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A ring theoretical proof in a factor category of indecomposable modules

Dedicated to Professor Mutsuo Takahashi on his 60th birthday

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Introduction.

We have widely extended the Krull-Remak-Schmidt-Azumaya's theorem in [4], [5], [6], [7], [8] and [11] and succeeded to prove some of main theorems by virtue of the theory of category ([12], Theorem 20.1 on page 30 and [13], Theorem 2). However, most statements in the above papers are related to modules, but not to categories. Thus, it is natural to expect to be able to prove all results in the frame of ring theory.

Recently, T. Ishii [10] succeeded to prove substantially the implication from i) to ii) in [11] in the frame of ring theory. Hence, the remaining is essentially Theorem 2 in [6].

In this short note, we shall give a ring theoretical proof of the above theorem by making use of an idea given in [10]. First, we shall translate a factor category induced from completely indecomposable modules into a category of semi-simple modules through equivalent functors, which gives a simpler proof of [4], Theorem 7, however it does not work on category of projectives or injectives (see [4] and [6]). Finally, we shall give a ring theoretical proof of [6], Theorem 2 by making great use of results in [10].

§ 1. Factor categories.

Throughout we shall assume that R is a ring with identity and all R-modules are unitary right R-modules. We shall denote the category of all R-modules by $\mathfrak{M}_R$. Let $\{T_\alpha\}_I$ be a set of R-modules. We shall define a full subadditive *category* $\mathfrak{T}$ *induced from* $\{T_\alpha\}_I$ (see [4]). Every objects in $\mathfrak{T}$ consist of all R-modules which are isomorphic to $\sum_K \oplus T_\delta$, where T_δ's are some members in $\{T_\alpha\}_I$ and the set of morphisms coincides with the set of R-homomorphisms. Let $\mathfrak{C}$ be an ideal in $\mathfrak{T}$ (see [2] or [9]). We define the *factor category* $\mathfrak{T}/\mathfrak{C}$ as follows: the objects in $\mathfrak{T}/\mathfrak{C}$ coincide with those in $\mathfrak{T}$ and $[T, T']_{\mathfrak{T}/\mathfrak{C}}=\mathrm{Hom}_R(T, T')/(\mathrm{Hom}_R(T, T')\cap\mathfrak{C})$ for T, T' in $\mathfrak{T}/\mathfrak{C}$.

An R-module M is called *completely indecomposable* if $\mathrm{End}_R(M)$ is a local ring. Now, we assume $\{A_\alpha\}_I$ is a set of completely indecomposable modules such that $A_\alpha \not\approx A_\beta$ if $\alpha \neq \beta$. Let $\mathfrak{A}$ be the category induced from $\{A_\alpha\}_I$. Every morphisms in $\mathfrak{A}$ can be expressed as column-summable matrices $(a_{\sigma\tau})$ (see [**1**] or [**4**]). We define an ideal $\mathfrak{J}'$ in $\mathfrak{A}$ as follows: $\mathfrak{J}' \cap [A, A'] = \{(a_{\sigma\tau}) \mid$ *any* $a_{\sigma\tau}$ *are not* R*-isomorphic for all* $\sigma, \tau\}$ (see [**4**]).

Since A_α is completely indecomposable, $\mathrm{End}_R(A_\alpha)/J(\mathrm{End}_R(A_\alpha))$ is a division ring Δ_α, where $J(\ )$ means the Jacobson radical of ( ). We put $\Delta = \prod_I \Delta_\alpha$, then Δ is a regular ring in the sense of Von Neumann and $\sum_I \oplus \Delta_\alpha$ is the socle of Δ. By $\mathfrak{S}$ we shall denote the category induced from $\{\Delta_\alpha\}_I$ in $\mathfrak{M}_\Delta$, where we regard Δ_α as a right Δ-module. It is clear that $\mathfrak{S}$ is a completely reducible and Grothendieck subcategory in $\mathfrak{M}_\Delta$. We note that for any morphism f in $\mathfrak{S}$ $\mathrm{Ker}\, f$ and $\mathrm{Im}\, f$ in $\mathfrak{S}$ coincide with those in $\mathfrak{M}_\Delta$.

First, we shall prove the following

THEOREM 1 ([**4**], Theorem 7). *Let* $\mathfrak{A}$, $\mathfrak{J}'$ *and* $\mathfrak{S}$ *be as above. Then* $\mathfrak{A}/\mathfrak{J}'$ *is equivalent to* $\mathfrak{S}$.

We need a lemma to prove it. Let $\{T_\alpha\}_I$ be a set of R-modules and $\mathfrak{T}$ the category induced from $\{T_\alpha\}_I$. We shall define a new category $\mathfrak{T}^*$. Let $M = \sum_K \oplus M_\gamma$ be an object in $\mathfrak{T}$, where $M_\gamma \approx T_\gamma$. We consider a pair $(M, \sum_K \oplus M_\gamma)$. We define $(M, \sum_K \oplus M_\gamma) \equiv (N, \sum_{K'} \oplus N_{\gamma'})$ if and only if $M = N$ and $\sum_K \oplus M_\gamma$, $\sum_{K'} \oplus N_{\gamma'}$ are the same decompositions of M. We put the set of morphisms $[(M, \sum_K \oplus M_\gamma), (N, \sum_{K'} \oplus N_{\gamma'})] = \mathrm{Hom}_R(M, N)$ (we should distinguish $[(M, \sum \oplus M_\gamma), (M', \sum \oplus M'_{\gamma'})]$ and $[(M, \sum \oplus M''_{\gamma''}), (M', \sum \oplus M'''_{\gamma'''})]$ if decompositions are different even though they are equal to $\mathrm{Hom}_R(M, M')$). Then we have an additive category $\mathfrak{T}^* = \{(M, \sum \oplus M_\gamma)\}$.

LEMMA 1. $\mathfrak{T}$ *is equivalent to* $\mathfrak{T}^*$.

PROOF. We shall define functors $S: \mathfrak{T} \to \mathfrak{T}^*$ and $U: \mathfrak{T}^* \to \mathfrak{T}$ such that SU (resp. US) is equivalent to $1_{\mathfrak{T}^*}$ (resp. $1_{\mathfrak{T}}$). For each M in $\mathfrak{T}$ we fix one decomposition: $M = \sum \oplus M_\alpha$; $M_\alpha \approx T_\alpha$ and we define

$$S(M) = (M, \sum_K \oplus M_\gamma).$$

Conversely, let $\boldsymbol{M}' = (M', \sum \oplus M'_\gamma)$ be in $\mathfrak{T}^*$. Put

$$U(\boldsymbol{M}') = \sum \oplus M'_\gamma = M'.$$

It is clear that S and U are functors and $US = 1_{\mathfrak{T}}$. Let $M' = U(\boldsymbol{M}') = \sum_L \oplus M^*_\varepsilon$ be the fixed decomposition of M'. Then $SU(\boldsymbol{M}') = (M', \sum_L \oplus M^*_\varepsilon)$. Put $\varphi_{\boldsymbol{M}'} = 1_{M'}$ $(\in [(M', \sum \oplus M'_\gamma), (M', \sum \oplus M^*_\varepsilon)])$. Then it is clear that SU and $1_{\mathfrak{T}^*}$ are naturally equivalent through $\{\varphi_{\boldsymbol{M}'}\}$.

Now, we assume $\{A_\alpha\}_I$ is a set of completely indecomposable modules such that $A_\alpha \not\approx A_\beta$ if $\alpha \neq \beta$. Let $\mathfrak{A}$ be the category induced from $\{A_\alpha\}_I$ in $\mathfrak{M}_R$. We can define the ideal $\mathfrak{J}'$ in $\mathfrak{A}^*$ as above. Since $\mathfrak{J}'$ does not depend on decompositions of M by [4], Corollary to Lemma 4, we have

COROLLARY. *$\mathfrak{A}/\mathfrak{J}'$ is equivalent to $\mathfrak{A}^*/\mathfrak{J}'$.*

PROOF OF THEOREM 1. We shall show an equivalence between $\mathfrak{A}^*/\mathfrak{J}'$ and $\mathfrak{S}^*$. Let $\boldsymbol{M}=(M, \sum_K \oplus M_\gamma)$ be an object in $\mathfrak{A}^*/\mathfrak{J}'$, where $M_\gamma \approx A_\gamma$. We put $M = \sum \oplus M_\gamma = \sum_I \sum_{J_\alpha} \oplus M_{\alpha\beta}$; $M_{\alpha\beta} \approx A_\alpha$. Let $\boldsymbol{N}=(N, \sum_I \sum_{J'_\alpha} \oplus N_{\alpha\beta'})$ be another object. We divide matrices in $\mathrm{Hom}_R(M, N)$ into $|I| \times |I|$ blocks, where $|I|$ means the cardinal number of I. Then it is clear from the decomposition above that every entries of matrix in every block except the diagonal consist of non isomorphisms and hence, those blocks belong to $\mathfrak{J}'$. We put $M(\alpha) = \sum_{J_\alpha} \oplus M_{\alpha\beta}$ and $S(\alpha) = \sum_{J_\alpha} \oplus \Delta_\alpha = \Delta_\alpha^{(J_\alpha)}$. Then we have a natural isomorphism

$$\mathrm{End}_R(M(\alpha))/\mathfrak{J}' \cap \mathrm{End}_R(M(\alpha)) \approx \mathrm{End}(S(\alpha)) \cdots\cdots\cdots\cdots\cdots (*),$$

(cf. [4]).

We define $T: \mathfrak{A}^*/\mathfrak{J}' \to \mathfrak{S}^*$ by setting $T(\boldsymbol{M}) = \sum_I \oplus \Delta_\alpha^{(J_\alpha)}$ and $T(\mathrm{Hom}_R(\boldsymbol{M}, \boldsymbol{N})/\mathfrak{J}' \cap \mathrm{Hom}_R(\boldsymbol{M}, \boldsymbol{N})) = \mathrm{Hom}_\Delta(T(\boldsymbol{M}), T(\boldsymbol{N}))$ via (*). Then T is a functor. Conversely, let $\boldsymbol{S}=(S, \sum_I \sum_{I_\alpha} \oplus \Delta_{\alpha\beta})$ be in $\mathfrak{S}^*$, where $\Delta_{\alpha\beta} \approx \Delta_\alpha$. We define

$$V(\boldsymbol{S}) = (M, \sum_I \oplus A_\alpha^{(I_\alpha)}).$$

Since $\Delta_\alpha \approx \mathrm{End}_\Delta(\Delta_{\alpha\beta}) \approx \mathrm{End}_R(A_\alpha)/J(\mathrm{End}_R(A_\alpha))$ and $J(\mathrm{End}_R(A_\alpha))$ consists of all non-isomorphisms, $V(\mathrm{Hom}_\Delta(\boldsymbol{S}, \boldsymbol{S}')) = \mathrm{Hom}_R(V(\boldsymbol{S}), V(\boldsymbol{S}'))/\mathfrak{J}' \cap \mathrm{Hom}_R(V(\boldsymbol{S}), V(\boldsymbol{S}'))$. V is also a functor: $\mathfrak{S}^* \to \mathfrak{A}^*/\mathfrak{J}'$. We have a natural isomorphism $\varphi_M: \sum_I \sum_{J_\alpha} \oplus M_{\alpha\beta} \to \sum_I \oplus A_\alpha^{(J_\alpha)}$ (and $\psi_S: \sum_I \sum_{I_\alpha} \oplus \Delta_{\alpha\beta} \to \sum_I \oplus \Delta_\alpha^{(I_\alpha)}$). Hence, $TV \sim 1_{\mathfrak{S}^*}$ and $VT \sim 1_{\mathfrak{A}^*/\mathfrak{J}'}$. Let $S_{\mathfrak{A}/\mathfrak{J}'}$ and $U_{\mathfrak{A}/\mathfrak{J}'}$ (resp. $S_{\mathfrak{S}}$ and $U_{\mathfrak{S}}$) be functors in Lemma 1. Then $X = U_{\mathfrak{S}} T S_{\mathfrak{A}/\mathfrak{J}'}$ and $Y = U_{\mathfrak{A}/\mathfrak{J}'} V S_{\mathfrak{S}}$ give the desired equivalence.

REMARK 1. The functor Y above does not depend on decompositions of M in $\mathfrak{A}$, however X does. If we fix another decomposition of M: $M = \sum_{K'} \oplus M'_{\alpha'}$ and define a functor $S'(M) = (M, \sum_{K'} \oplus M'_{\alpha'})$ in Lemma 1, then $1_{\mathfrak{A}/\mathfrak{J}'} \sim YX \sim YX'$. Hence, $X \sim X'$.

COROLLARY ([1]) (Uniqueness of decompositions). *Let $M = \sum_K \oplus M_\alpha = \sum_{K'} \oplus M'_{\alpha'}$ be two decompositions of M by completely indecomposable modules M_α and $M'_{\alpha'}$. Then there exists a one-to-one mapping φ of K onto K' such that $M_\alpha \approx M'_{\varphi(\alpha)}$ for all $\alpha \in K$* (cf. [3] and [4]).

PROOF. Let $M = \sum_I \sum_{J_\alpha} \oplus M_{\alpha\beta} = \sum_{I'} \sum_{J'_{\alpha'}} \oplus M'_{\alpha'\beta'}$. We may assume $I = I'$ since J_α or $J'_{\alpha'}$ may be empty. Then $X(M) = \sum_I \oplus \Delta_\alpha^{(J_\alpha)} \approx X'(M) = \sum_I \oplus \Delta_\alpha^{(J'_\alpha)}$ from

Remark 1. Hence $|J_\alpha|=|J'_\alpha|$ since $\mathfrak{S}$ is completely reducible.

From this corollary, we may understand that $X:\mathfrak{A}/\mathfrak{J}'\to\mathfrak{S}$ does not depend on decompositions of M.

REMARKS 2. Let X, Y, φ_M and ψ_S be as above. Then $X(\varphi_M)=1_{X(M)}$ and $Y(\psi_S)=1_{Y(S)}$ and hence, $XYX=X$ and $YXY=Y$.

3. Let $f:M\to N$ be in $\mathfrak{A}/\mathfrak{J}'$. Then $X(f)$ has the image I and the kernel K in $\mathfrak{M}_\Delta$ and $I=XY(I)$, $K=XY(K)$ from Remark 2. Therefore, $Y(I)$ and $Y(K)$ are the image and the kernel of f in $\mathfrak{A}/\mathfrak{J}'$, respectively. Let $M=N_1\oplus N_2$ in $\mathfrak{M}_R$ (N_i are not necessarily in $\mathfrak{A}/\mathfrak{J}'$) and e_i projections of M to N_i. Then [**5**], Proposition 2 and [**10**], Lemma 6 imply that there exist submodules N'_i of N_i such that N'_i are in $\mathfrak{A}/\mathfrak{J}'$ and $X(N'_i)=\operatorname{Im} X(e_i)$. We call them *dense submodules* [**5**].

§ 2. Exchange property.

Next, we shall give a proof of [**6**], Theorem 2 by means of the concept of locally direct summands in [**9**] and [**10**]. Let $\{T_\alpha\}_I$ be a set of R-modules. We take any countable subset $\{T_i\}_1^\infty$ of $\{T_\alpha\}_I$ and any set $\{f_i\}$ of non-isomorphisms $f_i:T_i\to T_{i+1}$. If for any above sets and any element t in T_1, there exists a number n, depending on t and given sets, such that $f_nf_{n-1}\cdots f_1(t)=0$, then we call the set $\{T_\alpha\}_I$ *locally semi-T-nilpotent*. We assume $\{A_\alpha\}_I$ is a set of completely indecomposable modules and we put $A=\sum_I\oplus A_\alpha$. Then we know from [**9**], Theorem 3.1.2 that $\mathrm{End}_R(A)\cap\mathfrak{J}'=J(\mathrm{End}_R(A))$ if and only if $\{A_\alpha\}_I$ is locally semi-T-nilpotent. Finally, let $M=N_1\oplus N_2$ be R-modules. If for any decomposition $M=\sum_K\oplus L_\alpha$ of M there exist submodules L'_α of L_α such that $M=N_1\oplus\sum_K\oplus L'_\alpha$, then we say N_1 *has the exchange property in* M (cf. [**5**]).

Now, we are ready to state [**6**], Theorem 2

THEOREM 2. *Let $\{M_\alpha\}_I$ be a set of completely indecomposable modules and $M=\sum_I\oplus M_\alpha$. Let $I=I_1\cup I_2$ be a partition of I and $N_i=\sum_{I_i}\oplus M_\alpha$. If $\{M_\alpha\}_{I_1}$ is locally semi-T-nilpotent, N_1 and N_2 have the exchange property in M.*

First, we shall prepare some fundamental properties of dense submodules (see Remark 3). We note that those properties below are trivial if we use the functors X and Y in the proof of Theorem 1. Let $M\supseteqq N$ be R-modules. By i_N we always denote, in the following, the inclusion of N into M.

We shall quote here [**10**], Corollaries 1 and 2 to Proposition 8.

LEMMA 2 ([**10**]). *Let M be in $\mathfrak{A}$. $M=\sum_I\oplus M_\alpha$ and e_α the projection of M onto M_α. Let N_α be a submodule of M_α, which is in $\mathfrak{A}$. Then N_α is a dense submodule of M_α if and only if there exists an R-homomorphism $p_\alpha:M\to N_\alpha$ such that $i_{M_\alpha}p_\alpha-e_\alpha\in\mathrm{End}_R(M)\cap\mathfrak{J}'$. In those cases for all $\alpha\in I$, $\sum\oplus N_\alpha$ is a* *dense submodule of M.*

We shall denote the fact $i_{M_\alpha}p_\alpha - e_\alpha \in \mathrm{End}_R(M)\cap\mathfrak{J}'$ by $i_{M_\alpha}p_\alpha \equiv e_\alpha$ (mod $\mathfrak{J}'$) or simply $i_{M_\alpha}p_\alpha \equiv e_\alpha$. We refer to [**10**] for other terminologies.

LEMMA 3. *Let M be an object in $\mathfrak{A}$ and N a direct summand of M. We assume that A_1, A_2 are in $\mathfrak{A}$, $A_1\oplus A_2$ is a dense submodule of N ond e a projection of M onto N. Then there exist R-homomorphisms $p_j: M\to A_j$ which satisfy* i) $e\equiv i_1p_1+i_2p_2$, ii) $p_ji_{j'}\equiv\delta_{jj'}1_{A_j}$ *and* iii) $p_je\equiv p_j$ $(ei_j\equiv i_j)$, *where $\delta_{jj'}$ is the Kronecker δ and $i_j=i_{A_j}$.*

PROOF. From Lemma 2 there exists an R-homomorphism $p: M\to A_1\oplus A_2$ such that $e\equiv ip$ and $pi\equiv 1_{A_1\oplus A_2}$, where $i=i_{A_1\oplus A_2}$. Since $e\equiv e^2\equiv ipe$, we may assume $pe=p$. Let i'_j, p'_j be the injections and projections of A_j in $A_1\oplus A_2$, respectively. Then $1_{A_1\oplus A_2}=i'_1p'_1+i'_2p'_2$. Put $p_j=p'_jp$, then $p_je=p_j$ and $i_1p_1+i_2p_2 \equiv i(i'_1p'_1+i'_2p'_2)p\equiv ip\equiv e$. $p_ji_j\equiv p'_jpi_j\equiv p'_jpii'_j\equiv p'_ji'_j\equiv 1_{A_j}$. The remaining parts are also trivial.

LEMMA 4. *Let M and N be as above. We assume $A_1\oplus A_2$ and $A_1\oplus A'_2$ are dense submodules of N. Then A_2 is R-isomorphic to A'_2 via p'_2i_2, where the A_i and A'_i are in $\mathfrak{A}$.*

PROOF. Let p_j, i_j be as above and put $\delta=p'_2i_2$. We shall show δ is isomorphic modulo $\mathfrak{J}'$. Let $\alpha: B\to A_2$ and $B\in\mathfrak{A}$. We assume $\delta\alpha\equiv 0$. Then $i_2\alpha\equiv ei_2\alpha\equiv(i_1p'_1+i'_2p'_2)i_2\alpha\equiv i_1p'_1i_2\alpha$, where $p'_1: M\to A_1\subseteq A_1\oplus A'_2$. Hence, $\alpha\equiv p_2i_2\alpha\equiv p_2i_1p'_1i_2\alpha \equiv 0$. Next, let $\beta: A'_2\to B$ and $\beta\delta\equiv 0$. Then $\beta p'_2\equiv\beta p'_2e\equiv\beta p'_2(i_1p_1+i_2p_2)\equiv\beta p'_2i_2p_2 \equiv 0$. Since p'_2 is epimorphic modulo $\mathfrak{J}'$, $\beta\equiv 0$. Therefore, δ is isomorphic modulo $\mathfrak{J}'$ by [**10**], Corollary 1 to Lemma 2. Then A_2 is R-isomorphic to A'_2 since A_2, A'_2 are in $\mathfrak{A}$ (cf. Corollary to Theorem 1 or [**3**]).

LEMMA 5. *Let M be in $\mathfrak{A}$ and $\{P_\alpha, Q_\beta\}_{I,J}$ a set of completely indecomposable modules. We assume $\sum_I\oplus P_\alpha$ and $\sum_J\oplus Q_\beta$ are dense submodules of M. Then for any subset I' of I there exists a subset J' of J such that $\sum_{I-I'}\oplus P_\alpha\oplus\sum_{J'}\oplus Q_\beta$ is also a dense submodule of M.*

PROOF. We put $P=\sum_I\oplus P_\alpha$ and $Q=\sum_J\oplus Q_\beta$. We may assume $I'\neq I$. Let α be in $I-I'$ and put $P^*=\sum_{\delta\neq\alpha}P_\delta$. Then there exist p_α, p^* such that $1_M\equiv i^*p^*+i_\alpha p_\alpha$ from Lemma 3. Furthermore, there exists an isomorphism modulo $\mathfrak{J}'$ $p_P: M\to P$ from Lemma 2. Let p be the projection of P onto P_α. Since $pp_Pi_Q\not\equiv 0$, there exists β in J such that $pp_Pi_\beta\not\equiv 0$, where $i_\beta=i_{Q_\beta}$ (cf. the definition of $\mathfrak{J}'$). Hence, pp_Pi_β is R-isomorphic from [**10**], Corollary 2 to Lemma 3. Now, we consider an external direct sum $Q_\beta\oplus P^*$ and put $j=(i_\beta, i^*): Q_\beta\oplus P^* \to M$. We shall show that j is monomorphic modulo $\mathfrak{J}'$. Let $\varphi=\binom{a}{b}: L\to Q_\beta\oplus P^*$ and $L\in\mathfrak{A}$. We assume $j\varphi\equiv 0$. Then $i_\beta a+i^*b\equiv 0\equiv pp_Pi_\beta a+pp_Pi^*b\equiv pp_Pi_\beta a+ pp_Pi_Pi^{*\prime}b\equiv pp_Pi_\beta a+pi^{*\prime}b\equiv pp_Pi_\beta a$, where $i^{*\prime}: P^*\to P$. Hence, since pp_Pi_β is isomorphic, $a\equiv 0$ and so $i^*b\equiv 0$. Since i^* is monomorphic modulo $\mathfrak{J}'$, $b\equiv 0$. Therefore, j is monomorphic modulo $\mathfrak{J}'$. Thus $\mathrm{Im}\, j=Q_\beta+P^*$ is an internal direct sum $Q_\beta+P^*$ in M and $Q_\beta\oplus\sum_{I'}\oplus P_\alpha(\subseteq Q_\beta\oplus P^*)$ is a locally direct summand in M from [10], Lemma 3. Now, we consider a set $\mathfrak{B}=\{T_K=\sum_{I'}\oplus P_\alpha\oplus\sum_K\oplus Q_\beta \mid K\subseteq J$ *and* T_K *is locally direct summand of* $M\}$. Then $\mathfrak{B}$ is not empty. We define a partial order $\geqq$ in $\mathfrak{B}$ by setting $T_K\geqq T_{K'}$ if and only if $K\supseteqq K'$. We can take a maximal element T in $\mathfrak{B}$ from Zorn's lemma. If T is not a dense submodule of M, we can take a dense submodule $T\oplus L$ of M (see [10], Remark 2). If we use the first argument on a component of L, we can find Q_β such that $T\oplus Q_\beta$ is in $\mathfrak{B}$, which contradicts the maximality of T. Therefore, T is a dense submodule.

COROLLARY. *Let* $M=N_1\oplus N_2$; $N_i\in\mathfrak{A}$ *and* $\sum_I\oplus P_\alpha$ *be as above. Then there exists a subset* I' *of* I *such that* $N_1\oplus\sum_{I'}\oplus P_\alpha$ *is a dense submodule of* M.

PROOF. Since N_i's are in $\mathfrak{A}$, $N_1\oplus N_2$ is itself a dense submodule of M.

PROOF OF THEOREM 2. First, we shall show that N_2 has the exchange property in M. Let $M=\sum_K\oplus Q_\gamma$ and $\sum_{J_\gamma}\oplus P_{\gamma\varepsilon}$ a dense submodule of Q_γ, where $P_{\gamma\varepsilon}$'s are completely indecomposable. Since $\sum_K\sum_{J_\gamma}\oplus P_{\gamma\varepsilon}$ is a dense submodule in M by Lemma 2, there exists a subset J'_γ of J_γ for each $\gamma\in K$ such that $N_2\oplus\sum_K\sum_{J'_\gamma}\oplus P_{\gamma\varepsilon}$ is dense in M from Corollary to Lemma 5. Let e_1 be the projection of $M=N_1\oplus N_2$ onto N_1 and $e_1: M\xrightarrow{p_1}N_1\xrightarrow{i_{N_1}}M$ and put $P'=\sum_K\sum_{J'_\gamma}\oplus P_{\gamma\varepsilon}$. Then $p_1 i_{P'}$ is isomorphic modulo $\mathfrak{J}'$ by Lemma 4. On the other hand, $\{M_\alpha\}_{I_1}$ is locally semi-T-nilpotent from the assumption. Hence, $p_1 i_{P'}$ is R-isomorphic and $M=P'\oplus\mathrm{Ker}\, p_1=P'\oplus N_2$. Finally, we shall show N_1 has the exchange property in M. Similarly to the above, we obtain subsets J^*_γ of J_γ such that $N_1\oplus\sum_K\sum_{J^*_\gamma}\oplus P_{\gamma\varepsilon}$ is dense in M. We put $P^*_\gamma=\sum_{J^*_\gamma}\oplus P_{\gamma\varepsilon}$, $P''_\gamma=\sum_{J''_\gamma}\oplus P_{\gamma\varepsilon}$ and $P^*=\sum_K\oplus P^*_\gamma$, where $J''_\gamma=J_\gamma-J^*_\gamma$. Then $N_1\approx\sum_K\oplus P''_\gamma$ from Lemma 4. Since $P''_\gamma\oplus P^*_\gamma\oplus\sum_{\alpha\neq\gamma}\sum_{J_\alpha}\oplus P_{\alpha\varepsilon}$ is dense in M, there exist R-homomorphisms $p''_\gamma: M\to P''_\gamma$ and $p_0: M\to P^*_\gamma\oplus\sum_{\alpha\neq\gamma}\sum_{J_\alpha}\oplus P_{\alpha\delta}$ such that

$$1_M\equiv i''_\gamma p''_\gamma+i_0 p_0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

where $i_0: P^*_\gamma\oplus\sum_{\alpha\neq\gamma}\sum_{J_\alpha}\oplus P_{\alpha\varepsilon}\to M$. Furthermore, since $\{P_{\gamma\varepsilon}\}_{J''_\gamma}$ is locally semi-T-nilpotent and $p''_\gamma i''_\gamma\equiv 1_{P''_\gamma}$, $M=P''_\gamma\oplus\mathrm{Ker}\, p''_\gamma$. We put $Q^*_\gamma=Q_\gamma\cap\mathrm{Ker}\, p''_\gamma$, then $Q_\gamma=P''_\gamma\oplus Q^*_\gamma$ and

$$M=\sum_K\oplus P''_\gamma\oplus\sum_K\oplus Q^*_\gamma \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

Let $Q^{*\prime}_\gamma$ be a dense submodule of Q^*_γ via inclusion $i^{*\prime}_\gamma$. Then $Q^{*\prime}_\gamma\oplus P''_\gamma$ and $P^*_\gamma\oplus P''_\gamma$ are dense submodules of Q_γ. $i^{*\prime}_\gamma\equiv(i''_\gamma p''_\gamma+i_0 p_0)i^{*\prime}_\gamma\equiv i_0 p_0 i^{*\prime}_\gamma$ from (1), since

$Q_\gamma^{*\prime} \subseteqq Q_\gamma^* \subseteqq \operatorname{Ker} p_\gamma''$. Put $\delta_\gamma = p_0 i_\gamma^{*\prime} = \binom{a}{b} : Q_\gamma^* \to P_\gamma^* \oplus \sum_{\alpha\neq\gamma}\sum_{J_\alpha} \oplus P_{\alpha\varepsilon}$. Then we have a commutative diagram modulo $\mathfrak{J}'$:

$$\begin{array}{ccccc} Q_\gamma^{*\prime} & \xrightarrow{\delta_\gamma} & P_\gamma^*\oplus\sum_{\alpha\neq\gamma}\sum_{J_\alpha}\oplus P_{\gamma\varepsilon} & \xrightarrow{i'=(i'',\, I)} & P_\gamma\oplus\sum_{\alpha\neq\gamma}\sum_{J_\alpha}\oplus P_{\alpha\varepsilon} \\ & {\scriptstyle i^{*\prime}}\searrow & \Big\downarrow{\scriptstyle i_0=(i_\gamma^*,\, i''')} & \swarrow{\scriptstyle i=(i_\gamma,\, i''')} & \\ & & M & & \end{array}$$

where i, i' etc. mean injections. Furthermore, let e_1 and e_2 be projections of $M = Q_\gamma \oplus (\sum_{\alpha\neq\gamma} Q_\alpha)$ onto each components. Then $0 \equiv e_2 i_\gamma^{*\prime} \equiv e_2 i i' \delta_\gamma \equiv e_2 i_\gamma i'' a + e_2 i''' b \equiv e_2 i''' b \equiv i''' b$, since $P_\gamma^* + Q_\gamma^{*\prime} \subseteqq Q_\gamma$ and $\sum_{\alpha\neq\gamma}\sum_{J_\alpha} \oplus P_{\alpha\varepsilon} \subseteqq \sum_{\alpha\neq\gamma} \oplus Q_\alpha$. Hence, $b \equiv 0$. Thus, we obtain a commutative diagram modulo $\mathfrak{J}'$:

$$\begin{array}{ccc} Q_\gamma^{*\prime} & \xrightarrow{\delta_\gamma} & P_\gamma^* \\ {\scriptstyle i_\gamma^{*\prime}}\searrow & & \swarrow{\scriptstyle i_\gamma^*} \\ & M & \end{array} \qquad \cdots\cdots\cdots\cdots\cdots\cdots \quad (3)$$

Now, since $P_\gamma'' \oplus P_\gamma^*$ and $P_\gamma'' \oplus Q_\gamma^{*\prime}$ are dense in Q_γ, there exists, from Lemmas 3 and 4, an R-homomorphism $p_\gamma^* : M \to P_\gamma^*$ such that $p_\gamma^* i_\gamma^* \equiv 1_{P_\gamma^*}$ and $p_\gamma^* i_\gamma^{*\prime}$ is isomorphic modulo $\mathfrak{J}'$. Hence, $\delta_\gamma \equiv p_\gamma^* i_\gamma^{*\prime}$ is isomorphic modulo $\mathfrak{J}'$. Put $\delta = \sum_K \delta_\gamma : \sum_K \oplus Q_\gamma^{*\prime} \to \sum_K \oplus P_\gamma^*$. Since $\sum_K \oplus Q_\gamma^{*\prime}$ is a dense submodule of $\sum \oplus Q_\gamma^*$, we have an R-homomorphism $p^{*\prime} : M \to \sum \oplus Q_\gamma^{*\prime}$ such that $1 - f \equiv i^{*\prime} p^{*\prime}$ from Lemma 2, where $i^{*\prime} : \sum \oplus Q_\gamma^{*\prime} \to M$ and $f : M \to \sum \oplus P_\gamma''$ is the projection in (2). On the other hand,

$$\sum \oplus P_\gamma^* \oplus N_1 \qquad \cdots\cdots\cdots\cdots\cdots\cdots \quad (4)$$

is a dense submodule of M. Hence, there exist R-homomorphisms $p^* : M \to \sum \oplus P_\gamma^*$ and $p_{N_1} : M \to N_1$ such that

$$1_M \equiv i^* p^* + i_{N_1} p_{N_1} \qquad \cdots\cdots\cdots\cdots\cdots\cdots \quad (5).$$

Since $\sum_K \oplus Q_\gamma^{*\prime} \subseteqq \sum_K \oplus Q_\gamma^* = \operatorname{Ker} f$, $f i^{*\prime} = 0$. Using this result, a fact that δ is isomorphic modulo $\mathfrak{J}'$ and (3), (4) and (5), we can prove similarly to Lemma 4 that $f i_{N_1}$ is isomorphic modulo $\mathfrak{J}'$. Hence, $f i_{N_1}$ is R-isomorphic and $M = N_1 \oplus \ker f = N_1 \oplus \sum_K \oplus Q_\gamma^*$.

REMARK 4. If we translate the last part of the proof of Theorem 2 into the category $\mathfrak{S}$ by functors X, Y in the proof of Theorem 1, then we obtain a much simpler proof than the above as follows:

Let M, Q_γ and $P_{\gamma\varepsilon}$ be as above and e_γ''s projections of M onto Q_γ. We shall use the same notations as above. Then $X(M) = \sum \oplus \operatorname{Im} X(e_\gamma)$, $\operatorname{Im} X(e_\gamma) = \sum_{J_\gamma} \oplus X(P_{\gamma\varepsilon})$ from Lemma 2. Since $\mathfrak{S}$ is completely reducible, $X(M) = X(N_1)$

$\oplus\sum_K\sum_{J^*_\gamma}\oplus X(P_{\gamma\varepsilon})$ and $\sum_K\sum_{J''_\gamma}\oplus X(P_{\gamma\varepsilon})\approx X(N_1)=\sum_{I_1}\oplus X(M_\alpha)$. Hence, $\{P_{\gamma\varepsilon}\}_{K,J'_\gamma}$ is locally semi-T-nilpotent. Let i_γ be the injection of P''_γ into M. Since X is equivalent, there exists an R-homomorphism $p_\gamma: M\to P''_\gamma$ such that $X(p_\gamma i_\gamma)=1_{X(P''_\gamma)}$ and $X(p_\gamma)|(\sum_{\delta\neq\gamma}\oplus X(P_\delta)\oplus X(P^*_\gamma))=0$. Hence, $M=P''_\gamma\oplus\mathrm{Ker}\,p_\gamma$, since $\{P_{\gamma\varepsilon}\}_{J'_\gamma}$ is locally semi-T-nilpotent. Let Q^*_γ and $Q^{*\prime}_\gamma$ be as before. Then $X(Q^{*\prime}_\gamma)\subseteqq \mathrm{Ker}\,X(p_\gamma)=\sum_{\delta\neq\gamma}\oplus X(P_\delta)\oplus X(P^*_\gamma)\subseteqq\sum_{\delta\neq\gamma}\oplus X(P_\delta)\oplus X(P_\gamma)$ and $X(P_\gamma)=X(P''_\gamma)\oplus X(Q^{*\prime}_\gamma)$. Hence, $X(Q^{*\prime}_\gamma)\subseteqq X(P^*_\gamma)$. On the other hand, $X(P_\gamma)=X(P''_\gamma)\oplus X(Q^{*\prime}_\gamma)=X(P''_\gamma)\oplus X(P^*_\gamma)$. Therefore, $X(P^*_\gamma)=X(Q^{*\prime}_\gamma)$. Now, let f be the projection of M onto $\sum_K\oplus P''_\gamma$ in (2). $X(M)=X(N_1)\oplus\sum_K\oplus X(P^*_\gamma)=X(N_1)\oplus\sum\oplus X(Q^{*\prime}_\gamma)$ and $X(M)=\sum\oplus X(P''_\gamma)\oplus\sum\oplus X(Q^{*\prime}_\gamma)$. Since $Q^{*\prime}_\gamma\subseteqq Q^*\subseteqq\mathrm{Ker}\,f$, $\mathrm{Ker}\,X(f)=\sum\oplus X(Q^{*\prime}_\gamma)$ and $X(f)|X(N_1)$ is Δ-isomorphic. Hence, $f|N_1: N_1\to\sum\oplus P''_\gamma$ is R-isomorphic and $M=N_1\oplus\sum_K\oplus Q^*_\gamma$.

REMARK 5. If we use directly the factor category $\mathfrak{A}/\mathfrak{J}'$, we have a further simpler proof than the above (see [6] and [7], the foot note).

References

[1] G. Azumaya, Correction and supplements to my paper concerning Krull-Remak-Schmidt's theorem, Nagoya Math. J., **1** (1950), 117–124.

[2] C. Ehresmann, Catégories et Structures, Dunod, Paris, 1965.

[3] S. Elliger, Interdirekte Summen von Moduln, J. Algebra, **18** (1971), 271–303.

[4] M. Harada and Y. Sai, On categories of indecomposable modules I, Osaka J. Math., **7** (1970), 323–344.

[5] M. Harada, On categories of indecomposable modules II, Osaka J. Math., **8** (1971), 309–321.

[6] M. Harada, Supplementary Remarks on categories of indecomposable modules, Osaka J. Math., **9** (1972), 49–55.

[7] M. Harada, On the exchange property in direct sums of indecomposable modules, to appear in Osaka J. Math.

[8] M. Harada, Note on categories of indecomposable modules, Publ. Dept. Math. Lyon, **9** (1972), 11–25.

[9] M. Harada, Applications of factor categories of completely indecomposable modules, Publ. Dept. Math. Lyon, (1974), 19–104.

[10] T. Ishii, On locally direct summands of modules, Osaka J. Math., **12** (1975), 473–482.

[11] H. Kanbara, On the Krull-Remak-Schmidt-Azumaya's theorem, Osaka J. Math., **9** (1972), 409–413.

[12] B. Mitchell, Theory of Categories, Academic Press, New York and London, 1965.

[13] Y. Sai, On regular categories, Osaka J. Math., **7** (1970), 301–306.

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