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A characterization of arithmetic Fuchsian groups

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§1. Introduction.

Let k be a totally real algebraic number field of degree n. Then k has n distinct embeddings φ_i $(1\leqq i\leqq n)$ into the real number field $\boldsymbol{R}$, where φ_1 is the identity. Let A be a quaternion algebra over k which is unramified at the place φ_1 and ramified at all other infinite places φ_i $(2\leqq i\leqq n)$. Then there exists an $\boldsymbol{R}$-isomorphism

$$\rho: A\underset{\boldsymbol{Q}}{\otimes}\boldsymbol{R}\longmapsto M_2(\boldsymbol{R})\oplus\boldsymbol{H}\oplus\cdots\oplus\boldsymbol{H}, \tag{1}$$

where $\boldsymbol{H}$ is the Hamilton quaternion algebra.

Denote by ρ_i the composite of $\rho|_A$ with the projection to the i-th factor. Then ρ_1 (resp. ρ_i $(2\leqq i\leqq n)$) is an isomorphism of A into $M_2(\boldsymbol{R})$ (resp. $\boldsymbol{H}$). By changing the indices suitably, for any element a of k we have

$$\rho_1(a\cdot 1_A)=a\cdot 1_2\,,\qquad \rho_i(a\cdot 1_A)=\varphi_i(a)\cdot 1_{\boldsymbol{H}}\quad(2\leqq i\leqq n)\,, \tag{2}$$

where 1_A, $1_{\boldsymbol{H}}$ and 1_2 are the unities of A, $\boldsymbol{H}$ and $M_2(\boldsymbol{R})$ respectively.

Denote by $\mathrm{tr}_A(\ )$ and $n_A(\ )$ (resp. $\mathrm{tr}_{\boldsymbol{H}}(\ )$ and $n_{\boldsymbol{H}}(\ )$) the reduced trace and the reduced norm of A (resp. $\boldsymbol{H}$). Then for any $\alpha\in A$, we have

$$\mathrm{tr}_A(\alpha)=\mathrm{tr}\,(\rho_1(\alpha))\,,\qquad \varphi_i(\mathrm{tr}_A(\alpha))=\mathrm{tr}_{\boldsymbol{H}}(\rho_i(\alpha))\quad(2\leqq i\leqq n)\,, \tag{3}$$

$$n_A(\alpha)=\det\,(\rho_1(\alpha))\,,\qquad \varphi_i(n_A(\alpha))=n_{\boldsymbol{H}}(\rho_i(\alpha))\quad(2\leqq i\leqq n)\,, \tag{4}$$

where $\mathrm{tr}\,(\ )$ and $\det\,(\ )$ are the trace and the determinant of $M_2(\boldsymbol{R})$ respectively.

Now take an order O of A and put

$$U=\{\varepsilon\in O\mid \varepsilon O=O \text{ and } n_A(\varepsilon)=1\}\,.$$

Then U is a group called the unit group of O of norm 1. Denote by $\Gamma(A, O)$ the image $\rho_1(U)$ of U under ρ_1. Then $\Gamma(A, O)$ is a discrete subgroup of $SL_2(\boldsymbol{R})$. The group $SL_2(\boldsymbol{R})$ operates on the upper half plane $H=\{z\in\boldsymbol{C}\mid\mathrm{Im}\,(z)>0\}$ in the following way:

$$SL_2(\boldsymbol{R})\ni g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}: z\longmapsto\frac{az+b}{cz+d}\,.$$

It is well-known that by the above operation $\Gamma(A, O)$ defines a Fuchsian group of the first kind i.e. a properly discontinuous group such that $H/\Gamma(A, O)$ is of finite volume. If we change the isomorphism ρ, $\Gamma(A, O)$ is transformed into a $GL_2(\boldsymbol{R})$-conjugate group.

DEFINITION. Let Γ be a discrete subgroup of $SL_2(\boldsymbol{R})$ such that H/Γ is of finite volume. Then we call Γ a Fuchsian group of the first kind. When Γ is commensurable with some $\Gamma(A, O)$, Γ is called *an arithmetic Fuchsian group* (cf. [**5**]). Moreover, if Γ is a subgroup of $\Gamma(A, O)$ of finite index, then we call Γ *a Fuchsian group derived from a quaternion algebra A*.

In this paper we shall prove the following theorem which gives a characterization of arithmetic Fuchsian groups Γ by the properties of the set $\mathrm{tr}(\Gamma) = \{\mathrm{tr}(\gamma) | \gamma \in \Gamma\}$.

THEOREM 1. *Let Γ be a Fuchsian group of the first kind. Then Γ is an arithmetic Fuchsian group if and only if Γ satisfies the following conditions* (I) *and* (II_1):

(I) *Let k_1 be the field $\boldsymbol{Q}(\mathrm{tr}(\gamma)|\gamma \in \Gamma)$ generated by the set $\mathrm{tr}(\Gamma)$ over the rational number field $\boldsymbol{Q}$. Then k_1 is an algebraic number field of finite degree, and $\mathrm{tr}(\Gamma)$ is contained in the ring O_{k_1} of integers of k_1.*

(II_1) *Let k_2 be the field $\boldsymbol{Q}((\mathrm{tr}(\gamma))^2|\gamma \in \Gamma)$ generated by the set $\{(\mathrm{tr}(\gamma))^2|\gamma \in \Gamma\}$ over $\boldsymbol{Q}$. Let φ be any isomorphism of k_1 into the complex number field $\boldsymbol{C}$ such that $\varphi|_{k_2} \neq$ the identity. Then $\varphi(\mathrm{tr}(\Gamma))$ is bounded in $\boldsymbol{C}$.*

In order to prove Theorem 1 we must prove first the following

THEOREM 2. *Let Γ be a Fuchsian group of the first kind. Then Γ is a Fuchsian group derived from a quaternion algebra if and only if Γ satisfies the condition* (I) *in Theorem* 1 *together with the following condition* (II_2):

(II_2) *Let φ be any isomorphism of $k_1 = \boldsymbol{Q}(\mathrm{tr}(\gamma)|\gamma \in \Gamma)$ into $\boldsymbol{C}$ such that $\varphi \neq$ the identity. Then $\varphi(\mathrm{tr}(\Gamma))$ is bounded in $\boldsymbol{C}$.*

REMARK. Theorem 2 is a generalization of a result in [**1**].

We shall first prove Theorem 2, in § 2. By making use of Theorem 2, we shall then prove Theorem 1, in § 3. Finally in § 4, it is shown that the conditions (I) and (II_1) are independent of each other.

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§ 2. Proof of Theorem 2.

In this section we shall prove Theorem 2.

2.1. Necessity of the conditions (I) and (II_2).

Let Γ be a subgroup of $\Gamma(A, O)$ of finite index. Then $k_1 = \boldsymbol{Q}(\mathrm{tr}(\gamma)|\gamma \in \Gamma)$ is contained in the center k of A. Therefore k_1 is totally real. Since $\mathrm{tr}_A(O)$

is contained in O_k, we see that $\mathrm{tr}(\Gamma)$ is contained in O_{k_1}. This shows that Γ satisfies the condition (I).

Now consider the case $n \geqq 2$. By (3), we see that $\varphi_i(\mathrm{tr}(\Gamma))$ is contained in $\mathrm{tr}_H(\rho_i(U))$ $(2 \leqq i \leqq n)$. On the other hand by (4) for any $\varepsilon \in U$ we have $n_H(\rho_i(\varepsilon)) = \varphi_i(n_A(\varepsilon)) = 1$ $(2 \leqq i \leqq n)$. Hence $\rho_i(U)$ is contained in the set $\boldsymbol{H}^{(1)} = \{x \in \boldsymbol{H} \mid n_H(x) = 1\}$. Since $\mathrm{tr}_H(\boldsymbol{H}^{(1)})$ coincides with the interval $[-2, 2]$, $\varphi_i(\mathrm{tr}(\Gamma))$ is bounded in $\boldsymbol{R}$ $(2 \leqq i \leqq n)$.

Finally we shall show that k_1 coincides with k. Suppose that k is a proper extension of k_1. Then there exists an isomorphism φ_i $(2 \leqq i \leqq n)$ such that $\varphi_i|_{k_1} =$ the identity. Using this φ_i, we see that $\mathrm{tr}(\Gamma)$ is contained in the interval $[-2, 2]$. This means that Γ contains no hyperbolic elements, which is a contradiction. Therefore, k_1 coincides with k. Thus we have shown that Γ satisfies the condition (II_2).

2.2. Sufficiency of the conditions (I) and (II_2).

PROPOSITION 1. *Let Γ be a Fuchsian group of the first kind. Let $A(\Gamma)$ be the vector space spanned by Γ over $k_1 = \boldsymbol{Q}(\mathrm{tr}(\gamma) \mid \gamma \in \Gamma)$ in $M_2(\boldsymbol{R})$. Then $A(\Gamma)$ is a quaternion algebra over k_1. Moreover, if Γ satisfies the condition* (I), *then the submodule $O(\Gamma)$ of $A(\Gamma)$ spanned by Γ over O_{k_1} is an order of $A(\Gamma)$.*

This proposition is proved in [**1**].

We shall now prove the following

PROPOSITION 2. *Let Γ be a Fuchsian group of the first kind. Assume that Γ satisfies the conditions* (I) *and* (II_2). *Then $k_1 = \boldsymbol{Q}(\mathrm{tr}(\gamma) \mid \gamma \in \Gamma)$ is totally real. Moreover, let φ be any isomorphism of k_1 into $\boldsymbol{R}$ such that $\varphi \neq$ the identity. Then $\varphi(\mathrm{tr}(\Gamma))$ is contained in the interval* $[-2, 2]$.

PROOF. Take any $\gamma \in \Gamma$. Let u and $1/u$ be the eigen-values of γ. Let φ be any isomorphism of k_1 into $\boldsymbol{C}$ such that $\varphi \neq$ the identity. Extend φ to an isomorphism ϕ of $k_1(u)$ into $\boldsymbol{C}$. We shall show that $|\phi(u)| = 1$. Suppose that $|\phi(u)| \neq 1$. Then by the inequality

$$|\varphi(\mathrm{tr}(\gamma^m))| = |(\phi(u))^m + 1/(\phi(u))^m| \geqq \big| |\phi(u)|^m - 1/|\phi(u)|^m \big|,$$

the set $\{\varphi(\mathrm{tr}(\gamma^m)) \mid m \in z\}$ is not bounded which contradicts (II_2). Therefore $|\phi(u)| = 1$. By the equations

$$\varphi(\mathrm{tr}(\gamma)) = \phi(u) + 1/\phi(u) = \phi(u) + \overline{\phi(u)},$$

$\varphi(\mathrm{tr}(\gamma))$ is a real number contained in the interval $[-2, 2]$. This shows that k_1 is totally real and that $\varphi(\mathrm{tr}(\Gamma))$ is contained in the interval $[-2, 2]$. q. e. d.

PROPOSITION 3. *Let Γ be a Fuchsian group of the first kind. Assume that Γ satisfies the conditions* (I) *and* (II_2). *Then*

$$A(\Gamma)\otimes_{\boldsymbol{Q}}\boldsymbol{R}\cong M_2(\boldsymbol{R})\oplus\boldsymbol{H}\oplus\cdots\oplus\boldsymbol{H}.$$

PROOF. In view of the proof of Proposition 1 in [1] by considering a suitable conjugate group of Γ, we may assume that Γ contains the following two elements:

$$\gamma_0=\begin{pmatrix} w & 0 \\ 0 & 1/w \end{pmatrix} \quad (w^2\neq 1), \qquad \gamma_1=\begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix} \quad (c_1\neq 0).$$

We shall show that $K=k_1(w)$ is a proper extension of k_1. If k_1 is a proper extension of $\boldsymbol{Q}$, then there exists an isomorphism ψ of K into $\boldsymbol{C}$ such that $\psi|_{k_1}\neq$ the identity. $\psi(w)$ and $1/\psi(w)$ are the roots of the equation $x^2-\psi(t_0)x+1=0$, where $t_0=\mathrm{tr}(\gamma_0)$. By Proposition 2 we have $|\psi(t_0)|<2$. Therefore $\psi(K)=\psi(k_1(w))$ is an imaginary field. On the other hand, by Proposition 2, $\psi(k_1)$ is a real field. It follows that K does not coincides with k_1.

If $k_1=\boldsymbol{Q}$, then t_0 is a rational integer such that $|t_0|>2$. Therefore the polynomial x^2-t_0x+1 is irreducible over $\boldsymbol{Q}$. This shows that K is a proper extension of k_1.

Consequently we have

$$\gamma_0=\begin{pmatrix} w & 0 \\ 0 & w' \end{pmatrix} \quad (w^2\neq 1), \qquad \gamma_1=\begin{pmatrix} a_1 & 1 \\ c_1 & a_1' \end{pmatrix} \quad (c_1\neq 0\in k_1),$$

and we see that

$$A(\Gamma)=A=\left\{\begin{pmatrix} a & b \\ b'c_1 & a' \end{pmatrix} \,\middle|\, a, b\in K\right\},$$

where a' is the k_1-conjugate of a.

LEMMA 1. *Let ψ be any isomorphism of $K=k_1(w)$ into $\boldsymbol{C}$ such that $\psi|_{k_1}\neq$ the identity. Then for any element $\gamma=\begin{pmatrix} a & b \\ b'c_1 & a' \end{pmatrix}$ of Γ we have the inequality $|\psi(a)|\leqq 1$.*

COROLLARY. *Let ψ be the same as in Lemma 1. Then we have $\psi(c_1)<0$, where $\gamma_1=\begin{pmatrix} a_1 & 1 \\ c_1 & a_1' \end{pmatrix}$.*

PROOF OF LEMMA 1. By Proposition 2 for any $\gamma=\begin{pmatrix} a & b \\ b'c_1 & a' \end{pmatrix}\in\Gamma$ we have the inequality $|\psi(\mathrm{tr}(\gamma\cdot\gamma_0^m))|\leqq 2$. Then we have

$$\psi(\mathrm{tr}(\gamma\cdot\gamma_0^m))=\psi(aw^m)+\psi(a'w'^m)=\psi(aw^m)+\overline{\psi(aw^m)}=2\,\mathrm{Re}\,(\psi(a)\cdot\psi(w^m)).$$

In view of the proof of Proposition 2, we see that $|\psi(w)|=1$. Since w is not a root of unity, the set $\{\psi(w)^m\,|\,m\in\boldsymbol{Z}\}$ is a dense subgroup of $\boldsymbol{C}^{(1)}=\{z\in\boldsymbol{C}\,|\,|z|=1\}$. Therefore we have $|\mathrm{Re}\,(\psi(a)\cdot z)|\leqq 1$, for any $z\in\boldsymbol{C}^{(1)}$. It follows that $|\psi(a)|\leqq 1$.

q. e. d.

PROOF OF COROLLARY. Applying Lemma 1 to γ_1 we see that $|\psi(a_1)|\leqq 1$.

By the equation

$$\det(\gamma_1)=a_1a_1'-c_1=1$$

we have

$$\psi(c_1)=\psi(a_1a_1')-1=|\psi(a_1)|^2-1\leqq 0\,.$$

By the fact that $c_1\neq 0$ we see that $\psi(c_1)<0$. q.e.d.

Let $\{\varphi_i\}$ $(1\leqq i\leqq n_1)$ be all distinct isomorphisms of k_1 into $\boldsymbol{R}$, where we assume that $\varphi_1=$ the identity. Extend φ_i to an isomorphism ψ_i of $K=k_1(w)$ into $\boldsymbol{C}$. Moreover we shall define an isomorphism Ψ_i of $A(\Gamma)$ into $M_2(\boldsymbol{C})$ in the following way:

$$\Psi_i:\alpha=\begin{pmatrix} a & b \\ b'c_1 & a' \end{pmatrix}\longmapsto \Psi_i(\alpha)=\begin{pmatrix} \psi_i(a) & \psi_i(b) \\ \psi_i(b'c_1) & \psi_i(a') \end{pmatrix}.$$

Then $A_i=\Psi_i(A(\Gamma))$ is a quaternion algebra over $\psi_i(k_1)$. By definition of ψ_i we see easily that

$$A(\Gamma)\otimes_{\boldsymbol{Q}}\boldsymbol{R}\cong\bigoplus_{i=1}^{n_1}(A_i\otimes_{\varphi_i(k_1)}\boldsymbol{R})\,.$$

Since we have

$$\psi_i(a')=\overline{\psi_i(a)}\qquad(2\leqq i\leqq n_1)\,,$$

we see that

$$A_i=\left\{\begin{pmatrix} a & b \\ \bar{b}\psi_i(c_1) & \bar{a} \end{pmatrix}\,\middle|\; a,\ b\in\psi_i(K)\right\}.$$

It follows from Corollary to Lemma 1 that

$$A_i\otimes_{\varphi_i(k_1)}\boldsymbol{R}\cong\boldsymbol{H}\qquad(2\leqq i\leqq n_1)\,.$$

This completes the proof of Proposition 3.

By Propositions 1, 2 and 3 k_1, $A(\Gamma)$ and $O(\Gamma)$ satisfy the assumptions in §1. Clearly, Γ is a subgroup of $\Gamma(A(\Gamma), O(\Gamma))$. Since both H/Γ and $H/\Gamma(A(\Gamma), O(\Gamma))$ are of finite volume, Γ is a subgroup of $\Gamma(A(\Gamma), O(\Gamma))$ of finite index. This shows that Γ is a Fuchsian group derived from a quaternion algebra.

§3. Proof of Theorem 1.

In this section we shall prove Theorem 1 by making use of Theorem 2.

3.1. Necessity of the conditions (I) and (II_1).

Let Γ be a Fuchsian group of the first kind. Denote by $\Gamma^{(2)}$ the subgroup of Γ generated by the set $\{\gamma^2|\gamma\in\Gamma\}$. Then $\Gamma^{(2)}$ is a normal subgroup of Γ such that $\Gamma/\Gamma^{(2)}$ is of exponent 2. Since Γ is finitely generated, $\Gamma/\Gamma^{(2)}$ is a finite abelian group of type $(2, 2, \cdots, 2)$. Therefore $\Gamma^{(2)}$ is a subgroup of Γ of finite index.

In view of the proof of Proposition 1 there exist two elements γ_0, and γ_1 of Γ such that $\{1_2, \gamma_0, \gamma_1, \gamma_0 \cdot \gamma_1\}$ is a basis of $A(\Gamma)$ over $k_1 = \boldsymbol{Q}(\mathrm{tr}(\gamma) | \gamma \in \Gamma)$. It is easy to see that we may assume that

$$\gamma_0 = \alpha^2, \qquad \gamma_1 = \beta^2,$$

where α and β are hyperbolic elements of Γ. Since $\mathrm{tr}(\alpha^2) \neq 0$, by the equation

$$\beta^4 - \mathrm{tr}(\beta^2) \cdot \beta^2 + 1 = 0,$$

either $\mathrm{tr}(\alpha^2 \cdot \beta^2)$ or $\mathrm{tr}(\alpha^2 \cdot \beta^4)$ is non-zero. Therefore without loss of generality we may assume that

$$\mathrm{tr}(\alpha^2 \cdot \beta^2) \neq 0. \tag{5}$$

PROPOSITION 4. *Let Γ be a Fuchsian group of the first kind. Denote by $\Gamma^{(2)}$ the subgroup of Γ generated by the set $\{\gamma^2 | \gamma \in \Gamma\}$. Let $k_2 = \boldsymbol{Q}((\mathrm{tr}(\gamma))^2 | \gamma \in \Gamma)$ and $k_2' = \boldsymbol{Q}(\mathrm{tr}(\gamma) | \gamma \in \Gamma^{(2)})$. Then k_2 coincides with k_2'.*

PROOF. Take any basis of $A(\Gamma)$ over k_1 of the form $\{1_2, \alpha^2, \beta^2, \alpha^2 \cdot \beta^2\}$ where α and β are elements of Γ satisfying (5). Let A_0 be the vector space spanned by $\{1_2, \alpha^2, \beta^2, \alpha^2 \cdot \beta^2\}$ over k_2. We shall show that A_0 is a quaternion algebra over k_2 and that A_0 coincides with $A(\Gamma^{(2)})$. The multiplication table of the algebra $A(\Gamma)$ with respect to the basis $\{1_2, \alpha^2, \beta^2, \alpha^2 \cdot \beta^2)$ is as follows:

	1_2	α^2	β^2	$\alpha^2 \cdot \beta^2$
1_2	1_2	α^2	β^2	$\alpha^2 \cdot \beta^2$
α^2	α^2	α^4	$\alpha^2 \cdot \beta^2$	$\alpha^4 \cdot \beta^2$
β^2	β^2	$\beta^2 \cdot \alpha^2$	β^4	$\beta^2 \cdot \alpha^2 \cdot \beta^2$
$\alpha^2 \cdot \beta^2$	$\alpha^2 \cdot \beta^2$	$\alpha^2 \cdot \beta^2 \cdot \alpha^2$	$\alpha^2 \cdot \beta^4$	$(\alpha^2 \cdot \beta^2)^2$

For any $\gamma \in \Gamma$ we have

$$\mathrm{tr}(\gamma^2) = (\mathrm{tr}(\gamma))^2 - 2.$$

It implies that k_2 is contained k_2'. It is easy to see that, α^4, β^4, $\alpha^4 \cdot \beta^2$ and $\alpha^2 \cdot \beta^4$ are all contained in A_0.

LEMMA 2. *Let δ_1 and δ_2 be two elements of Γ. Then $\mathrm{tr}(\delta_1^2 \cdot \delta_2^2)$ is contained in k_2.*

PROOF. We have

$$\begin{aligned}\mathrm{tr}(\delta_1^2 \cdot \delta_2^2) &= \mathrm{tr}((\mathrm{tr}(\delta_1) \cdot \delta_1 - 1_2)(\mathrm{tr}(\delta_2) \cdot \delta_2 - 1_2)) \\ &= \mathrm{tr}(\delta_1)\,\mathrm{tr}(\delta_2)\,\mathrm{tr}(\delta_1 \cdot \delta_2) - (\mathrm{tr}(\delta_1))^2 - (\mathrm{tr}(\delta_2))^2 + 2.\end{aligned}$$

On the other hand, using the equation

$$\mathrm{tr}(\delta_1)\,\mathrm{tr}(\delta_2)=\mathrm{tr}(\delta_1\cdot\delta_2)+\mathrm{tr}(\delta_1\cdot\delta_2^{-1}),$$

we obtain

$$(\mathrm{tr}(\delta_1\cdot\delta_2^{-1}))^2=(\mathrm{tr}(\delta_1))^2(\mathrm{tr}(\delta_2))^2+(\mathrm{tr}(\delta_1\cdot\delta_2))^2 -2\,\mathrm{tr}(\delta_1)\,\mathrm{tr}(\delta_2)\,\mathrm{tr}(\delta_1\cdot\delta_2).$$

It implies that $\mathrm{tr}(\delta_1)\,\mathrm{tr}(\delta_2)\,\mathrm{tr}(\delta_1\cdot\delta_2)$ is contained in k_2. Hence $\mathrm{tr}(\delta_1^2\cdot\delta_2^2)$ is contained in k_2. q. e. d.

LEMMA 3. *Let α, β be the same as in the definition of A_0. Then an element γ of $A(\Gamma)$ is contained in A_0 if and only if* $\mathrm{tr}(\gamma)$, $\mathrm{tr}(\gamma\cdot\alpha^{-2})$, $\mathrm{tr}(\gamma\cdot\beta^{-2})$ *and* $\mathrm{tr}(\gamma\cdot\beta^{-2}\cdot\alpha^{-2})$ *are all contained in* $k_2=\boldsymbol{Q}((\mathrm{tr}(\gamma))^2\,|\,\gamma\in\Gamma)$.

PROOF. Let γ be any element of $A(\Gamma)$. Then we have

$$\gamma=x_0 1_2+x_1\alpha^2+x_2\beta^2+x_3\alpha^2\cdot\beta^2,$$

where x_i $(0\leqq i\leqq 3)$ belongs to the field k_1.

Multiplying γ by α^{-2}, β^{-2} and $\beta^{-2}\cdot\alpha^{-2}$ respectively and taking the traces, we have the equations

$$\mathrm{tr}(\gamma)=2x_0+\mathrm{tr}(\alpha^2)x_1+\mathrm{tr}(\beta^2)x_2+\mathrm{tr}(\alpha^2\cdot\beta^2)x_3,$$

$$\mathrm{tr}(\gamma\cdot\alpha^{-2})=\mathrm{tr}(\alpha^2)x_0+2x_1+\mathrm{tr}(\alpha^2\cdot\beta^{-2})x_2+\mathrm{tr}(\beta^2)x_3,$$

$$\mathrm{tr}(\gamma\cdot\beta^{-2})=\mathrm{tr}(\beta^2)x_0+\mathrm{tr}(\alpha^2\cdot\beta^{-2})x_1+2x_2+\mathrm{tr}(\alpha^2)x_3,$$

$$\mathrm{tr}(\gamma\cdot\beta^{-2}\cdot\alpha^{-2})=\mathrm{tr}(\alpha^2\cdot\beta^2)x_0+\mathrm{tr}(\beta^2)x_1+\mathrm{tr}(\alpha^2)x_2+2x_3.$$

Put

$$D=\begin{pmatrix} 2 & \mathrm{tr}(\alpha^2) & \mathrm{tr}(\beta^2) & \mathrm{tr}(\alpha^2\cdot\beta^2) \\ \mathrm{tr}(\alpha^2) & 2 & \mathrm{tr}(\alpha^2\cdot\beta^{-2}) & \mathrm{tr}(\beta^2) \\ \mathrm{tr}(\beta^2) & \mathrm{tr}(\alpha^2\cdot\beta^{-2}) & 2 & \mathrm{tr}(\alpha^2) \\ \mathrm{tr}(\alpha^2\cdot\beta^2) & \mathrm{tr}(\beta^2) & \mathrm{tr}(\alpha^2) & 2 \end{pmatrix}.$$

Then

$$D\cdot\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} \mathrm{tr}(\gamma) \\ \mathrm{tr}(\gamma\cdot\alpha^{-2}) \\ \mathrm{tr}(\gamma\cdot\beta^{-2}) \\ \mathrm{tr}(\gamma\cdot\beta^{-2}\cdot\alpha^{-2}) \end{pmatrix}.$$

Now we shall show that the matrix D is contained in the group $GL_4(k_2)$. By Lemma 2 we see that D belongs to $M_2(k_2)$. Considering x_i $(0\leqq i\leqq 3)$ as variables, we can express the norm form of $A(\Gamma)$ in the following way:

$$
\begin{aligned}
n_{A(\Gamma)}(\gamma) &= x_0^2+\mathrm{tr}\,(\alpha^2)x_0x_1+\mathrm{tr}\,(\beta^2)x_0x_2+\mathrm{tr}\,(\alpha^2\cdot\beta^2)x_0x_3 \\
&\quad +x_1^2+\mathrm{tr}\,(\alpha^2\cdot\beta^{-2})x_1x_2+\mathrm{tr}\,(\beta^2)x_1x_3+x_2^2+\mathrm{tr}\,(\alpha^2)x_2x_3+x_3^2 \\
&= \frac{1}{2}(x_0,\, x_1,\, x_2,\, x_3)\cdot D\cdot\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.
\end{aligned}
$$

Since $A(\Gamma)$ is a quaternion algebra over k_1, the norm form of $A(\Gamma)$ is non-degenerate. Hence $\det(D)$ is non-zero. This shows that D is contained in the group $GL_4(k_2)$. It follows that γ is contained in A_0 if and only if $\mathrm{tr}\,(\gamma)$, $\mathrm{tr}\,(\gamma\cdot\alpha^{-2})$, $\mathrm{tr}\,(\gamma\cdot\beta^{-2})$ and $\mathrm{tr}\,(\gamma\cdot\beta^{-2}\cdot\alpha^{-2})$ are all contained in k_2. q. e. d.

We shall show that $\beta^2\alpha^2$ is contained in A_0. By Lemma 2 $\mathrm{tr}\,(\beta^2\alpha^2)$ and $\mathrm{tr}\,(\beta^2\alpha^2\beta^{-2}\alpha^{-2})$ $(=\mathrm{tr}\,((\beta^2\alpha\beta^{-2})^2\cdot\alpha^{-2}))$ are contained in k_2. Applying Lemma 3 to $\beta^2\alpha^2$ we see that $\beta^2\alpha^2$ is contained in A_0. It follows from this that $\alpha^2\cdot\beta^2\alpha^2$ and $\beta^2\alpha^2\cdot\beta^2$ are also contained in A_0. Thus we have shown that A_0 is an algebra over k_2 such that

$$A_0\otimes_{k_2}k_1=A(\Gamma)\,.$$

It is clear by definition that A_0 is contained in $A(\Gamma^{(2)})$. Take any $\gamma\in\Gamma$. We shall show that γ^2 is contained in A_0. By Lemma 2 $\mathrm{tr}\,(\gamma^2)$, $\mathrm{tr}\,(\gamma^2\alpha^{-2})$ and $\mathrm{tr}\,(\gamma^2\beta^{-2})$ are all contained in k_2. Considering the assumption (5), by the equations

$$
\begin{aligned}
&\mathrm{tr}\,(\gamma^2\beta^{-2}\alpha^{-2})=\mathrm{tr}\,(\alpha^2\beta^2\gamma^{-2})=\mathrm{tr}\,(\alpha^2\beta^2\gamma^{-1})\,\mathrm{tr}\,(\gamma)-\mathrm{tr}\,(\alpha^2\beta^2)\,,\\
&(\mathrm{tr}\,(\alpha^2\beta^2\gamma))^2=(\mathrm{tr}\,(\alpha^2\beta^2)\,\mathrm{tr}\,(\gamma)-\mathrm{tr}\,(\alpha^2\beta^2\gamma^{-1}))^2\\
&\qquad=(\mathrm{tr}\,(\alpha^2\beta^2))^2(\mathrm{tr}\,(\gamma))^2+(\mathrm{tr}\,(\alpha^2\beta^2\gamma^{-1}))^2\\
&\qquad\quad-2\,\mathrm{tr}\,(\alpha^2\beta^2)\,\mathrm{tr}\,(\alpha^2\beta^2\gamma^{-1})\,\mathrm{tr}\,(\gamma)\,,
\end{aligned}
$$

we see that $\mathrm{tr}\,(\gamma^2\beta^{-2}\alpha^{-2})$ is contained in k_2. We can apply Lemma 3 to γ^2. Hence we see that γ^2 is contained in A_0. It follows that $\Gamma^{(2)}$ is contained in A_0. In particular, $\mathrm{tr}\,(\Gamma^{(2)})$ is contained in k_2. Therefore k_2' is contained in k_2. Thus we have shown that k_2' coincides with k_2. This completes the proof of Proposition 4. By the way since $A(\Gamma^{(2)})$ is a quaternion algebra over k_2' $(=k_2)$, we see that $A_0=A(\Gamma^{(2)})$. q. e. d.

PROPOSITION 5. *Let Γ be an arithmetic Fuchsian group commensurable with $\Gamma(A, O)$ where A and O are the same as in* § 1. *Then $k_2=Q((\mathrm{tr}\,(\gamma))^2|\gamma\in\Gamma)$ coincides with the center k of A and $A(\Gamma^{(2)})$ coincides with $\rho_1(A)$.*

PROOF. By the assumption there exists a subgroup Γ_1 of both Γ and $\Gamma(A, O)$ of finite index. By 2.1. § 2 we see that k coincides with the field

$\boldsymbol{Q}(\mathrm{tr}(\gamma)|\gamma\in\Gamma_1)$. Moreover by Proposition 1 we see that $A(\Gamma_1)=\rho_1(A)$. We may take Γ_1 as a normal subgroup of Γ. Take any $\gamma\in\Gamma$. Then γ induces an automorphism φ_γ of Γ_1 defined as follows:

$$\varphi_\gamma : \Gamma_1 \ni \alpha \longmapsto \gamma^{-1}\alpha\gamma \in \Gamma_1 .$$

φ_γ can be extended to an automorphism φ_γ of $A(\Gamma_1)=\rho_1(A)$ in a natural way which is the identity of the center $k\cdot 1_2$ of $\rho_1(A)$. By the Skolem-Noether's Theorem there exists an invertible element δ_0 of A such that for any $\alpha\in\rho_1(A)$ we have

$$\varphi_\gamma(\alpha)=\rho_1(\delta_0)^{-1}\cdot\alpha\cdot\rho_1(\delta_0) .$$

Since we have

$$\rho_1(A)\otimes_k\boldsymbol{R}\cong M_2(\boldsymbol{R}) ,$$

we have the expression

$$\gamma=a\cdot\rho_1(\delta_0) ,$$

where a is a non-zero real number. By the equation

$$1=\det(\gamma)=\alpha^2\det(\rho_1(\delta_0))=a^2 n_A(\delta_0) ,$$

a^2 is a non-zero element of k. Hence γ^2 is contained in $\rho_1(A)$. It follows that $\Gamma^{(2)}$ is contained in $\rho_1(A)$. Therefore, $A(\Gamma^{(2)})$ is contained in $\rho_1(A)$ and k_2 is contained in k.

It is clear that

$$A(\Gamma^{(2)})\otimes_{k_2}k\cong\rho_1(A) .$$

By the assumption (1) of A, k_2 coincides with k and that $A(\Gamma^{(2)})=\rho_1(A)$. This completes the proof of Proposition 5.

We shall show that (I) and (II$_1$) are necessary conditions. Let Γ be an arithmetic Fuchsian group commensurable with $\Gamma(A, O)$. Take any $\gamma\in\Gamma$. Then γ^m is contained in $\rho_1(O)$ for some positive integer m. Let u and $1/u$ be the eigen-values of γ. Then u^m and $1/u^m$ are the eigen-values of γ^m. Since $\mathrm{tr}(\gamma^m)$ is contained in O_k, u^m and $1/u^m$ are algebraic integers. Hence u and $1/u$ are also algebraic integers. It follows that $\mathrm{tr}(\gamma)$ is contained in O_{k_1}. This shows that Γ satisfies the condition (I).

Let φ be any isomorphism of k_1 into $\boldsymbol{C}$ such that $\varphi|_{k_2}\neq$ the identity. Then by Proposition 5 k_2 coincides with k and hence $\varphi|_{k_2}=\varphi_i$ for some i $(2\leqq i\leqq n)$. Extend φ to an isomorphism ψ of $k_1(u)$ in to $\boldsymbol{C}$. Since γ^m belongs to $\rho_1(A)$, $\varphi(\mathrm{tr}(\gamma^m))$ is contained in the interval $[-2, 2]$. Since $\psi(u^m)$ and $1/\psi(u^m)$ are the roots of the equation

$$x^2-\psi(\mathrm{tr}(\gamma^m))x+1=0 ,$$

we have $|\psi(u^m)|=1$. Hence we have $|\psi(u)|=1$. It follows from the equations

$$\varphi(\mathrm{tr}(\gamma))=\phi(u)+1/\phi(u)=\phi(u)+\overline{\phi(u)}$$

that $\varphi(\mathrm{tr}(\gamma))$ is contained in the interval $[-2, 2]$. This shows that $\varphi(\mathrm{tr}(\Gamma))$ is bounded. Therefore, Γ satisfies the condition (II_1).

3.2. Sufficiency of the conditions (I) and (II_1).

Let Γ be a Fuchsian group of the first kind satisfying the conditions (I) and (II_1). By Proposition 4 we see easily that $\Gamma^{(2)}$ satisfies the conditions (I) and (II_2) in Theorem 2. By Theorem 2 $\Gamma^{(2)}$ is a Fuchsian group derived from a quaternion algebra. Since $\Gamma^{(2)}$ is a subgroup of Γ of finite index, Γ is an arithmetic Fuchsian group. This completes the proof of Theorem 1.

REMARK. In view of the proof of Theorem 1 Γ is an arithmetic Fuchsian group if and only if $\Gamma^{(2)}$ is a Fuchsian group derived from a quaternion algebra.

§ 4. Independency of the conditions (I) and (II_1).

In this section we shall show that the conditions (I) and (II_1) in our Theorem are independent of each other. First we shall give an example of a Fuchsian group which satisfies the condition (I) but does not satisfy the condition (II_1).

For any rational integer q such that $q \geqq 7$ put $\lambda=2\cos(\pi/q)$. Then the field $k_\lambda=\boldsymbol{Q}(\lambda)$ is a totally real algebraic number field of degree $1/2\cdot\varphi(2q)$, where $\varphi(\ )$ is the Euler function. Let $\Gamma(\lambda)$ be the subgroup of $SL_2(\boldsymbol{R})$ generated by the following two elements:

$$S=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T_\lambda=\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}.$$

$\Gamma(\lambda)$ is introduced by E. Hecke in [3] and is shown to be a Fuchsian group of the first kind. It is easy to see that $k_\lambda=\boldsymbol{Q}(\mathrm{tr}(\gamma)\,|\,\gamma\in\Gamma(\lambda))$. Since λ is contained in O_{k_λ}, $\Gamma(\lambda)$ is a subgroup of $SL_2(O_{k_\lambda})$. Therefore, we have $\mathrm{tr}(\Gamma(\lambda))\subset O_{k_\lambda}$. It follows that $\Gamma(\lambda)$ satisfies the condition (I). Since $\Gamma^{(2)}(\lambda)$ contains $(ST_\lambda)^2T_\lambda^{2m}$ for any rational integer m, we see that $\Gamma(\lambda)$ does not satisfy the condition (II_1).

Now we shall construct a Fuchsian group which satisfies the condition (II_1) but does not satisfy the condition (I). For this purpose we make use of the arithmetic Fuchsian group $\Gamma(A, O)$ defined in § 1. Let k, A, O and $\Gamma(A, O)$ be the same as in § 1. We assume that $k\neq\boldsymbol{Q}$. Then A is a division quaternion algebra over k. Hence $H/\Gamma(A, O)$ is compact (cf. e.g. [2]). It follows that $\Gamma(A, O)$ does not contain any parabolic elements. Since $\Gamma(A, O)$ is a finitely generated subgroup of $SL_2(\boldsymbol{R})$, by Lemma 8 in [4] there exists a torsion-free subgroup Γ of $\Gamma(A, O)$ of finite index. It follows that Γ is generated by $2g$ $(g\geqq 2)$ hyperbolic elements

$$\{\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_g, \beta_g\},$$

which satisfy the unique fundamental relation

$$\alpha_1\cdot\beta_1\cdot\alpha_1^{-1}\cdot\beta_1^{-1}\cdots\alpha_g\cdot\beta_g\cdot\alpha_g^{-1}\cdot\beta_g^{-1}=I_2.$$

By considering a suitable conjugate group instead of Γ, we may assume that

$$\beta_1=\begin{pmatrix} w & 0 \\ 0 & w' \end{pmatrix}, \qquad \alpha_1=\begin{pmatrix} a_1 & b_1 \\ c_1b_1' & a_1' \end{pmatrix} \quad (k \ni c_1 \neq 0),$$

and that

$$A(\Gamma)=A=\left\{\begin{pmatrix} a & b \\ c_1b' & a' \end{pmatrix} \,\middle|\, a, b \in K\right\}.$$

For any non-zero real number u we put

$$\alpha(u)=\alpha_1\cdot\begin{pmatrix} u & 0 \\ 0 & 1/u \end{pmatrix}.$$

Then

$$\lim_{u\to 1}\alpha(u)=\alpha(1)=\alpha_1.$$

Let Γ_u be the subgroup of $SL_2(\boldsymbol{R})$ generated by $2g$ elements

$$\{\alpha(u), \beta_1, \alpha_2, \beta_2, \cdots, \alpha_g, \beta_g\},$$

which satisfy the following relation:

$$\alpha(u)\cdot\beta_1\alpha(u)^{-1}\cdot\beta_1^{-1}\cdot\alpha_2\beta_2\cdot\alpha_2^{-1}\cdot\beta_2^{-1}\cdots\alpha_g\cdot\beta_g\cdot\alpha_g^{-1}\cdot\beta_g^{-1}=I_2.$$

Since H/Γ is compact, we can apply to Γ the theory of small deformations which is proved by A. Weil in [**6**]. Therefore, there exists a neighbourhood V of 1 in $\boldsymbol{R}$ such that for any u in V Γ_u is a Fuchsian group of the first kind.

Now we impose on Γ_u the following condition:

$$K \ni u \quad \text{and} \quad uu'=1.$$

Then Γ_u is contained in A. It follows that Γ_u satisfies the condition (II_1).

By the relation $\mathrm{tr}(\alpha(u))=\mathrm{tr}_{K/k}(a_1u)$, where $\mathrm{tr}_{K/k}(\ )$ means the trace map of K to k, if $\mathrm{tr}_{K/k}(a_1u)$ is not contained in O_k, then Γ_u does not satisfy the condition (I). We need the following

LEMMA 4. *There exists a sequence* $\{u_m\}$ *which satisfies the following conditions:*

(i) u_m *is contained in* K *and* $u_m\cdot u_m'=1$,

(ii) $\lim_{m\to\infty} u_m=1$,

(iii) $\mathrm{tr}_{K/k}(u_m)$ *is not contained in* O_k.

PROOF. Let u be an element of K such that $u\cdot u'=1$. Then by Hilbert's Theorem 90 we can find an element v of K such that $u=v/v'$. Put

$$d_1=(\mathrm{tr}\,(\beta_1))^2-4=(w-w')^2\,.$$

Then $K=k(w)=k(\sqrt{d_1})$. Since v can be expressed as follows:

$$v=(1+\sqrt{d_1}\,x)y\,,$$

where x and y are elements of k, we have

$$u=\frac{1+d_1x^2+2x\sqrt{d_1}}{1-d_1x^2}\,. \tag{6}$$

Since $\mathrm{tr}_{K/k}(a_1)=\mathrm{tr}\,(\alpha_1)$ is contained in O_k, $\mathrm{tr}_{K/k}(a_1u)$ is contained in O_k if and only if $\mathrm{tr}_{K/k}(a_1(u+1))$ is so. By (6), we have

$$\mathrm{tr}_{K/k}(a_1(u+1))=\frac{2\,\mathrm{tr}_{K/k}(a_1)+2(a_1-a_1')\sqrt{d_1}\cdot x}{1-d_1\cdot x^2}\,. \tag{7}$$

Since k is a totally real algebraic number field of degree $n\geqq 2$, we can find an element x_0 of O_k such that $0<|x_0|<1$. For any positive integer m, put

$$u_m=\frac{1+d_1x_0^{2m}+2x_0^m\cdot\sqrt{d_1}}{1-d_1x_0^{2m}}\,.$$

Then we see easily that $\{u_m\}$ satisfies the conditions (i) and (ii).

Since $n_{k/\boldsymbol{Q}}(x_0)$ is a non-zero rational integer, there exists an index i such that $|\varphi_i(x_0)|>1$. Therefore, we have

$$\begin{aligned}\lim_{m\to\infty} n_{k/\boldsymbol{Q}}(\mathrm{tr}_{K/k}(a_1(u_m+1)))&=2^n\lim_{m\to\infty}\prod_{i=1}^{n}\frac{\varphi_i(\mathrm{tr}_{K/k}(a_1))+\varphi_i((a_1-a_1')\sqrt{d_1})\varphi_i(x_0)^m}{1-\varphi_i(d_1)\cdot\varphi_i(x_0)^{2m}}\\&=0\,.\end{aligned}$$

On the other hand, by (7) we have

$$\lim_{m\to\infty}\mathrm{tr}_{K/k}(a_1(u_m+1))=2\,\mathrm{tr}_{K/k}(a_1)\neq 0\,.$$

This implies that for any sufficiently large m, $\mathrm{tr}_{K/k}(a_1u_m)$ is not contained in O_k. This completes the proof of Lemma 4.

By this lemma we can give an example of a Fuchsian group which satisfies the condition (II_1) but does not satisfy the condition (I).

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