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A class of infinitesimal generators of one-dimensional Markov processes

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In this note we consider operators $\mathfrak{A}$ of the form

$$(\mathfrak{A}f)(x)=(D_m D_x f)(x)+b(x)(D_x f)(x)+ \\ +\int_0^1 (f(y)-f(x)-(y-x)(D_x f)(x))\frac{n_x(dy)}{\varphi_x(y)}, \qquad x\in[0,1] \tag{1}$$

in spaces of continuous functions over the interval $[0, 1]$ (for the properties of m, b, n_x and the definition of φ_x see the beginning of 2.). It is shown, that $\mathfrak{A}$ restricted by two boundary conditions

$$\Phi_0(f)=0\,, \qquad \Phi_1(f)=0 \tag{2}$$

of Feller-Ventcel-type (see (13)) is the infinitesimal generator of a strongly continuous nonnegative contraction (s. c. n. c.) semigroup in the subspace of $C_{[0,1]}$, which is defined by the boundary conditions (2).

Similar results (in cases without boundary conditions) can be found in [**1**]. As in [**1**] (or [**2**]) we use a perturbation type argument, but here it does not consist in a "smallness" condition on the perturbing operator B (with respect to the unperturbed operator A), but in the compactness of the operator $B(\lambda I-A)^{-1}$ ($\lambda>0$) (see theorem 1 below).

To avoid technical complications, we consider only the case of a strongly increasing and continuous function m in (1). The general case of arbitrary nondecreasing m can be treated similarly (comp. [**1**])*).

1. In this section we consider a Banach space $\mathfrak{B}$ with a certain fixed semi-inner product $[f, g]$, $f, g\in\mathfrak{B}$ ([**3**], IX. 8). An operator A in $\mathfrak{B}$ is called dissipative (with respect to $[f, g]$), if

$$\mathrm{Re}\,[Af, f]\leqq 0 \qquad \text{for all} \quad f\in\mathfrak{D}(A)\,.$$

The following theorem is a slight modification of the Hille-Yosida theorem for contraction semigroups (comp. [**3**], theorem IX. 8).

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THEOREM 1. *Let* $\mathfrak{B}_0$ *be a (closed) subspace of the Banach space* $\mathfrak{B}$, A_0 *a linear operator from* $\mathfrak{D}_0 \subset \mathfrak{B}_0$, $\bar{\mathfrak{D}}_0 = \mathfrak{B}_0$, *into* $\mathfrak{B}$ *with the property, that for a certain* $\lambda > 0$ *the operator* $\lambda I - A_0$ *maps* $\mathfrak{D}_0$ *bijective onto* $\mathfrak{B}$. *Suppose, B is a linear operator in* $\mathfrak{B}$ *with* $\mathfrak{D}(B) \supset \mathfrak{D}_0$ *and the properties:*

(i) $A_0 + B$ *is dissipative.*

(ii) $B(\lambda I - A_0)^{-1}$ *is compact in* $\mathfrak{B}$.

Then the operator $\lambda I - A_0 - B$ *maps* $\mathfrak{D}_0$ *bijective onto* $\mathfrak{B}$. *For* $\mathfrak{D}_1 := (\lambda I - A_0 - B)^{-1} \mathfrak{B}_0$ *we have* $\bar{\mathfrak{D}}_1 = \mathfrak{B}_0$ *and the operator* $(A_0 + B)|_{\mathfrak{D}_1}$ *generates a strongly continuous contraction semigroup in* $\mathfrak{B}_0$.

PROOF. Condition (i) yields (see [3], IX. 8)

$$\|(\lambda I - A_0 - B) f\| \geqq \lambda \|f\| \qquad (f \in \mathfrak{D}_0). \tag{3}$$

Therefore $(I - B(\lambda I - A_0)^{-1}) g = 0$ implies $g = 0$, and from condition (ii) it follows, that $I - B(\lambda I - A_0)^{-1}$ maps $\mathfrak{B}$ bijective onto $\mathfrak{B}$. The identity

$$\lambda I - A_0 - B = (I - B(\lambda I - A_0)^{-1})(\lambda I - A_0)$$

gives immediately the first statement, and from (3) we have

$$\|(\lambda I - A_0 - B)^{-1}\| \leqq \frac{1}{\lambda}. \tag{4}$$

An argument from [3], IX. 8 shows that (4) holds true for all $\lambda > 0$. Finally, from the Hille-Yosida theorem and [4], theorem 12.2.4 we get the desired result.

In the following we take $\mathfrak{B} = \boldsymbol{C}_{[0,1]}$, the Banach space of all real continuous functions on $[0, 1]$, with the following semi-inner product: Choose for each $g \in \boldsymbol{C}_{[0,1]}$ a point x_g with the property $g(x_g) = \max_{x \in [0,1]} g(x)$ and define for $f, g \in \boldsymbol{C}_{[0,1]}$:

$$[f, g] := f(x_g) g(x_g). \tag{5}$$

A linear operator A in $\boldsymbol{C}_{[0,1]}$ is said to satisfy the maximum principle, if $f \in \mathfrak{D}(A)$, $f(x_0) = \max_{x \in [0,1]} f(x) \geqq 0$ imply $(Af)(x_0) \leqq 0$. An operator A, which satisfies the maximum principle, is dissipative (with respect to the semi-inner product (5)).

2. Let m be a strongly increasing continuous function on $[0, 1]$. We consider the second order generalized differential operator $D_m D_x$ as defined e.g. in [3] or [5]. As m is continuous, this operator is well defined and we have $D_x f \in \boldsymbol{C}_{[0,1]}$ if $f \in \mathfrak{D}(D_m D_x)$. The operator A_0:

$$\mathfrak{D}(A_0) = \mathfrak{D}_0 := \{f \in \boldsymbol{C}_{[0,1]} : f \in \mathfrak{D}(D_m D_x),\ D_m D_x f \in \boldsymbol{C}_{[0,1]},$$

$$(D_x f)(0) = (D_x f)(1) = 0\} \tag{6}$$

$$A_0 f := D_m D_x f \qquad (f \in \mathfrak{D}_0)$$

is the infinitesimal generator of a s. c. n. c. semigroup in $C_{[0,1]}$.

We define a family of functions φ_x, $x\in[0,1]$, by

$$\varphi_x(y):=\int_x^y(y-s)dm(s), \qquad y\in[0,1].$$

As m is strongly increasing, for each $\varepsilon>0$ there exists a $\gamma_\varepsilon>0$, such that

$$\varphi_x(y)\geqq\gamma_\varepsilon \quad \text{if} \quad |x-y|\geqq\varepsilon, \quad x,y\in[0,1].$$

Further, let n_x, $x\in[0,1]$, be a family of nonnegative measures on $[0,1]$ with the following properties:

(a) $n_x([0,1])\leqq K<\infty \qquad (x\in[0,1])$.

(b) $\xi\to x$ implies $n_\xi\to n_x$ *-weakly ($x,\xi\in[0,1]$), that is

$$\int_0^1 f(y)n_\xi(dy)\to\int_0^1 f(y)n_x(dy) \quad \text{for all } f\in C_{[0,1]}.$$

(c) $\sup_{x\in[0,1]}\int_{|x-y|\leqq\delta,y\in[0,1]} n_x(dy)\to 0 \quad \text{if } \delta\downarrow 0$.

Condition (c) implies $n_x(\{x\})=0$ for all $x\in[0,1]$.

We shall consider restrictions by boundary conditions of the following operator $\mathfrak{A}$ on $\mathfrak{D}(D_mD_x)$:

$$(\mathfrak{A}f)(x)=(D_mD_xf)(x)+b(x)(D_xf)(x)$$
$$+\int_0^1(f(y)-f(x)-(y-x)(D_xf)(x))\frac{n_x(dy)}{\varphi_x(y)}. \tag{7}$$

Here b is a continuous function on $[0,1]$.

THEOREM 2. *Suppose m, n_x ($x\in[0,1]$) and b have the properties mentioned above. Then the restriction of $\mathfrak{A}$ to $\mathfrak{D}_0$ is the infinitesimal generator of a s.c.n.c. semigroup in $C_{[0,1]}$.*

PROOF. We shall show that the operators A_0 in (6) and B:

$$(Bf)(x):=\int_0^1(f(y)-f(x)-(y-x)(D_xf)(x))\frac{n_x(dy)}{\varphi_x(y)}+b(x)(D_xf)(x) \ (f\in\mathfrak{D}_0)$$

satisfy the conditions of Theorem 1 with $\mathfrak{B}=\mathfrak{B}_0=C_{[0,1]}$.

The operator B maps $\mathfrak{D}_0$ into $C_{[0,1]}$. Indeed, with

$$f(x)=\alpha+\int_0^x(x-s)\varphi(s)dm(s), \qquad \varphi\in C_{[0,1]},\ \int_0^1\varphi dm=0,$$

and

$$h_x(y):=\begin{cases}\varphi_x(y)^{-1}\int_x^y(y-s)\varphi(s)dm(s) & y\neq x\\ \varphi(x) & y=x\end{cases} \qquad (x,y\in[0,1])$$

we have

$$|(Bf)(x)-(Bf)(\xi)|\leqq\left|\int_0^1 h_x(y)(n_x(dy)-n_\xi(dy))\right|$$
$$+\int_0^1|h_x(y)-h_\xi(y)|n_\xi(dy)+|b(x)f(x)-b(\xi)f(\xi)|\quad(x,\xi\in[0,1]).$$

As h_x is continuous, by condition (b) the first term on the right hand side tends to zero if $\xi\rightarrow x$. The same is obvious for the third term. It will follow for the second term, if we show that $\xi\rightarrow x$ also implies

$$\max_{y\in[0,1]}|h_x(y)-h_\xi(y)|\longrightarrow 0.$$

Given $\varepsilon>0$, choose $\delta(\varepsilon)$ with the property $|\varphi(s)-\varphi(t)|\leqq\varepsilon$ if $|s-t|\leqq\delta(\varepsilon)$, $s,t\in[0,1]$. Then for all ξ with $|x-\xi|\leqq\frac{\delta(\varepsilon)}{4}$ and all y with $|x-y|\leqq\frac{\delta(\varepsilon)}{2}$ we have $|h_x(y)-h_\xi(y)|\leqq 3\varepsilon$, and the desired result follows without difficulty.

It is easy to see, that A_0+B satisfies the maximum principle and therefore (i).

With two solutions ψ, χ of the equation $D_mD_xf-\lambda f=0$,

$$\psi(0;\lambda)=1,\qquad(D_x\psi)(0;\lambda)=0;$$
$$\chi(1;\lambda)=1,\qquad(D_x\chi)(1;\lambda)=0,$$

the resolvent $(\lambda I-A_0)^{-1}$, $\lambda>0$, admits a representation

$$((\lambda I-A_0)^{-1}f)(x)=\int_0^1 G(x,s;\lambda)f(s)\,dm(s), \tag{8}$$

$$G(x,s;\lambda):=\begin{cases}c(\lambda)\psi(x;\lambda)\chi(s;\lambda) & x\leqq s\\ c(\lambda)\chi(x;\lambda)\psi(s;\lambda) & x>s\end{cases},$$

where $c(\lambda)$ is holomorphic in the right half plane. It follows

$$\begin{aligned}(B(\lambda I-A_0)^{-1}f)(x)=c(\lambda)\int_0^1 n_x(dy)\frac{1}{\varphi_x(y)}\Big\{&\lambda\int_x^y(y-s)\psi(s)\,dm(s)\int_x^1\chi f\,dm\\ &+\lambda\int_x^y(y-s)\chi(s)\,dm(s)\int_0^x\psi f\,dm+\psi(y)\int_x^y(\chi(y)-\chi(s))f(s)\,dm(s)\\ &-\chi(y)\int_x^y(\psi(y)-\psi(s))f(s)\,dm(s)\Big\}\\ &+b(x)c(\lambda)\Big\{(D_x\chi)(x)\int_0^x\psi f\,dm+(D_x\psi)(x)\int_x^1\chi f\,dm\Big\}.\end{aligned} \tag{9}$$

We define $n_x^{(\varepsilon)}(\Gamma):=n_x(\Gamma\backslash(x-\varepsilon,x+\varepsilon))$ ($\varepsilon>0$, Γ-measurable subset of $[0,1]$, $x\in[0,1]$). Denote the right hand side of (9) (with n_x replaced by $n_x^{(\varepsilon)}$) by $(Kf)(x)$ ($(K^{(\varepsilon)}f)(x)$ resp.). The functions $K^{(\varepsilon)}f$ belong to the space $\boldsymbol{B}_{[0,1]}$ of all bounded measurable functions on [0, 1]. The operator $S:(Sf)(x)=\int_x^1 \chi f\,dm$ is compact in $C_{[0,1]}$, and the operators T_1, T_2:

$$(T_1 g)(x):=\int_0^1 \frac{n_x^{(\varepsilon)}(dy)}{\varphi_x(y)} g(y), \qquad (T_2 g)(x):=\int_0^1 \frac{n_x^{(\varepsilon)}(dy)}{\varphi_x(y)} g(x)$$

from $C_{[0,1]}$ into $B_{[0,1]}$ are bounded. This implies the compactness of $K^{(\varepsilon)}$ as a mapping from $C_{[0,1]}$ into $B_{[0,1]}$. Furthermore, by (c) we have $\|K^{(\varepsilon)}-K\|\to 0$ if $\varepsilon\downarrow 0$, therefore K is compact as a mapping from $C_{[0,1]}$ into $B_{[0,1]}$, and the compactness of $B(\lambda I-A_0)^{-1}$ in $C_{[0,1]}$ follows.

It remains to show, that the strongly continuous semigroup generated by $A+B$ is nonnegative. But this follows e. g. from [6], Theorem 2.8.

Before extending Theorem 2 to more general boundary conditions, we shall study certain initial and boundary problems for the equation

$$\mathfrak{A}f-\lambda f=0 \qquad (\lambda>0). \tag{10}$$

LEMMA 3. *Under the conditions of Theorem 2 the following holds:*

a) (10) *with boundary conditions* $f(0)=f(1)=0$ *has only the trivial solution* $f=0$;

b) (10) *with boundary conditions* $f(0)=0$, $f(1)=1$ *has a unique solution in* $C_{[0,1]}$; *this solution is nonnegative.*

If we additionally supppose $\operatorname{supp} n_x\supset[x, 1]$*) *for all* $x\in[0, 1]$, *we have*

c) (10) *with initial condition* $f(0)=(D_x f)(0)=0$ *has only the trivial solution* $f=0$;

d) (10) *with initial condition* $f(0)=1$, $(D_x f)(0)=0$ (*or* $f(0)=0$, $(D_x f)(0)=1$) *has a unique solution in* $C_{[0,1]}$.

PROOF. A function f, satisfying (10), cannot have a positive absolute maximum (or a negative absolute minimum) in (0, 1). Indeed, $f(x_0)=\max_{x\in[0,1]} f(x)$, $0<x_0<1$, implies $(D_x f)(x_0)=0$, $(D_m D_x f)(x_0)\leqq 0$ and from (10) we get $f(x_0)=0$. Therefore the maximum and minimum of f are at the endpoints of [0, 1], that is in cases $f(0)=0$ or $f(1)=0$ the function f is of constant sign, and in case a) we have $f=0$.

To prove c) suppose $f(0)=(D_x f)(0)=0$. If $F:=\{x: f(x)\neq 0\}\neq\emptyset$, consider $x_0=\inf F$. Then we have

$$(D_m D_x f)(x_0)+\int_{x_0}^1 f(y)\frac{n_{x_0}(dy)}{\varphi_{x_0}(y)}=0,$$

and if e. g. $f\geqq 0$, then $(D_m D_x f)(x_0)\geqq 0$ and $\operatorname{supp} n_{x_0}\supset[x_0, 1]$ imply $f=0$, a contradiction.

With $f(x)=\beta+\beta' x+\int_0^x (x-s)\varphi(s)dm(s)$ equation (10) becomes

*) Apparently, the condition $\operatorname{supp} n_x\supset[x, 1]$ is only for technical reason.

$$\varphi(x)+b(x)\int_0^x \varphi\, dm+\int_0^1\int_x^y (y-s)\varphi(s)dm(s)\frac{n_x(dy)}{\varphi_x(y)}-\lambda\int_0^x (x-s)\varphi(s)dm(s) \quad (11)$$
$$=-b(x)\beta'+\lambda\beta+\lambda\beta' x.$$

As in the proof of Theorem 2 it can be shown, that the left hand side of (11) has the form $(I+G)\varphi$ with a compact operator G in $C_{[0,1]}$. Therefore, to prove existence and uniqueness of f in case d) it remains to show, that the homogeneous equation corresponding to (11) has only the trivial solution. But this was shown in c). The proof of b) is similar.

In the following we have to deal with the solutions f_0, f_1 of equation (10) with boundary conditions

$$f_0(0)=1,\ f_0(1)=0; \qquad f_1(0)=0,\ f_1(1)=1.$$

Then

$$F(x):=1-f_0(x)-f_1(x)>0 \qquad (0<x<1). \tag{12}$$

Indeed, $F(0)=F(1)=0$, and

$$(D_mD_xF)(x)+b(x)(D_xF)(x)+\int_0^1 (F(y)-F(x)-(y-x)(D_xF)(x))\frac{n_x(dy)}{\varphi_x(y)}$$
$$-\lambda F(x)=-\lambda$$

implies, that F cannot have a nonpositive minimum on $(0, 1)$.

If $f\in\mathfrak{D}(D_mD_x)$ we define (see e.g. [7], II. 5)

$$\begin{aligned}\Phi_0(f)&:=\kappa_0 f(0)+\int_0^1 \frac{f(0)-f(x)}{x}\, dq_0(x)+\sigma_0(\mathfrak{A}f)(0)\,,\\ \Phi_1(f)&:=\kappa_1 f(1)+\int_0^1 \frac{f(1)-f(x)}{1-x}\, dq_1(x)+\sigma_1(\mathfrak{A}f)(1)\,.\end{aligned} \tag{13}$$

Here $\mathfrak{A}$ is again given by (7), the constants κ_0, κ_1, σ_0, σ_1 are nonnegative, q_0 and q_1 are (nonnegative) measures on $[0, 1]$ and $\kappa_i+\sigma_i+\int_0^1 dq_i>0$ $(i=0, 1)$. It is understood that

$$\left.\frac{f(0)-f(1)}{x}\right|_{x=0}=-(D_xf)(0)\,, \qquad \left.\frac{f(1)-f(x)}{1-x}\right|_{x=1}=(D_xf)(1)\,.$$

Let

$$\mathfrak{D}_{\Phi_0,\Phi_1}:=\{f\in\mathfrak{D}(D_mD_x):\Phi_0(f)=\Phi_1(f)=0\}\,.$$

As is well known (see [7]), $\overline{\mathfrak{D}_{\Phi_0,\Phi_1}}=C_{[0,1]}$ if and only if

$$\sigma_i>0 \quad\text{or}\quad q_i(\{r_i\})>0 \quad\text{or}\quad \int_{0+0}^{1-0}\frac{dq_i(x)}{|x-i|}=\infty \qquad \text{for}\quad i=0, 1\,. \tag{14}$$

THEOREM 4. *Suppose the equations* $\Phi_0(f)=0$, $\Phi_1(f)=0$ *are not equivalent to* $f(0)=f(1)$. *Then, if* (14) *is fulfilled, the restriction* A *of the operator* $\mathfrak{A}$ *to*

$\mathfrak{D}_{\Phi_0,\Phi_1}$ *is the infinitesimal generator of a s.c.n.c. semigroup in* $\boldsymbol{C}_{[0,1]}$. *If* (14) *is not fulfilled, the restriction* A *of* $\mathfrak{A}$ *to*

$$\mathfrak{D}_1 := \{f \in \mathfrak{D}_{\Phi_0,\Phi_1} : \mathfrak{A}f \in \overline{\mathfrak{D}_{\Phi_0,\Phi_1}}\}$$

is the infinitesimal generator of a s.c.n.c. semigroup in $\overline{\mathfrak{D}_{\Phi_0,\Phi_1}}$.

PROOF. By $R_\lambda^{(0)}$, $\lambda>0$, we denote the resolvent of the restriction of $\mathfrak{A}$ to $\mathfrak{D}_0$ (see Theorem 2). If $g \in \boldsymbol{C}_{[0,1]}$, the solution f of $\lambda f-\mathfrak{A}f=g$ has the form

$$f(x)=(R_\lambda^{(0)}g)(x)+C_0f_0(x)+C_1f_1(x).$$

In the same way as in [7], Lemma II. 5.5 it follows, that C_0 and C_1 are uniquely determined by the boundary conditions $\Phi_0(f)=\Phi_1(f)=0$ (here we have to make use of (12)).
Let us now suppose $\sigma_0\sigma_1>0$. Then $\mathfrak{A}$, restricted to $\mathfrak{D}_{\Phi_0,\Phi_1}$, satisfies the maximum principle. Therefore, by [6], Theorem 2.8, it generates a s.c.n.c. semigroup in $\boldsymbol{C}_{[0,1]}$.

If $\sigma_0\sigma_1=0$, e.g. $\sigma_0=\sigma_1=0$, we define operators A_ε ($\varepsilon \geqq 0$):

$$\mathfrak{D}(A_\varepsilon) := \{f \in \mathfrak{D}(D_mD_x) : \Phi_0(f)+\varepsilon(\mathfrak{A}f)(0)=\Phi_1(f)+\varepsilon(\mathfrak{A}f)(1)=0\},$$

$$A_\varepsilon f := \mathfrak{A}f \quad \text{if} \quad f \in \mathfrak{D}(A_\varepsilon).$$

Their resolvents exist by the first part of the proof and depend continuously on ε (in the strong operator topology). As A_ε is the infinitesimal generator of a s.c.n.c. semigroup in $\boldsymbol{C}_{[0,1]}$ it has the properties

$$f \in \boldsymbol{C}_{[0,1]},\ f \geqq 0 \ \Rightarrow\ (\lambda I-A_\varepsilon)^{-1}f \geqq 0\,;\ \|(\lambda I-A_\varepsilon)^{-1}\| \leqq \frac{1}{\lambda} \quad (\lambda>0).$$

If $\varepsilon \downarrow 0$, the same relations hold true for the resolvent of $A_0=A$, and the statement follows from [7], Theorem I.1.1 and [4], Theorem 12.2.4.

Finally let us mention, that the boundary conditions (2) satisfying (14) are the most general ones, which turn $\mathfrak{A}$ into the infinitesimal generator of a s.c.n.c. semigroup in $\boldsymbol{C}_{[0,1]}$ (see the proof of [7], Theorem II.5.2).

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