity is an "open" condition, F is t-regular in the neighborhood of L_1^{2r-1} in L_1^{2r+1}. Outside this neighborhood, $\hat{L}_1^{2r}$ is a submanifold of L_1^{2r+1}. Therefore we can make F t-regular to $\hat{L}_1^{2r} - L_1^{2r-1}$ by deforming F by homotopy outside the neighborhood of L_1^{2r-1}. Then $F^{-1}(\hat{L}_1^{2r})$ is a $\tilde{\boldsymbol{Z}}_m$-manifold regularly embedded in W^{n+1}.

By constructions of I and II, we readily see that these maps are inverses to each other. The proof for the bordism groups is similar.

REMARK. Let $T_m = S^1 \underset{m}{\cup} e^2$ be the Moore space. We may regard T_m as the 2-skeleton $\hat{L}_1^2$ of $K(\boldsymbol{Z}_m, 1)$. The natural map

$$T_m = \hat{L}_1^2 \longrightarrow K(\boldsymbol{Z}_m, 1)$$

defines a natural transformation from Sullivan's $\boldsymbol{Z}_m$-manifold theory to our $\tilde{\boldsymbol{Z}}_m$-manifold theory (see [7]).

§5. Surgery on $\tilde{\boldsymbol{Z}}_m$-manifolds.

Let $\hat{X}^n$ be a $\tilde{\boldsymbol{Z}}_m$-manifold. A normal map of degree one is the following diagram:

$$\text{(5a)} \qquad \begin{array}{ccc} \nu_{\hat{M}} & \xrightarrow{\hat{b}} & \xi \\ \downarrow & & \downarrow \\ \hat{M}^n & \xrightarrow{\hat{f}} & \hat{X}^n \end{array}$$

where $\hat{b}$ is a bundle map of vector bundles covering the $\tilde{\boldsymbol{Z}}_m$-map $\hat{f}$ of degree one. As in the case of usual manifolds, we can define normal cobordism classes of normal maps of degree one, which is denoted by $N(\hat{X})$.

Starting from the normal map given by diagram (5a), we obtain the following diagram by regularization:

$$\text{(5b)} \qquad \begin{array}{ccc} \nu_{\overline{\overline{M}}} & \xrightarrow{\overline{\overline{b}}} & \overline{\overline{\xi}} \\ \downarrow & & \downarrow \\ \overline{\overline{M}}^{n+1} & \xrightarrow{\overline{\overline{f}}} & \overline{\overline{X}}^{n+1} \end{array}$$

where $\bar{\bar{\xi}}$ is the pull-back of ξ by the retraction $\bar{\bar{X}} \to \hat{X}$ and $\bar{\bar{b}}$ is an extension of $\hat{b}$. Diagram (5b) defines a normal map of degree one into the manifold $\bar{\bar{X}}^{n+1}$. Hence this construction defines a map

$$\Phi : N(X) \longrightarrow N(\bar{\bar{X}})$$

where $N(\bar{\bar{X}})$ is the set of normal cobordism classes of normal maps of degree one into the manifold $\bar{\bar{X}}^{n+1}$ in the usual sense.

Conversely, let us start from a normal map of $\bar{\bar{X}}^{n+1}$:

$$\text{(5c)} \qquad \begin{array}{ccc} \nu_W & \xrightarrow{\ B\ } & \zeta \\ \downarrow & & \downarrow \\ W^{n+1} & \xrightarrow{\ F\ } & \bar{\bar{X}}^{n+1} . \end{array}$$

Make F t-regular to $\hat{X}^n \subset \bar{\bar{X}}^{n+1}$ as in the proof of Theorem 4.3. Then $\hat{M}^n = F^{-1}(\hat{X}^n)$ is regularly embedded in W^{n+1} and hence we have $\nu_{\hat{M}} = \nu_W | \hat{M}$. Let $\hat{f} = F | \hat{M}$, $\xi = \zeta | \hat{X}$, and $\hat{b} = B | \nu_{\hat{M}}$, then we get diagram (5a). This construction gives rise to a map

$$\Psi : N(\bar{\bar{X}}) \longrightarrow N(\hat{X}) .$$

It is clear that Φ and Ψ are inverses to each other. Therefore we have a bijective correspondence:

$$N(\hat{X}) \approx N(\bar{\bar{X}}) .$$

It is well known that $N(\bar{\bar{X}})$ can be identified with $[\bar{\bar{X}}, G/O]$ (see e.g. [**12**]). Hence we obtain

PROPOSITION 5.1. *We have a bijective correspondence*

$$N(\hat{X}^n) \approx [\hat{X}^n, G/O] .$$

DEFINITION. Let $\varepsilon = h$ or s. A $\tilde{Z}_m$-map $\hat{f} : \hat{M}^n \to \hat{X}^n$ of $\tilde{Z}_m$-manifolds is called an ε-smoothing of $\hat{X}^n$ if $\hat{f}$ is an ε-homotopy equivalence of pairs $(\hat{M}^n, \delta\hat{M}) \simeq (\hat{X}^n, \delta\hat{X})$.

DEFINITION. Two ε-smoothings $\hat{f}_i : \hat{M}_i^n \to \hat{X}^n$ $(i=0, 1)$ are called concordant if there exists an ε-smoothing

$$\hat{F} : \hat{W}^{n+1} \longrightarrow \hat{X}^n \times I$$

with

$$\partial\hat{W} = \hat{M}_0 \cup \hat{M}_1 \quad \text{and} \quad \hat{f}_i = \hat{F} | \hat{M}_i .$$

The set of concordance classes of ε-smoothings of $\hat{X}^n$ is denoted by $hS^{\varepsilon}(\hat{X})$.

Let $\hat{f}: \hat{M}^n \to \hat{X}^n$ be an ε-smoothing of $\hat{X}^n$ and g be its homotopy inverse. Then we have a normal map:

$$\begin{array}{ccc} \nu_{\hat{M}} & \longrightarrow & g^*\nu_{\hat{M}} \\ \downarrow & \hat{f} & \downarrow \\ \hat{M}^n & \longrightarrow & \hat{X}^n \end{array}$$

whose normal cobordism class is called the normal invariant of $\hat{f}$. Thus we obtain a map

$$\eta: hS^{\varepsilon}(\hat{X}^n) \longrightarrow [\hat{X}^n, G/O].$$

Let the object (X^{2n}, T) define the $\tilde{\boldsymbol{Z}}_m$-manifold $\hat{X}^{2n}$.

THEOREM 5.2. *Let $\hat{X}^{2n}$ be a $\tilde{\boldsymbol{Z}}_m$-manifold with $\pi_1(X)=\pi_1(\partial X)=\{1\}$. Then we have the following exact sequence valid for $n \geqq 3$:*

$$hS^{\varepsilon}(\hat{X}^{2n}) \xrightarrow{\eta} [\hat{X}, G/O] \xrightarrow{\theta} Q_{2n} \qquad (\varepsilon=h, s)$$

where Q_{2n} is $\boldsymbol{Z}_2$ when m is even and is the trivial group when m is odd.

PROOF. Let $n=2k+1$. Take a normal map $\hat{f}: \hat{M}^{4k+2} \to \hat{X}^{4k+2}$. By Theorem 3.7, we can make $\delta\hat{f}: \delta\hat{M} \to \delta\hat{X}$ into an ε-equivalence by surgery. Then we have a surgery problem $f:(M, \partial M) \to (X, \partial X)$ with $f|\partial M$ an ε-equivalence. Define $\theta(\hat{f})=\theta(f) \in \boldsymbol{Z}_2$, the Kervaire obstruction. We can construct a normal cobordism $F: N^{4k+2} \to \delta\hat{X} \times I$ such that $\partial N = M_0 \cup M_1$, $M_0=\delta\hat{M}$, $F|M_0=\delta\hat{f}$, and $F|M_1$ is also an ε-equivalence. Then extend this cobordism in the neighborhood of $\delta\hat{M}$ in $\hat{M}$. Denote by $\tilde{M}_0$, $\tilde{M}_1$ and $\tilde{N}$ the natural m-fold coverings of M_0, M_1 and N respectively. Then the manifold $M'=M \underset{\partial M=M_0}{\cup} \tilde{N}$ gives a $\tilde{\boldsymbol{Z}}_m$-manifold $\hat{M}'$ and a normal map $\hat{f}': \hat{M}' \to \hat{X}$ which is normally cobordant to $\hat{f}$. Since Kervaire invariants are multiplied by m under coverings, $\theta(\hat{f}')$ can be made zero if m is odd. When m is even, $\theta(\hat{f})=\theta(\hat{f}')$ is a well-defined element in $\boldsymbol{Z}_2$.

Let $n=2k$. Take a normal map $\hat{f}: \hat{M}^{4k} \to \hat{X}^{4k}$. Define $\theta(\hat{f})=d'\theta(\delta\hat{f})$, the surgery obstruction for $\delta\hat{f}: \delta\hat{M} \to \delta\hat{X}$. This is always zero when m is odd. Suppose that this obstruction vanishes, we have an ε-equivalence at $\delta\hat{X}$. The remaining problem is to compute the index obstruction of $f:(M, \partial M) \to (X, \partial X)$ keeping $\partial f = f|\partial M$ fixed. If this index obstruction, say σ, is not zero in $L_{4k}(1)$, we choose an element $\sigma' \in L^{\varepsilon}_{4k}(\boldsymbol{Z}_m)$ with $\tau(\sigma')=-\sigma$ by Lemma 3.1 of Wall. Letting σ' act on $\delta\hat{f}: \delta\hat{M} \to \delta\hat{X}$ we obtain a normal map $\hat{f}': \hat{M}' \to \hat{X}$ with $\delta\hat{f}'$ an ε-equivalence. Then the normal map $f':(M', \partial M') \to (X, \partial X)$ has zero index obstruction by the additivity of index. This completes the proof.

REMARK. Let the object (X^{4k+2}, T) define the $\tilde{\boldsymbol{Z}}_m$-manifold $\hat{X}$ with $\pi_1(X) = \pi_1(\partial X)=\{1\}$. When m is even, we can construct a $\tilde{\boldsymbol{Z}}_2$-manifold $\bar{X}$ by restricting the $\boldsymbol{Z}_m$-action to the subgroup $\boldsymbol{Z}_2 \subset \boldsymbol{Z}_m$. Then $\bar{X}$ is a non-orientable manifold and we have a natural projection $\rho: \bar{X} \to \hat{X}$ which is a homeomorphism on $\bar{X} - \delta\bar{X}$ and an $(m/2)$-fold covering on $\delta\bar{X}$. The proof of Theorem 5.2 shows that we have a commutative diagram

$$\begin{array}{ccc} [\hat{X}^{4k+2}, G/O] & \xrightarrow{\theta} & \boldsymbol{Z}_2 \\ & {\scriptstyle\rho^*}\searrow \quad \nearrow{\scriptstyle c} & \\ & [\bar{X}^{4k+2}, G/O] & \end{array}$$

where c is the Kervaire obstruction map.

Let m be even and consider the natural inclusions $i: \hat{L}^{4k-2} \to L^{4k-1}$ and $j: L^{4k-1} \to \hat{L}^{4k}$ where $\hat{L}^{4k} = \hat{L}^{4k}(m; p_1, \cdots, p_{2k-2}, p_{2k-1})$, $L^{4k-1} = L^{4k-1}(m; p_1, \cdots, p_{2k-2}, p_{2k-1})$ and $\hat{L}^{4k-2} = \hat{L}^{4k-2}(m; p_1, \cdots, p_{2k-2})$.

LEMMA 5.3. *We have the following commutative diagram*

$$\begin{array}{ccc} [\hat{L}^{4k}, G/O] & \xrightarrow{\theta} & \\ \downarrow{\scriptstyle j^*} & & \\ [L^{4k-1}, G/O] & \xrightarrow{d'\theta} & \boldsymbol{Z}_2 \\ \downarrow{\scriptstyle i^*} & & \\ [\hat{L}^{4k-2}, G/O] & \xrightarrow{\theta} & \end{array}$$

PROOF. $d'\theta j^* = \theta$ is clear by the proof of Theorem 5.2. Let $f: L^{4k-1} \to G/O$ be a normal map. Then $f | L^{4k-3}$ is representable by an ε-equivalence by Theorem 3.7. The surgery obstruction $\theta(f) \in L^{\varepsilon}_{4k-1}(\boldsymbol{Z}_m)$ comes from a class $x \in L_{4k-1}(\boldsymbol{Z})$ as in the proof of Theorem 3.7. On the other hand, $\hat{L}^{4k-2} - L^{4k-3}$ gives the splitting of $L^{4k-1} - L^{4k-3}$ which induces the isomorphism $L_{4k-1}(\boldsymbol{Z}) \cong L_{4k-2}(1) \cong \boldsymbol{Z}_2$. By this identification we have $d'\theta(f) = x = \theta(i^*(f))$.

LEMMA 5.4. *Let m be even, then*

$$\text{(i)} \qquad \theta: [\hat{L}^{2n}(m; p_1, \cdots, p_{n-1}), G/O] \longrightarrow \boldsymbol{Z}_2$$

and

$$\text{(ii)} \qquad d'\theta: [L^{4k-1}(m; p_1, \cdots, p_{2k-1}), G/O] \longrightarrow \boldsymbol{Z}_2$$

are surjective.

PROOF. By Lemma 5.3, it is enough to show that

$$\theta: [\hat{L}^{4k}(m; p_1, \cdots, p_{2k-1}), G/O] \longrightarrow \boldsymbol{Z}_2$$

is surjective. Take an integer p_{2k} satisfying

$$p_1 \cdots p_{2k-1} p_{2k} s \equiv 1 \pmod{m},$$

then we have a homotopy equivalence

$$\Sigma_s^{4k+1}/T_s \longrightarrow L^{4k+1}(m\,; p_1, \cdots, p_{2k-1}, p_{2k})$$

by Proposition 2.1. This example defines a normal invariant

$$f\colon L^{4k+1}(m\,; p_1, \cdots, p_{2k-1}, p_{2k}) \longrightarrow G/O$$

such that $\theta(f|\hat{L}^{4k}(m\,; p_1, \cdots, p_{2k-1})) = d'\theta(f|L^{4k-1}(m\,; p_1, \cdots, p_{2k-1}))$ is non-zero by Lemma 2.3. This completes the proof.

§ 6. Free Z_m-actions on homotopy spheres.

Making use of the results developed so far, we shall determine homotopy spheres which admit free Z_m-actions. We have the commutative diagram below with exact rows

$$\text{(A)}\quad \begin{array}{ccccccc} L_{2n}^{\varepsilon}(Z_m) & \xrightarrow{\omega} & hS^{\varepsilon}(L_q^{2n-1}) & \xrightarrow{\eta} & [L_q^{2n-1}, G/O] & \xrightarrow{\theta'} & Z_2 \text{ or } 0 \\ \downarrow \tau & & \downarrow \kappa & & \downarrow \pi_q^* & & \\ L_{2n}(1) & \xrightarrow{\omega} & \Theta_{2n-1} & \xrightarrow{\eta} & \pi_{2n-1}(G/O) & \longrightarrow & 0 \end{array}$$

where τ is the transfer map, κ takes the universal covering, $\pi_q\colon S^{2n-1} \to L_q^{2n-1} = L^{2n-1}(m\,; q, 1, \cdots, 1)$ is the natural projection and the map θ' is equal to $d'\theta$ if m, n are even and is trivial otherwise.

Now we are in position to state our main theorem. We shall work in the category of h-smoothings and h-equivalences though all the results hold similarly for the "simple" category.

THEOREM 6.1. *A homotopy sphere Σ^{2n-1} ($n \geqq 3$) admits a free Z_m-action of type q if and only if its normal invariant $\eta(\Sigma^{2n-1})$ belongs to the subgroup*

$$\text{Image}\,\{\pi_q^*\colon [L_q^{2n-1}, G/O] \longrightarrow \pi_{2n-1}(G/O)\}$$

of $\pi_{2n-1}(G/O)$.

As a direct corollary, we can give the solution of Orlik's conjecture in a more detailed version.

COROLLARY 6.2. *Every homotopy sphere Σ^{2n-1} ($n \geqq 3$) that bounds a parallelizable manifold admits a free Z_m-action of type q for any m and q.*

In the statement of the theorem above, the necessity of the condition is apparent. We shall show its sufficiency.

PROOF OF THEOREM 6.1 WHEN m IS ODD:

Let Σ^{2n-1} be a homotopy sphere whose normal invariant $\eta(\Sigma)$ belongs to

Image π_q^*. In this case, since the map $\eta: hS(L_q^{2n-1})\to[L_q^{2n-1}, G/O]$ is surjective, there exists a homotopy smoothing $f: M^{2n-1}\to L_q^{2n-1}$ satisfying $\eta(\Sigma)=\pi_q^*\eta(M^{2n-1})$. The universal cover $\kappa(M)=\tilde{M}$ and Σ have the same normal invariants in $\pi_{2n-1}(G/O)$ by commutativity of the diagram (A). Hence there exists an element $\lambda\in L_{2n}(1)$ with $\lambda * M=\Sigma$. Since the transfer map τ is surjective when m is odd, there exists an element $\lambda'\in L_{2n}(\boldsymbol{Z}_m)$ with $\tau(\lambda')=\lambda$. Then the universal cover of the homotopy smoothing $\lambda' * M$ is diffeomorphic to Σ^{2n-1}. This completes the proof when m is odd.

From now on we assume that m is even. Then the proof of Theorem 6.1 can be deduced by the following two lemmas.

LEMMA 6.3. *If* $\eta_0\in$ *Image* π_q^*, *then there exists a homotopy smoothing* $h: M^{2n-1}\to L_q^{2n-1}$ *with* $\eta_0=\pi_q^*\eta(M^{2n-1})$.

LEMMA 6.4. *If a homotopy sphere* Σ_0^{2n-1} *admits a free* $\boldsymbol{Z}_m$*-action of type* q, *then* $\Sigma_0^{2n-1}\,\#\,\Sigma^{2n-1}$ *admits a free* $\boldsymbol{Z}_m$*-action of type* q *for any* $\Sigma^{2n-1}\in bP_{2n}$.

PROOF OF LEMMA 6.3. If n is odd, then any normal map $f: L_q^{2n-1}\to G/O$ is obtained as the normal invariant of a homotopy smoothing by Theorem 3.7. Hence in this case the assertion follows. When n is even, take a normal map $f: L_q^{2n-1}\to G/O$ with $\eta_0=\pi_q^*(f)$. Suppose that $\theta'(f)=0$, then f is the normal invariant of a homotopy smoothing of L_q^{2n-1} as before. Let $\theta'(f)\neq 0$. There exists a normal map $g: \hat{L}_q^{2n}\to G/O$ with $\theta(g)\neq 0$ by Lemma 5.4(i). Consider the normal map

$$f'=f+(g\,|\,L_q^{2n-1}):\ L_q^{2n-1}\longrightarrow G/O$$

where addition is given by the H-space structure (Whitney sum) of G/O. Then we have $\pi_q^*(f')=\pi_q^*(f)=\eta_0$ since

$$[\hat{L}_q^{2n}, G/O]\xrightarrow{j^*}[L_q^{2n-1}, G/O]\xrightarrow{\pi_q^*}\pi_{2n-1}(G/O)$$

is exact where j is the inclusion. According to Lemma 5.3 and the remark after Theorem 5.2, we see that the map

$$\theta'=d'\theta:\ [L_q^{2n-1}, G/O]\longrightarrow \boldsymbol{Z}_2$$

can be calculated as

$$[L_q^{2n-1}, G/O]\xrightarrow{i^*}[\hat{L}_q^{2n-2}, G/O]\xrightarrow{\rho^*}[P^{2n-2}, G/O]\xrightarrow{c}\boldsymbol{Z}_2.$$

Therefore θ' is a homomorphism since the Kervaire obstruction map c is a homomorphism by the primitivity of Sullivan's k-class ([**11**], [**13**]). Hence we have $\theta'(f')=0$ and there exists a homotopy smoothing $M^{2n-1}\to L_q^{2n-1}$ with $\eta(M)=f'$ satisfying the condition $\eta_0=\pi_q^*\eta(M)$.

PROOF OF LEMMA 6.4. When n is even, surjectivity of the transfer map

$\tau: L_{2n}(\boldsymbol{Z}_m) \to L_{2n}(1)$ implies the assertion by chasing the diagram (A). Let $n = 2k+1$. Put

$$\hat{X}^{4k+2} = \Sigma_0^{4k+1} \times_{\boldsymbol{Z}_m} C(m)$$

where $C(m)$ is a cone on m-points, i.e. $\hat{X}$ is the mapping cylinder of the natural projection $\pi: \Sigma_0 \to \Sigma_0/\boldsymbol{Z}_m$. Then $\hat{X}^{4k+2}$ is a $\tilde{\boldsymbol{Z}}_m$-manifold with boundary $\partial\hat{X} = \Sigma_0$. We have a cofibration

$$\Sigma_0 \xrightarrow{\pi} \Sigma_0/\boldsymbol{Z}_m = \delta\hat{X} \xrightarrow{e} \hat{X}/\partial\hat{X}.$$

Similar results hold for the surgery theory of $(\hat{X} \operatorname{rel} \partial\hat{X})$ as in the case of closed $\tilde{\boldsymbol{Z}}_m$-manifolds. Then we have the following commutative diagram where all rows and columns are exact:

$$\text{(B)} \qquad \begin{array}{ccccccc}
 & & hS(\hat{X} \operatorname{rel} \partial\hat{X}) & \xrightarrow{\eta} & [\hat{X}/\partial\hat{X}, G/O] & \xrightarrow{\theta} & \boldsymbol{Z}_2 \\
 & & \downarrow & & \downarrow e^* & & \\
L_{4k+2}(\boldsymbol{Z}_m) & \xrightarrow{\omega} & hS(\Sigma_0/\boldsymbol{Z}_m) & \xrightarrow{\eta} & [\Sigma_0/\boldsymbol{Z}_m, G/O] & \longrightarrow & 0 \\
\downarrow \tau & & \downarrow \kappa & & \downarrow \pi^* & & \\
L_{4k+2}(1) & \xrightarrow{\omega} & hS(\Sigma_0) & \xrightarrow{\eta} & [\Sigma_0, G/O] & \longrightarrow & 0.
\end{array}$$

According to the remark after Theorem 5.2, we have a commutative diagram

$$\begin{array}{ccc}
[\hat{L}_q^{4k+2}, G/O] & \theta & \\
h^* \downarrow & & \searrow \\
 & & \boldsymbol{Z}_2 \\
 & & \nearrow \\
[\hat{X}/\partial\hat{X}, G/O] & \theta &
\end{array}$$

where $h: \hat{X}/\partial\hat{X} \to \hat{L}_q^{4k+2}$ is a homotopy equivalence. Hence by Lemma 5.4 (i), there exists $f \in [\hat{X}/\partial\hat{X}, G/O]$ with $\theta(f) \neq 0$. Since we can perform surgery on $f|\delta\hat{X}$ by Theorem 3.7, f is represented by a normal map $\hat{g}: \hat{M}^{4k+2} \to \hat{X}^{4k+2}$ such that $\delta\hat{g}: \delta\hat{M} \to \delta\hat{X}$ is a homotopy equivalence. Then M^{4k+2} is a parallelizable manifold with Kervaire invariant $\neq 0$ and its boundary is the disjoint union of Σ_0^{4k+1} and the universal cover of $\delta\hat{M}$. Therefore the universal cover of $\delta\hat{M}$ is diffeomorphic to $\Sigma_0^{4k+1} \# \Sigma_K^{4k+1}$ where Σ_K^{4k+1} is the Kervaire sphere. Thus the proof is complete.

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