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Tensor products of $C(X)$-spaces and their conjugate spaces

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For any locally compact (Hausdorff) space X, we denote by $C(X)$ and $C_0(X)$ the Banach algebra of all bounded continuous functions on X and the ideal of those $f \in C(X)$ which vanish at infinity, respectively. Thus the conjugate space $C_0(X)'$ of $C_0(X)$ can be identified with the space $M(X)$ of all bounded regular measures on X. Now let $X_1, \cdots, X_N$ be finitely many locally compact spaces, and X the product space thereof. Given a Banach space B, we consider

$$V_0(X) \hat{\otimes} B = C_0(X_1) \hat{\otimes} \cdots \hat{\otimes} C_0(X_N) \hat{\otimes} B\,,$$

the (complete) projective tensor product of $C_0(X_1), \cdots, C_0(X_N)$, and B (cf. [**10**]). Notice that the Banach space $V_0(X) \hat{\otimes} B$ can be regarded as a linear subspace of $C(X:B)$, the space of all B-valued bounded continuous functions on X.

The main purpose of this paper is to prove that, under a certain condition on B', the space $(V_0(X) \hat{\otimes} B)'$ has a natural decomposition which is similar to the well-known decomposition $M(X) = M_c(X) + M_d(X)$. As a special case of this result it is shown that $M(X)$ is norm-dense in $V_0(X)'$ if and only if all except at most one X_j are residual (i. e., contain no perfect sets). We also give an application of the latter result to the study of Fourier restriction algebras.

Let $V_0(X) \hat{\otimes} B$ be as above. Then $V_0(X) \hat{\otimes} B$ has a natural Banach $V(X)$-module structure, where $V(X) = C(X_1) \hat{\otimes} \cdots \hat{\otimes} C(X_N) \subset C(X)$:

$$(\phi F)(x) = \phi(x) F(x) \qquad (\phi \in V(X),\ F \in V_0(X) \hat{\otimes} B,\ x \in X)\,.$$

We define the product $\phi P \in (V_0(X) \hat{\otimes} B)'$ of a $\phi \in V(X)$ and a $P \in (V_0(X) \hat{\otimes} B)'$ by setting

$$\langle F, \phi P \rangle = \langle \phi F, P \rangle \qquad \forall F \in V_0(X) \hat{\otimes} B\,.$$

Notice that the imbedding $V_0(X) \subset V(X)$ is isometric. We also define the X-support of P, $S_X(P)$, to be the smallest closed subset S of X such that $\langle F, P \rangle = 0$ whenever $F \in V_0(X) \hat{\otimes} B$ and $F = 0$ on some neighborhood of S (cf. [**5**; p. 31]).

DEFINITIONS. Let $P \in (V_0(X) \hat{\otimes} B)'$ be given.

(a) We call P *point-mass-like* if $S_X(P)$ is either a singleton or empty.

(b) We call P *discrete* if it belongs to the closed linear span of all point-mass-like elements in $(V_0(X)\hat{\otimes}B)'$.

(c) We say that P is *continuous* at a point $x\in X$ if to each $\varepsilon>0$ there corresponds a neighborhood W of x such that

$$\phi\in V(X) \text{ and } \operatorname{supp}\phi\subset W \;\Rightarrow\; \|\phi P\|\leqq\varepsilon\|\phi\|_{V(X)}.$$

The element P is called *continuous* (on X) if it is continuous at every point of X.

Finally we introduce the following property of a Banach space A:

$$(\mathcal{P})\qquad\left\{\begin{array}{l}\text{For any sequence } (P_n)_1^\infty \text{ of elements of } A \text{ with norms } \geqq 1\\ \text{and any } 0<R<\infty \text{ there exist finitely many complex}\\ \text{numbers } \alpha_1, \alpha_2, \cdots, \alpha_n \text{ of absolute values } \leqq 1 \text{ such that}\\ \qquad \|\alpha_1P_1+\alpha_2P_2+\cdots+\alpha_nP_n\|_A>R.\end{array}\right.$$

Our main result is stated as follows.

THEOREM 1. *Let B be a Banach space whose conjugate space B' has Property $(\mathcal{P})$, and let $P\in(V_0(X)\hat{\otimes}B)'$ be given.*

(i) *P can be uniquely written as $P=P_c+P_d$, where $P_c\in(V_0(X)\hat{\otimes}B)'$ is continuous and $P_d\in(V_0(X)\hat{\otimes}B)'$ is discrete. Moreover, $\|P_d\|\leqq\|P\|$.*

(ii) *There exists a unique family $\{P_x: x\in X\}\subset(V_0(X)\hat{\otimes}B)'$, with $S_X(P_x)\subset\{x\}$ $\forall x\in X$, such that*

$$\lim_{\mathcal{F}}\|P_d-\sum_{x\in E}P_x\|=0.$$

Here $\mathcal{F}$ denotes the directed family of all finite product subsets E of X.

To prove this, we need a lemma.

LEMMA 1. *Let B be as in Theorem 1. Let also $P\in(V_0(X)\hat{\otimes}B)'$ and $x\in X$ be given. Then there exists a unique $P_x\in(V_0(X)\hat{\otimes}B)'$ with the following property: to each $0<\varepsilon<1$ there corresponds a neighborhood W of x such that $\|\phi P-P_x\|\leqq\varepsilon\|\phi\|_{V(X)}$ whenever $\phi\in V(X)$,* supp $\phi\subset W$, *and* $\phi(x)=1$.

PROOF. Write $x=(x_1, x_2, \cdots x_N)$,

$$E_j=E_j(x)=X_1\times\cdots\times X_{j-1}\times\{x_j\}\times X_{j+1}\times\cdots\times X_N,$$

and $E=E(x)=E_1\cup\cdots\cup E_N$.

We first prove that given $\varepsilon>0$ there exists a neighborhood U of x such that

$$\phi\in V(X) \text{ and } \operatorname{supp}\phi\subset U\backslash E \;\Rightarrow\; \|\phi P\|\leqq\varepsilon\|\phi\|_{V(X)}. \tag{1}$$

Suppose this is false. Then there exists $\varepsilon>0$ such that (1) does not hold for any neighborhood U of x. We shall construct a sequence $(\phi^{(n)})_1^\infty$ of elements of $V_0(X)$ as follows. Put $\phi^{(0)}=0$, and suppose that $\phi^{(0)}, \cdots, \phi^{(n-1)}$ have been defined for some natural number n so that $\operatorname{supp}\phi^{(k)}$ is compact and is disjoint from E $(0\leqq k<n)$. Choose any compact (product) neighborhood $U=U^{(n)}=U_1\times\cdots\times U_n$ of x such that

$$(2)\qquad U_j\cap\pi_j[\operatorname{supp}\phi^{(k)}]=\emptyset \qquad (1\leqq j\leqq N,\ 0\leqq k<n).$$

Here each π_j is the natural projection from X onto X_j. Since (1) is assumed not to hold, we can find a $\psi=\psi^{(n)}\in V(X)$ such that

$$(3)\qquad \operatorname{supp}\psi\subset(\operatorname{int}U)\backslash E,\ \|\psi\|_{V(X)}<1, \text{ and } \|\psi P\|>\varepsilon.$$

By (2) and the definition of $V(X)$, we may assume that ψ has the form $\psi=\psi_1\otimes\cdots\otimes\psi_N$ with $\psi_j\in C_0(X_j)$, $1\leqq j\leqq N$. Therefore, by (3) and the definition of $V_0(X)\hat{\otimes}B$, there exists an element

$$F^{(n)}=f_1^{(n)}\otimes\cdots\otimes f_N^{(n)}\otimes b^{(n)}\in V_0(X)\hat{\otimes}B$$

such that

$$(4)\qquad \operatorname{supp}F^{(n)}\subset U\backslash E,\ |\langle F^{(n)},P\rangle|>\varepsilon,$$

$$(5)\qquad \|f_j^{(n)}\|_\infty=1=\|b^{(n)}\|_B \qquad (1\leqq j\leqq N).$$

Set $\phi^{(n)}=f_1^{(n)}\otimes\cdots\otimes f_N^{(n)}$, which completes the induction.

We now prove that

$$(6)\qquad \|\sum_{k=1}^n\alpha_k\phi^{(k)}\|_{V_0(X)}\leqq 1$$

for all $n\in\boldsymbol{N}$, and all complex numbers $\alpha_1,\alpha_2,\cdots,\alpha_n$ of absolute values $\leqq 1$. First choose any complex numbers β_k with $\beta_k^N=\alpha_k$, $1\leqq k\leqq n$, and notice that $f_j^{(1)},f_j^{(2)},\cdots,f_j^{(n)}$ have disjoint supports by (2) and (4), $1\leqq j\leqq N$. Since $|\beta_k|\leqq 1$, it follows from (5) that

$$(7)\qquad \|\sum_{k=1}^n\omega_k\beta_k f_j^{(k)}\|_\infty\leqq 1 \qquad \forall\omega_k\in\boldsymbol{C},\ |\omega_k|\leqq 1,\ 1\leqq k\leqq n$$

for all j. On the other hand, we have

$$(8)\qquad \begin{cases}\displaystyle\sum_{k=1}^n\alpha_k\phi^{(k)}\\ \displaystyle =N^{-n}\sum_\omega\Big(\sum_{k=1}^n\omega_k\beta_k f_1^{(k)}\Big)\otimes\cdots\otimes\Big(\sum_{k=1}^n\omega_k\beta_k f_N^{(k)}\Big),\end{cases}$$

where the last sum is taken over all n-tuples $\omega=(\omega_1,\omega_2,\cdots,\omega_n)$ of complex numbers with $\omega_k^N=1$ $(1\leqq k\leqq n)$. We conclude from (7) and (8) that (6) holds.

Now define a $\Phi_k\in B'$ by setting

$$(9)\qquad \langle b,\Phi_k\rangle=\langle\phi^{(k)}\otimes b,P\rangle \qquad \forall b\in B$$

for each $k=1,2,\cdots$. Since $F^{(k)}=\phi^{(k)}\otimes b^{(k)}$, we have $\|\Phi_k\|_{B'}>\varepsilon$ by (4), (5) and

(9). Since B' has Property ($\mathscr{P}$), it follows that there are finitely many complex numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ of absolute values $\leqq 1$ and an element $b \in B$, with norm $\leqq 1$, such that

(10) $$|\langle b, \sum_{k=1}^{n} \alpha_k \Phi_k \rangle| > \|P\|.$$

We infer from (9) and (10) that

(11) $$|\langle (\sum_{k=1}^{n} \alpha_k \phi^{(k)}) \otimes b, P \rangle| > \|P\|,$$

which contradicts (6) since b has norm $\leqq 1$. We have thus established (1).

Next we prove that given $\varepsilon > 0$, there exists a neighborhood W of x such that

(12) $$\operatorname{supp} \phi \subset W_\varepsilon \text{ and } \phi(x) = 0 \Rightarrow \|\phi P\| \leqq \varepsilon \|\phi\|_{V(X)}$$

whenever $\phi \in V(X)$. Notice that this is an easy consequence of (1) if $N=1$. So, assume that $N \geqq 2$ and the desired conclusion is true with N replaced by $N-1$. Given $\varepsilon > 0$, choose a compact neighborhood U_ε of x as in (1). Also fix any $\psi_\varepsilon \in V(X)$ such that $\operatorname{supp} \psi_\varepsilon \subset U_\varepsilon$ and $\|\psi_\varepsilon\|_{V(X)} = 1 = \psi_\varepsilon$ in some neighborhood $V_\varepsilon \subset U_\varepsilon$ of x. Let $\mathscr{K}$ be the directed family of all compact subsets of $X \backslash E = (X_1 \backslash \{x_1\}) \times \cdots \times (X_N \backslash \{x_N\})$. With each $K \in \mathscr{K}$ we shall associate an element $\phi_K \in V(X)$ such that $\|\phi_K\|_{V(X)} = 1 = \phi_K$ on K and $(\operatorname{supp} \phi_K) \cap E = \emptyset$. Then

$$\|\phi_K \psi_\varepsilon P\| \leqq \varepsilon \|\phi_K \psi_\varepsilon\|_{V(X)} \leqq \varepsilon$$

by (1). Therefore, for each fixed $\varepsilon > 0$, the net $\{\phi_K \psi_\varepsilon P : K \in \mathscr{K}\}$ has a weak-* cluster point $Q_\varepsilon \in (V_0(X) \hat{\hat{\otimes}} B)'$ with $\|Q_\varepsilon\| \leqq \varepsilon$. It is easy to see that $R_\varepsilon = \psi_\varepsilon P - Q_\varepsilon$ is supported by E. Moreover, we claim that R_ε has a decomposition of the form $R_\varepsilon = R_1 + \cdots + R_N$, where the X-support of R_j is contained in E_j $(1 \leqq j \leqq N)$. In fact, first consider the elements of $(V_0(X) \hat{\hat{\otimes}} B)'$ of the form $(f_1 \otimes 1 \otimes \cdots \otimes 1) R_\varepsilon$ with $f_1 \in C_0(X_1)$ and $\|f_1\|_\infty = 1 = f_1(x_1)$. Let R_1 be any weak-* cluster point of such elements as $\operatorname{supp} f_1$ approaches x_1. Then obviously $R_\varepsilon - R_1$ is supported by $E_2 \cup \cdots \cup E_N$. It suffices to repeat this process with R_ε and x_1 replaced by $R_\varepsilon - R_1$ and x_2, respectively, and so on. Notice that each R_j can be regarded as an element of $(V_0(Y_j) \hat{\hat{\otimes}} B)'$, where $Y_j = X_1 \times \cdots \times X_{j-1} \times X_{j+1} \times \cdots \times X_N$. It follows from the inductive hypothesis that the required condition holds for every R_j, and hence for R_ε. Finally we choose a neighborhood $W_\varepsilon \subset V_\varepsilon$ of x so that (12) holds with P replaced by R_ε. If $\phi \in V(X)$ and $\operatorname{supp} \phi \subset W_\varepsilon$, then $\phi \psi_\varepsilon = \phi$ and so

$$\begin{aligned} \|\phi P\| &= \|\phi \psi_\varepsilon P\| = \|\phi R_\varepsilon + \phi Q_\varepsilon\| \\ &\leqq \varepsilon \|\phi\|_{V(X)} + \|\phi\|_{V(X)} \|Q_\varepsilon\| \leqq 2\varepsilon \|\phi\|_{V(X)}. \end{aligned}$$

This establishes (12) with ε replaced by 2ε.

Now let $\varepsilon>0$ be given, and let W_ε be any neighborhood of x as in (12). If $\phi=\phi'$ and $\phi''\in V(X)$ satisfy supp $\phi\subset W_\varepsilon$ and $\phi(x)=1$, then

$$\|\phi'P-\phi''P\|=\|(\phi'-\phi'')P\|\leqq\varepsilon(\|\phi'\|_{V(X)}+\|\phi''\|_{V(X)}) \tag{13}$$

by (12). Since $\varepsilon>0$ is arbitrary and W_ε can be taken arbitrarily small, it follows from (13) that there exists a point-mass-like element $P_x\in(V_0(X)\hat{\otimes}B)'$ such that

$$\|\phi P-P_x\|\leqq\varepsilon(\|\phi\|_{V(X)}+1)\leqq 2\varepsilon\|\phi\|_{V(X)}$$

whenever $\phi\in V(X)$, $\phi(x)=1$, and supp $\phi\subset W_\varepsilon$. This completes the proof, since the uniqueness of P_x is obvious.

PROOF OF THEOREM 1. Let B and $\mathcal{F}$ be as in Theorem 1, and let $P\in(V_0(X)\hat{\otimes}B)'$ be given. With each $x\in X$ we associate a point-mass-like element $P_x\in(V_0(X)\hat{\otimes}B)'$ as in Lemma 1.

We first prove that

$$\|\sum_{x\in E}P_x\|\leqq\|P\| \qquad \forall E\in\mathcal{F}\,. \tag{1}$$

Fix any $E\in\mathcal{F}$. Given a neighborhood U of E, we can find a $\phi\in V_0(X)$ such that supp $\phi\subset U$, $\|\phi\|_{V(X)}=1$, and $\phi=1$ on E, since E is a compact product set. If U is sufficiently small and ϕ is as above, then we have by Lemma 1

$$\|\phi P-\sum_{x\in E}P_x\|<\varepsilon\,,$$

where ε is an arbitrary, but preassigned, real positive number. Since $\|\phi P\|\leqq\|P\|$, this establishes (1).

To complete the proof, it clearly suffices to confirm that the net $\sum_E P_x$, $E\in\mathcal{F}$, converges to some element of $(V_0(X)\hat{\otimes}B)'$. (Then the other assertions of the theorem can be proved very easily.) Notice that each P_x is written as $P_x=\delta_x\otimes\Phi_x$ for a unique $\Phi_x\in B'$, where δ_x is the unit point-mass at x.

Let $(X_j)_d$ be the set X_j with the discrete topology, and $Y_j=(X_j)_d\cup\{p_j\}$ its one-point compactification $(1\leqq j\leqq N)$. We consider

$$V(Y)\hat{\otimes}B=C(Y_1)\hat{\otimes}\cdots\hat{\otimes}C(Y_N)\hat{\otimes}B\,.$$

By the above remark, we can identify each P_x with $\delta_x\otimes\Phi_x\in(V(Y)\hat{\otimes}B)'$. Then the linear span of all point-mass-like elements in $(V_0(X)\hat{\otimes}B)'$ can be isometrically imbedded in $(V(Y)\hat{\otimes}B)'$. Therefore (1) assures that the net under consideration has a weak-* cluster point $Q\in(V(Y)\hat{\otimes}B)'$.

Suppose for a moment that Q is discrete and let $\varepsilon>0$ be given. Then there exists a finitely supported element $R\in(V(Y)\hat{\otimes}B)'$ such that $\|Q-R\|<\varepsilon$. We can define the restriction R' of R to $X\subset Y$ in the obvious way. If $E\in\mathcal{F}$

contains the Y-support of R', then we have

$$\|\sum_{x\in E}Q_x-R'\|=\|\sum_{x\in E}(Q-R)_x\|\leqq\|Q-R\|<\varepsilon\,. \tag{2}$$

This follows from (1) with X and P replaced by Y and $Q-R$, respectively. On the other hand, it is obvious that $Q_x=P_x$ for all $x\in X$, since every point of X is isolated in Y. Therefore (2) implies that the net $\sum_E P_x$, $E\in\mathscr{F}$, forms a Cauchy net in $(V_0(Y)\hat{\otimes}B)'$ and hence in $(V_0(X)\hat{\otimes}B)'$. This completes the proof, provided that Q is discrete.

Consequently, in order to reach the desired conclusion, it suffices to prove that *every* $Q\in(V(Y)\hat{\otimes}B)'$ is discrete. We do this by induction on N. Fix Q and $\varepsilon>0$. Since Y is totally disconnected, it follows from Lemma 1 that there exists a clopen neighborhood $U=U_1\times\cdots\times U_N$ of $p=(p_1,\cdots,p_N)\in Y$ such that

$$\|\xi_U Q-Q_p\|<\varepsilon\,, \tag{3}$$

where ξ_U denotes the characteristic function of U. Write

$$Y^j=Y_1\times\cdots\times Y_{j-1}\times(Y_j\backslash U_j)\times Y_{j+1}\times\cdots\times Y_N$$

for $1\leqq j\leqq N$. These sets are clopen in Y and cover $Y\backslash U$. Therefore we can write $(1-\xi_U)Q=R_1+\cdots+R_N$, where $R_j\in(V(Y)\hat{\otimes}B)'$ has Y-support$\subset Y^j$, $1\leqq j\leqq N$. Notice that each $Y_j\backslash U_j$ is a finite set, since p_j is the only one (possible) accumulation point in Y_j. If $N=1$, this implies that $(1-\xi_U)Q$ is finitely supported. If $N\geqq 2$ and if we assume the result for $N-1$, it follows that every R_j is a finite sum of discrete elements and is therefore a discrete element. Finally, we have

$$\|Q-(Q_p+R_1+\cdots+R_N)\|=\|\xi_U Q-Q_p\|<\varepsilon \tag{4}$$

by (3). Since $\varepsilon>0$ is arbitrary, this yields the desired conclusion.

THEOREM 2. *Suppose that at least one of the spaces X_j is infinite. Then the linear span of all continuous and discrete elements of $(V_0(X)\hat{\otimes}B)'$ is dense in $(V_0(X)\hat{\otimes}B)'$ if and only if B' satisfies $(\mathscr{P})$.*

PROOF. One direction of the above assertion is a trivial consequence of Theorem 1. To prove the non-trivial part, we may assume $N=1$.

Suppose that B' does not satisfy $(\mathscr{P})$, but that the linear span of all discrete and continuous elements is dense in $(C_0(X)\hat{\otimes}B)'$. Then there exist a finite constant C and a sequence $(\Phi_k)_1^\infty$ of elements of B' such that

$$\|\Phi_k\|_{B'}\geqq 1\quad\forall k\in\boldsymbol{N},\quad\text{and}\quad\|\sum_{k=1}^n\alpha_k\Phi_k\|_{B'}\leqq C\sup_k|\alpha_k| \tag{1}$$

for all finite sequences $\alpha_1,\cdots,\alpha_n$ of complex numbers. The space X contains a countable set $E=\{x_k\}_1^\infty$ of distinct elements such that every x_k is isolated in $\bar{E}$.

Define

$$P_n=\sum_{k=1}^{n}\delta_{x_k}\otimes\Phi_k\in(C_0(X)\hat{\otimes}B)' \tag{2}$$

for all $n\in\boldsymbol{N}$. It is an easy consequence of (1) that $(P_n)_1^\infty$ is a bounded sequence in $(C_0(X)\hat{\otimes}B)'$. Let $P\in(C_0(X)\hat{\otimes}B)'$ be any weak-* cluster point of $(P_n)_1^\infty$. Obviously P is supported by $\bar{E}$, and

$$\text{the } X\text{-support of } P-P_n\subset\bar{E}\setminus\{x_k\}_1^n \tag{3}$$

for all n. By one of the assumptions, there exist a continuous element Q and a discrete element $R\in(V_0(X)\hat{\otimes}B)'$ such that $\|P-Q-R\|<1/3$. We may assume that the X-support of Q is contained in a finite set $F\subset X$. Choose any $m\in N$ so that $F\cap E\subset\{x_k\}_1^m$, and let R' be the "restriction" of R to $F\cap E$. Since Q is a continuous element, it follows from (3) that

$$\|P_n-R'\|\leqq 1/3 \qquad \forall n\geqq m\,. \tag{4}$$

The proof of this fact is similar to that of (1) in the proof of Theorem 1. But (4) implies

$$\begin{aligned}\|\Phi_n\|_{B'}=\|\delta_{x_n}\otimes\Phi_n\|&=\|P_n-P_{n-1}\|\\ &\leqq\|P_n-R'\|+\|P_{n-1}-R'\|\leqq 2/3\end{aligned}$$

for all $n>m+1$. This contradicts (1), and the proof is complete.

The following result must be well-known. Since we do not know any adequate reference about it, we give a complete proof.

LEMMA 2. *Let $(S,\mathscr{B},\lambda)$ be a measure space, and $M(S)=M(S,\mathscr{B})$ the Banach space of all countably additive complex measures on $\mathscr{B}$. Then $M(S)$ and all the spaces $L^p=L^p(S,\mathscr{B},\lambda)$, $1\leqq p<\infty$, have Property $(\mathscr{P})$.*

PROOF. Let $1\leqq p<\infty$, and $f_1,\cdots,f_n\in L^p$. Let also $\Omega=\Omega_n$ be the set of all n-tuples $\varepsilon=(\varepsilon_1,\cdots,\varepsilon_n)$ of ±1. For any function ϕ on Ω, define

$$\mathscr{E}(\phi)=2^{-n}\sum_{\varepsilon\in\Omega}\phi(\varepsilon)\,.$$

Then we have

$$(\mathscr{E}|\sum_{k=1}^{n}\varepsilon_kf_k|^p)^{1/p}\leqq C_p\mathscr{E}|\sum_{k=1}^{n}\varepsilon_kf_k| \tag{1}$$

for some absolute constant C_p depending only on p (see Theorem (8.4) of Chap. V of [**13**: p, 213]); we need (1) only for $p=2$.

First suppose $1\leqq p\leqq 2$. Then we have

$$\sum_{k=1}^{n}|f_k|^p\leqq n^{(2-p)/2}(\sum_{k=1}^{n}|f_k|^2)^{p/2} \tag{2}$$

by Hölder's inequality. Hence

$$n^{-(p-1)/p}\sum_{k-1}^{n}\|f_k\|_p \leqq \Big(\sum_{k=1}^{n}\int|f_k|^p d\lambda\Big)^{1/p} \qquad \text{by Hölder}$$

$$\leqq n^{(2-p)/2p}\Big\{\int\Big(\sum_{k=1}^{n}|f_k|^2\Big)^{p/2}d\lambda\Big\}^{1/p} \qquad \text{by (2)}$$

$$= n^{(2-p)/2p}\Big\{\int\Big(\mathcal{E}\Big|\sum_{k=1}^{n}\varepsilon_k f_k\Big|^2\Big)^{p/2}d\lambda\Big\}^{1/p}$$

$$\leqq C_2 n^{(2-p)/2p}\Big\{\int\Big(\mathcal{E}\Big|\sum_{k=1}^{n}\varepsilon_k f_k\Big|\Big)^p d\lambda\Big\}^{1/p} \qquad \text{by (1)}$$

$$\leqq C_2 n^{(2-p)/2p}\mathcal{E}\Big\|\sum_{k=1}^{n}\varepsilon_k f_k\Big\|_p \qquad \text{by Minkowski}.$$

Therefore, we have

(3) $$n^{-1/2}\sum_{k=1}^{n}\|f_k\|_p \leqq C_2\Big\|\sum_{k=1}^{n}\varepsilon_k f_k\Big\|_p$$

for at least one $\varepsilon\in\Omega$, provided that $1\leqq p\leqq 2$.

Next suppose $2\leqq p<\infty$. Using the inequality $\|\cdot\|_{l^p}\leqq\|\cdot\|_{l^2}$, we then have

$$n^{-(p-1)/p}\sum_{k=1}^{n}\|f_k\|_p \leqq \Big(\int\sum_{k=1}^{n}|f_k|^p d\lambda\Big)^{1/p}$$

$$\leqq \Big\{\int\Big(\sum_{k=1}^{n}|f_k|^2\Big)^{p/2}d\lambda\Big\}^{1/p} = \Big\{\int\Big(\mathcal{E}\Big|\sum_{k=1}^{n}\varepsilon_k f_k\Big|^2\Big)^{p/2}d\lambda\Big\}^{1/p}$$

$$\leqq \Big\{\int\mathcal{E}\Big|\sum_{k=1}^{n}\varepsilon_k f_k\Big|^p d\lambda\Big\}^{1/p} \qquad \text{by Hölder}$$

$$= \Big\{\mathcal{E}\int\Big|\sum_{k=1}^{n}\varepsilon_k f_k\Big|^p d\lambda\Big\}^{1/p}.$$

Hence $2\leqq p<\infty$ imply

(4) $$n^{-(p-1)/p}\sum_{k=1}^{n}\|f_k\|_p \leqq \Big\|\sum_{k=1}^{n}\varepsilon_k f_k\Big\|_p$$

for at least one $\varepsilon\in\Omega$.

By (3) and (4), all the spaces L^p, $1\leqq p<\infty$, have Property ($\mathcal{P}$). That $M(S)$ has Property ($\mathcal{P}$) follows from the result for $p=1$ combined with the Radon-Nikodym Theorem. This completes the proof.

THEOREM 3. *Let $X=X_1\times\cdots\times X_N$ be as before ($N\geqq 1$). Then each of the following conditions implies the others:*

(i) *All except at most one X_j are residual.*

(ii) *$M(X)$ is dense in $V_0(X)'$.*

(iii) *$V_0(X)'$ has Property ($\mathcal{P}$).*

PROOF. If $N=1$, there is nothing to prove, since then (iii) is a special case of Lemma 2. So suppose $N\geqq 2$.

We first confirm the implication (i) $\Rightarrow$ (ii). Without loss of generality, assume that $X_1, X_2, \cdots, X_{N-1}$ are residual. Put $Y=X_1\times\cdots\times X_{N-1}$ and $B=C_0(X_N)$, so that $V_0(X)=V_0(Y)\hat{\otimes}B$ isometrically. Then the only continuous element of $(V_0(Y)\hat{\otimes}B)'$ is the zero element, since Y is residual and the Y-support of any continuous element has no isolated point. On the other hand, $B'=M(X_N)$ has Property ($\mathcal{P}$) by Lemma 2. It follows from Theorem 1 that the set of all discrete elements is dense in $(V_0(Y)\hat{\otimes}B)'$. This establishes (ii), since it is trivial that every point-mass-like element of $(V_0(Y)\hat{\otimes}B)'=V_0(X)'$ is given by a measure in $M(X)$.

Suppose now that at least two of the spaces X_j, say, X_1 and X_2, contain perfect sets. We want to prove that then neither (ii) nor (iii) holds. Take a compact perfect set $K_j\subset X_j$ for $j=1, 2$, and put $K=K_1\times K_2$. Then we can imbed $V(K)'$ into $V_0(X)'$ isometrically. If $N=2$, this is trivial; if $N>2$, choose any point $x\in X_3\times\cdots\times X_N$ and identify K with $K\times\{x\}$ in the obvious way. Notice that if $M(X)$ is given the norm of $V_0(X)'$, then $\mu\to\mu|_K$ (or $\mu\to\mu|_{K\times\{x\}}$) is a norm-decreasing mapping from $M(X)$ into $V(K)'$. Therefore, if $M(X)$ were dense in $V_0(X)'$, then $M(K)$ would be dense in $V(K)'$. Now let $\boldsymbol{T}$ be the circle group, and let $\phi_j: K_j\to\boldsymbol{T}$ be any continuous surjection ($j=1, 2$). Then the product mapping $\phi=\phi_1\times\phi_2: K\to\boldsymbol{T}^2$ induces an isometric homomorphism $f\to f\circ\phi: V(\boldsymbol{T}^2)\to V(K)$ (see [**5**; Theorem 4.1]). Therefore we shall regard $V(\boldsymbol{T}^2)$ as a closed subalgebra of $V(K)$. Let

$$A(\boldsymbol{T})\xrightarrow{M}V(\boldsymbol{T}^2)\xrightarrow{P}A(\boldsymbol{T})\tag{1}$$

be the mappings defined in [**2**]: $(Mf)(x, y)=f(x+y)$ and $(Pg)(x)=\int_T g(x-y, y)dy$. Then M is an isometric homomorphism, P is a norm-decreasing mapping, and $P\circ M=$identity. Consequently we have two isometric imbeddings $A(\boldsymbol{T})\subset V(\boldsymbol{T}^2)\subset V(K)$. By Corollary 3.13 of [**1**: p. 35], there exists a $\Phi\in PM(\boldsymbol{T})=A(\boldsymbol{T})'$ such that

$$\|\Phi-\mu\|_{PM}>1 \qquad \forall\mu\in M(\boldsymbol{T}).\tag{2}$$

Let $\tilde{\Phi}\in V(K)'$ be any norm-preserving extension of Φ, and $\nu\in M(K)$. If we denote by $\mu\in PM(\boldsymbol{T})$ the restriction of ν to $A(\boldsymbol{T})$ as a functional, then obviously $\mu\in M(\boldsymbol{T})$, and we have

$$\|\tilde{\Phi}-\nu\|_{V(K)'}\geqq\|\Phi-\mu\|_{PM}>1\tag{3}$$

by (2). Therefore $M(K)$ is not dense in $V(K)'$. By one of the above remarks, this implies that $M(X)$ is not dense in $V_0(X)'$. Hence (ii) $\Rightarrow$ (i), and we have established the equivalence of (i) and (ii).

Next we prove that $V_0(X)'$ does not have Property ($\mathcal{P}$) under the assumption given in the above paragraph. After imbedding $V(\boldsymbol{T}^2)$ into $V(K)$ as above, we take any net $\{L_\alpha\}$ of norm-decreasing linear mappings from $V(K)$ into $V(\boldsymbol{T}^2)$ such that

$$\lim_\alpha \|L_\alpha f - f\|_{V(Y)} = 0 \qquad f \in V(\boldsymbol{T}^2)\,; \tag{4}$$

such a net exists (cf. [**5**; p. 28]). Let L'_α be the adjoint mapping of L_α. Since every L'_α has norm $\leqq 1$, there exists a norm-decreasing linear mapping $L' : V(\boldsymbol{T}^2)' \to V(K)'$ such that

$$\lim_\beta \langle f, L'_\beta \Phi\rangle = \langle f, L'\Phi\rangle \qquad \forall f \in V(K) \text{ and } \forall \Phi \in V(\boldsymbol{T}^2)' \tag{5}$$

for some subnet $\{L_\beta\}$ of $\{L_\alpha\}$. Since the imbedding $V(\boldsymbol{T}^2) \subset V(K)$ is isometric, we infer from (4) and (5) that L' is an isometry. On the other hand, it is trivial that $P' : PM(\boldsymbol{T}) \to V(\boldsymbol{T}^2)'$ is an isometry. Therefore, all the mappings

$$PM(\boldsymbol{T}) \xrightarrow{P'} V(\boldsymbol{T}^2)' \xrightarrow{L'} V(K)' \subsetneq V_0(X)'$$

are isometries. Since $PM(\boldsymbol{T}) \cong l^\infty(\boldsymbol{Z})$ does not have Property $(\mathcal{P})$, it follows that $V_0(X)'$ does not have $(\mathcal{P})$, either. Here $\boldsymbol{Z}$ denotes the group of integers. This establishes the implication (iii) $\Rightarrow$ (i).

It only remains to prove (i) $\Rightarrow$ (iii). Consider

$$C_0(\boldsymbol{Z}) \hat{\otimes} V_0(X) = C_0(\boldsymbol{Z}) \hat{\otimes} C_0(X_1) \hat{\otimes} \cdots \hat{\otimes} C_0(X_N)\,. \tag{6}$$

If we assume (i), it follows from the implication (i) $\Rightarrow$ (ii) that $M(\boldsymbol{Z} \times X)$ is dense in $(C_0(\boldsymbol{Z}) \hat{\otimes} V_0(X))'$. Therefore $V_0(X)'$ must have Property $(\mathcal{P})$ by Theorem 2.

This completes the proof.

COROLLARY 1. *Suppose that all the spaces X_j, $1 \leqq j \leqq N$, are residual. Then the second conjugate space of $V_0(X)$ is isometrically isomorphic to the Banach space of all $f \in l^\infty(X)$ such that*

$$\|f\|_{\mathfrak{N}} = \sup_E \|f\|_{V(E)} < \infty\,.$$

Here the supremum is taken over all finite product subsets E of X.

PROOF. Notice that $M(X) = M_d(X)$ is dense in $V_0(X)'$ by hypothesis and Theorem 3.

Given $F \in V_0(X)''$, define an $f \in l^\infty(X)$ by setting $f(x) = \langle \delta_x, F\rangle$ for all $x \in X$. Since $M_d(X)$ is dense in $V_0(X)'$, F is completely determined by f, and we have

$$\begin{aligned} \|F\| &= \sup_E \{|\langle \mu, F\rangle| : \mu \in M(E) \text{ and } \|\mu\|_{V(E)'} \leqq 1\} \\ &= \sup_E \left\{\left|\int f d\mu\right| : \mu \in M(E) \text{ and } \|\mu\|_{V(E)'} \leqq 1\right\} \\ &= \sup_E \|f\|_{V(E)} = \|f\|_{\mathfrak{N}}\,. \end{aligned}$$

The converse part is obvious, and this completes the proof.

Notice that for any locally compact spaces X_j, a function $f \in l^\infty(X)$ is a multiplier of $V_0(X)$ if and only if f belongs to $V_0(X)$ locally at every point of X and $\|f\|_{\mathfrak{M}} < \infty$. Moreover, if f is a multiplier of $V_0(X)$, then the multiplier norm of f is equal to $\|f\|_{\mathfrak{M}}$. (See [**12**: Lemma 1.1] and [**6**: Theorem 4.5].) Therefore Theorems 1, 3 and Corollary 1 yield the following.

COROLLARY 2. *Suppose that all the spaces X_j, $1 \leqq j \leqq N$, are discrete. Then we have:*

(a) *For each $\Phi \in V_0(X)'$,*

$$\lim_{\mathfrak{F}} \|\Phi - \sum_{x \in E} \langle \xi_{\{x\}}, \Phi \rangle \delta_x\| = 0 .$$

(b) *$V_0(X)''$ is isometrically isomorphic to the Banach space of all multipliers of $V_0(X)$.*

Now let G be a LCA group, Γ its character group, and $A(\Gamma)$ the Fourier algebra on Γ (cf. [**4**]). For any closed subset X of Γ, $A(X)$ denotes the Fourier restriction algebra $A(\Gamma)|_X$ with the natural quotient norm. Let $\bar{X}$ be the closure of X in $\bar{\Gamma}$, the Bohr compactification of Γ. We consider $A_d(\Gamma) = M_d(G)\hat{} \cong A(\bar{\Gamma})$, $A_d(X) = A_d(\Gamma)|_X \cong A(\bar{X})$, and $A_0(X) = A_d(X) \cap C_0(X)$.

COROLLARY 3. *Suppose that G is compact, and that $X_1, X_2, \cdots, X_N$ $(N \geqq 1)$ are finitely many, disjoint subsets of Γ with dissociate union. Put $X = X_1 \cdot X_2 \cdot \cdots \cdot X_N \subset \Gamma$, and identify X with the product space of the X_j, $1 \leqq j \leqq N$.*

(a) *Then $A(X) = V_0(X)$ and $A_0(X) \subset A(X)$.*

(b) *$B(X) = M(G)\hat{}|_X$ is (isomorphic to) the second conjugate space of $A(X)$.*

(c) *If $\phi \in L^\infty(G)$ and* supp $\hat{\phi} \subset X$, *then*

$$\lim_{\mathfrak{F}} \|\phi - \sum_{\gamma \in E} \hat{\phi}(\gamma)\gamma\|_\infty = 0 ,$$

where $\mathfrak{F}$ denotes the directed family of all finite subsets E of X of the form $E = E_1 \cdot E_2 \cdot \cdots \cdot E_N$ with $E_j \subset X_j$ for $1 \leqq j \leqq N$.

PROOF. That $A(X) = V_0(X)$ is an easy consequence of Theorem 3.2 in [**3**]. Since the proof is quite routine, we omit it. To prove $A_0(X) \subset A(X)$, first notice that $A_d(X) \subset V(X)$ by the definition of $A_d(X)$. Let Y_j be the one-point compactification of X_j, $1 \leqq j \leqq N$, and $Y = Y_1 \times \cdots \times Y_N$. Then $C_0(X) \subset C(Y)$, and $V(Y) \subset V(X)$ with obvious identifications. On the other hand, we have $C_0(X) \cap V(X) \subset V(Y)$ by Theorem 4.3 in [**5**]. Therefore

$$A_0(X) \subset C_0(X) \cap V(X) = C_0(X) \cap V(Y) ,$$

so that $A_0(X) \subset A(X)$, since evidently $V_0(X) = C_0(X) \cap V(Y)$. This establishes (a).

Notice that $A(X)'$ is $L^\infty_X(G) = \{\phi \in L^\infty(G) : \text{supp}\, \hat{\phi} \subset X\}$, as is well-known.

Therefore part (c) is an easy consequence of part (a) combined with Corollary 2.

Part (b) follows from part (c), because $B(X)$ is the conjugate space of $C_X(G)=C(G)\cap L_X^\infty(G)$ for any $X\subset\Gamma$.

Now let $\varepsilon>0$ be given. A closed subset K of G is said to be a K_ε-set if to each $f\in C(K)$ with $|f|=1$ there correspond a character $\gamma\in\Gamma$ and a complex number $c\in \boldsymbol{T}=\{|z|=1\}$ such that $|f(x)-c\gamma(x)|\leqq\varepsilon$ for all $x\in K$. Although the following result is similar to Varopoulos' Theorem 4.4.1 in [**11**: p. 78], his proof does not work in our case.

PROPOSITION 1. *Let $E_1,\cdots,E_N$ be disjoint compact subsets of a LCA group G whose union is a K_ε-set for some $0<\varepsilon<(2/N)\sin(\sqrt{6}-2)$, and let $E=E_1+\cdots+E_N\subset G$. Then E is a set of bounded synthesis for $A(G)$.*

PROOF. The curious restriction for $\varepsilon>0$ is used only to assure that every point x of E has a unique expression of the form $x=x_1+\cdots+x_N$ with $x_j\in E_j$ $(1\leqq j\leqq N)$, and that there exists a $\phi\in A(\boldsymbol{T})$ such that

$$\|\phi\|_{A(\boldsymbol{T})}=\sum_{m=-\infty}^{\infty}|\hat{\phi}(m)|=C<1\,, \qquad \text{and} \tag{1}$$

$$\phi(z)=z-1 \qquad \text{if} \quad z\in\boldsymbol{T} \text{ and } |z-1|<N\varepsilon\,. \tag{2}$$

For the latter fact, we refer the reader to Remark (b) at the end of [**9**].

We prove the above assertion only for $N=2$, since the proof for the general case is similar. We also assume that all the sets E_j are totally disconnected, since we are only interested in this case. (However, if some of the sets E_j contain non-trivial connected sets, then the proof becomes very complicated.)

For $i=1, 2$ and $n\in\boldsymbol{N}$, let $E_i=E_{i1}\cup\cdots\cup E_{in}$ be any partition of E_i into disjoint clopen subsets. Choose and fix $2n$ points $x_j\in E_{1j}$ and $y_j\in E_{2j}$, $1\leqq j\leqq n$. We define a linear mapping $L: PM(E)\to M_d(E)$ by setting

$$LP=\sum_{j,k=1}^{n}\hat{P}_{jk}(1)\delta_{x_j+y_j} \qquad \forall P\in PM(E)\,, \tag{3}$$

where $P_{jk}\in PM(E)$ is the part of $P\in PM(E)$ carried by $E_{1j}+E_{2k}$. Notice that the sets $E_{1j}+E_{2k}$ $(1\leqq j,\ k\leqq n)$ are disjoint by the above remark.

We then claim that $\|LP\|_{PM}\leqq(1-C)^{-1}\|P\|_{PM}$ for all $P\in PM(E)$, where C is as in (1). To prove this, let $\|L\|$ be the norm of L as an operator on $PM(E)$, and notice that

$$\widehat{LP}(\gamma^{-1})=\sum_{j,k=1}^{n}\gamma(x_j+y_j)\widehat{P_{jk}}(1) \qquad \forall\gamma\in\Gamma \tag{3$'$}$$

for all $P\in PM(E)$. Fix an arbitrary $\gamma\in\Gamma$. Since E_1 and E_2 are disjoint and their union is a K_ε-set, there exist $\chi\in\Gamma$ and $\alpha=c^2\in\boldsymbol{T}$ such that

$$\sup\{|\gamma(x_j+y_k)-\alpha\chi(x+y)|: x\in E_{1j},\ y\in E_{2k}\}<2\varepsilon \tag{4}$$

for all $1 \leqq j,\ k \leqq n$. It follows from (2) with $N=2$ and (4) that for each pair (j, k) we have

$$\gamma(x_j+y_k)-\alpha\chi=\alpha\chi\{\bar{\alpha}\gamma(x_j+y_k)\bar{\chi}-1\}$$
$$=\sum_{m=-\infty}^{\infty}\hat{\phi}(m)\alpha^{1-m}\gamma^m(x_j+y_k)\chi^{1-m}$$

on some neighborhood of $E_{1j}+E_{2k}$. Therefore

$$(5)\qquad |\widehat{LP}(\gamma^{-1})-\alpha\hat{P}(\chi^{-1})|=|\sum_{j,k=1}^{n}\langle\gamma(x_j+y_k)-\alpha\chi,\ P_{jk}\rangle|$$
$$\leqq\sum_{m=-\infty}^{\infty}|\hat{\phi}(m)|\cdot|\sum_{j,k=1}^{n}\langle\gamma^m(x_j+y_k)\chi^{1-m},\ P_{jk}\rangle|$$
$$=\sum_{m=-\infty}^{\infty}|\hat{\phi}(m)|\cdot|L(\chi^{1-m}P)^{\wedge}(\gamma^{-m})|$$
$$\leqq\sum_{m=-\infty}^{\infty}|\hat{\phi}(m)|\cdot\|L\|\cdot\|P\|_{PM}\leqq C\|L\|\cdot\|P\|_{PM}\,.$$

Hence

$$(6)\qquad |\widehat{LP}(\gamma^{-1})|\leqq(1+C\|L\|)\|P\|_{PM}\,.$$

Since $\gamma\in\Gamma$ and $P\in PM(E)$ are arbitrary, (6) implies $\|L\|\leqq 1+C\|L\|$. Since $C<1$, we conclude $\|L\|\leqq(1-C)^{-1}$.

To complete the proof, it suffices to show that given $P\in PM(E)$ and $\gamma\in\Gamma$, $\widehat{LP}(\gamma^{-1})$ approaches $P(\gamma^{-1})$ as the partitions $\{E_{ij}\}_j$ of E_i become finer and finer. Notice that $\|\phi\|_{A(\boldsymbol{T})}$ can be made arbitrarily small if we require (2) for a sufficiently small $\varepsilon>0$ (cf. Lemma 1 of [**7**: p. 290]). Therefore we can do this easily by arguing as in (5) with $\alpha=1$ and $\chi=\gamma$ after replacing $\phi\in A(\boldsymbol{T})$ by other suitable functions in $A(\boldsymbol{T})$.

This completes the proof.

COROLLARY 4. *Suppose that G is compact, and that $X_1, \cdots, X_N$ are finitely many, disjoint subsets of Γ whose union is a K_ε-set for some $0<\varepsilon<(2/N)\sin(\sqrt{6}-2)$. If we put $X=X_1\cdot X_2\cdot\cdots\cdot X_N\subset\Gamma$, then $A(X)=A_0(X)$ and $\bar{X}$ is a set of bounded synthesis for the algebra $A(\bar{\Gamma})=A_d(\Gamma)$.*

PROOF. By hypothesis and Theorem 3.1 of [**12**], we have $A_d(X)=V(X)$ and $A(X)=V_0(X)$. Since $V_0(X)=C_0(X)\cap V(X)$ as was observed in the proof of Corollary 3, we have $A(X)=A_0(X)$.

It is easy to prove that under our hypothesis the sets $\bar{X}_1, \cdots, \bar{X}_N$ are disjoint and their union is an extremally disconnected K_ε-set in $\bar{\Gamma}$. This, combined with Proposition 1, completes the proof.

COROLLARY 5. *Let G and $X\subset\Gamma$ be as in Corollary 4. Suppose $N\geqq 2$ and every X_j is infinite. Then X contains a subset E such that*

(i) $A(E)\subset A_0(E)\subset B_0(E)\equiv B(E)\cap C_0(E)$.

(ii) $A_0(E)$ (*resp.* $B_0(E)$) *contains a function* f *such that* $\Phi\circ f\notin A(E)$ (*resp.* $\Phi\circ f\notin A_0(E)$) *for all non-constant entire functions* Φ.

PROOF. This is an easy consequence of Theorem 2 and its proof in [8]. We omit the details.

REMARKS. Let $X=X_1\times\cdots\times X_N$ and B be as before.

(I) If B' satisfies ($\mathscr{P}$), then the set of all compactly supported elements is dense in $(V_0(X)\otimes B)'$. The proof is similar to that of Lemma 1.

(II) Suppose that B' satisifies ($\mathscr{P}$), $P\in(C_0(X)\hat{\otimes}B)'$, and $E\subset X$ is closed. Then there exists a unique $P_E\in(C_0(X)\hat{\otimes}B)'$, with $S_X(P_E)\subset E$, having the following property: to each $\varepsilon>0$ there corresponds a neighborhood W of E such that $\|\phi P-P_E\|\leqq\varepsilon\|\phi\|_\infty$ whenever $\phi\in C(X)$, $\phi=1$ on E, and supp $\phi\subset W$.

(III) Suppose $N=2$. Applying (II) twice, we conclude that given $P\in V_0(X)'$ and $E=E_1\times E_2\subset X$ closed, there exists a unique $P_E\in V_0(X)'$, with supp $P_E\subset E$, having the following property: to each $\varepsilon>0$ there corresponds a neighborhood W of E such that $\|\phi P-P_E\|\leqq\varepsilon\|\phi\|_{V(X)}$ whenever $\phi\in V(X)$, $\phi=1$ on E, and supp $\phi\subset W$. However, no analog of this holds if $N\geqq 3$, all the spaces X_j are infinite, and at least two of them contain perfect sets.

(IV) Under the hypothesis of Corollary 4, the set of all accumulation points of X in Γ is a set of synthesis.

(V) All the results in this paper were obtained in the last year of the author's sojourn at Kansas State University (1972-1974).

References

[1] C.F. Dunkl and D.E. Ramirez, Topics in harmonic analysis, Appleton-Century-Crofts, New York, 1971.

[2] S.C. Herz, Remarques sur la note précédente de M. Varopoulos, C.R. Acad. Sci. Paris, **260** (1965), 6001-6004.

[3] E. Hewitt and H. Zuckerman, Singular measures with absolutely continuous convolution squares, Proc. Cambridge Philos. Soc., **62** (1966), 399-420.

[4] W. Rudin, Fourier analysis on groups, Interscience Tracts in Pure and Appl. Math., no. 12, Interscience, New York, 1962.

[5] S. Saeki, The ranges of certain isometries of tensor products of Banach spaces, J. Math. Soc. Japan, **23** (1971), 27-39.

[6] S. Saeki, Homomorphisms of tensor algebras, Tôhoku Math. J., **23** (1971), 173-199.

[7] S. Saeki, Tensor products of Banach algebras and harmonic analysis, Tôhoku Math. J., **24** (1972), 281-299.

[8] S. Saeki, On restriction algebras of tensor algebras, J. Math. Soc. Japan, **25** (1973), 506-522.

[9] S. Saeki, Infinite tensor products in Fourier algebras, submitted to Tôhoku Math. J.,

[10] L. Schwartz, Produits tensoriels topologiques d'espaces vectoriels topologiques.

Espaces vectoriels topologiques nucléaires. Applications, Faculté des Sciences de Paris (1953–1954).
[11] N. Th. Varopoulos, Tensor algebras and harmonic analysis, Acta Math., **119** (1967), 51–112.
[12] N. Th. Varopoulos, Tensor algebras over discrete spaces, J. Functional Analysis, **3** (1969), 321–335.
[13] A. Zygmund, Trigonometric series, Vol. I, Cambridge University Press, New York, 1959.

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