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On embeddings of spaces into ANR and shapes

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§ 1. Introduction.

Shapes of compact metric spaces were introduced by K. Borsuk [3]. He generalized in [2] and [4] this concept to general metric spaces by defining weak shapes and positions. The notions of shapes or weak shapes of spaces give classifications of spaces coarser than homotopy type and they are determined by circumstance under which the space is embedded into an AR as a closed set. In this paper we shall show that a given metric space X is embedded into an AR with a convenient structure for investigating shape theoretical properties of X. By making use of this embedding, for a locally compact metric space X, it is shown that there is a locally compact Δ-space whose weak shape is equal to X's. In case X is compact this fact has been proved in [12] by Mardešić-Segal approach to shape [13]. However the compactness of a space is essential in Mardešić-Segal approach and we can not use it for our case. The concept of fundamental skeletons of a space is introduced. Every Δ-space has fundamental skeletons, but it is known that there is an AR which does not have fundamental skeletons. Finally a partial answer to a problem concerning position raised by Borsuk [4] is given.

Throughout this paper all of spaces are metric and maps are continuous. By an AR and an ANR we mean always an AR for metric spaces and an ANR for metric spaces, and by dimension we imply the covering dimension.

§ 2. Embedding of spaces into ANR.

Let X and Y be metric spaces and let $f: X \rightarrow Y$ be a continuous map. We define a *metrizable mapping cylinder* $M(X, Y, f)$ as follows. It is obtained by identifying points $(x, 1) \in X \times \{1\} \subset X \times I$ and $f(x) \in Y$ for $x \in X$ in a topological sum $X \times I \cup Y$, where $I = [0, 1]$. Let $p: X \times I \cup Y \rightarrow M(X, Y, f)$ be a quotient map. We denote $p(x, t)$ for $(x, t) \in X \times I$ by (x, t) and $p(y)$ for $y \in Y$ by y simply. We consider X and Y as subsets of $M(X, Y, f)$ (X is identified with the set $\{(x, 0): x \in X\}$). We give $M(X, Y, f)$ the following topology. A point (x, t), $x \in X$ and $0 \leqq t < 1$, has a neighborhood system consisting of all sets of the form $U \times V$, where U and V range over neighborhoods of x and t

in X and $[0, 1)=\{t: 0 \leqq t<1\}$ respectively. For a point $y \in Y$, the collection $\{W \cup f^{-1}(W) \times\{t: 1/n<t<1\}: W$ is a neighborhood of y in Y and $n=1, 2, \cdots\}$ forms a neighborhood base at y in $M(X, Y, f)$. Obviously $M(X, Y, f)$ is metrizable and it contains X and Y as closed sets. If Y consists of one point, then we obtain a *metrizable cone* $C(X)$ over X. The following theorem is essentially due to Bothe [**5**].

THEOREM 1. *Let X be a finite dimensional metric space. Then there is an* ANR $M(X)$ *satisfying the following conditions.*

(1) *$M(X)$ contains X as a closed set.*

(2) *$w(M(X))=w(X)$, where $w(X)$ is the weight of X.*

(3) $\dim M(X)=\dim X+1$.

(4) *If X is complete, then $M(X)$ is complete.*

(5) *If X is locally compact, then $M(X)$ is locally compact.*

PROOF. Choose a sequence of locally finite open covers $\{\mathfrak{U}_n: n=1, 2, \cdots\}$ of X such that order of $\mathfrak{U}_n \leqq \dim X+1$, $\mathfrak{U}_{n+1} \overset{*}{>} \mathfrak{U}_n$ for each n and mesh $\mathfrak{U}_n \rightarrow 0$ $(n \rightarrow \infty)$. Here we mean by $\mathfrak{U}>\mathfrak{V}$ (resp. $\mathfrak{U} \overset{*}{>} \mathfrak{V}$) that $\mathfrak{U}$ is a refinement (resp. star refinement) of $\mathfrak{V}$. By K_n we denote the nerve of $\mathfrak{U}_n$ with metric topology. Take a vertex v of K_{n+1} and let V be the element of $\mathfrak{U}_{n+1}$ corresponding to v. Let $\sigma(v)$ be the closed simplex of K_n which is spanned by vertices corresponding to all elements of $\mathfrak{U}_n$ containing V. Map v to the barycenter of $\sigma(v)$. By extending linearly this map we obtain a map $\pi_{n n+1}: K_{n+1} \rightarrow K_n$ which is a simplicial map from K_{n+1} into the barycentric subdivision of K_n. The inverse sequence $\{K_n, \pi_{n n+1}\}$ is called a *barycentric system* on the sequence $\{\mathfrak{U}_n\}$ by Isbell [**8**]. The limit space $\varprojlim \{K_n\}$ is equal to a completion X^* of X (cf. [**14**, Theorem 14.4] and [**8**, Lemma 33]). Let $\mu_n: X^* \rightarrow K_n$ be the projection and put $\pi_n=\mu_n | X$, $n=1, 2, \cdots$. By $M(K_{n+1}, K_n, \pi_{n n+1})$ denote a metrizable mapping cylinder. Consider a topological sum $N=\bigcup_{n=1}^{\infty} M(K_{n+1}, K_n, \pi_{n n+1})$. For each n, by identifying $K_{n+1} \times\{0\}$ of $M(K_{n+1}, K_n, \pi_{n n+1})$ and K_{n+1} of $M(K_{n+2}, K_{n+1}, \pi_{n+1 n+2})$ in N we obtain a metrizable space M in which each $M(K_{n+1}, K_n, \pi_{n n+1})$ has a proper topology as a closed set. Since $\pi_{n n+1}$ is piecewise linear, M is a cell complex. Put $M(X)=M \cup X$. Give $M(X)$ the following topology: M is open in $M(X)$ and has its proper topology. Take $x \in X$. For $n=1, 2, \cdots$, let V be an open star containing $\pi_n(x)$ in K_n. For $m>n$, consider an open set $(\pi_{n m})^{-1} V \times[0, 1)$ of $M(K_{m+1}, K_m, \pi_{m m+1})$, where $\pi_{n m}=\pi_{n n+1} \cdots \pi_{m-1 m}$. The collection of the sets of the form $(\pi_n^{-1}(V) \cap X) \cup \bigcup_{m=n+1}^{\infty}(\pi_{n m})^{-1} V \times[0, 1)$, where V ranges over open stars containing $\pi_n(x)$ in K_n, $n=1, 2, \cdots$, forms a neighborhood base of x in $M(X)$. Obviously $M(X)$ is a metrizable space with $\dim M(X)=\dim X+1$ and contains X as a closed set. For each n, let M_n be the subspace

$\bigcup_{m=1}^{n+1} M(K_{m+1}, K_m, \pi_{m\,m+1})$ of $M(X)$, where $M_1=K_1$. Define $\nu_n: M(X)\to M_n$ by putting

$$\nu_n(x)=\pi_n(x)\,, \qquad x\in X\,,$$

$$\nu_n(x, t)=\pi_{n\,m+1}(x)\,, \qquad (x, t)\in M(K_{m+1}, K_m, \pi_{m\,m+1})\,, \quad m\geqq n\,,$$

$$\nu_n(x, t)=(x, t)\,, \qquad (x, t)\in M(K_{m+1}, K_m, \pi_{m\,m+1})\,, \quad m<n\,.$$

Obviously ν_n is continuous. Let U be an open set of M_n and let $W=\nu_n^{-1}(U)$. It is easy to show that U is a strong deformation retract of W. Thus we can know that $M(X)$ is locally contractible. Since $M(X)$ is finite dimensional, $M(X)$ is an ANR by [**9**, Theorem 1]. If X is complete, then we can choose a sequece of covers $\{\mathfrak{U}_n\}$ used in the construction of $M(X)$ such that X is equal to $X^*=\underleftarrow{\lim}\{K_n\}$. It is easy to know that $M(X)$ has a complete $\aleph_0$ system of open coverings in the sense of Frolik. From [**6**, Theorem 2.4] follows the completeness of $M(X)$. Finally, let X be locally compact. If we choose a locally finite open cover $\mathfrak{U}_n$, $n=1, 2, \cdots$, such that each member of $\mathfrak{U}_n$ has a compact closure, then $X=\underleftarrow{\lim}\{K_n\}$ and $M(X)$ is locally compact. By the construction of $M(X)$ it is obvious that $w(M(X))=w(X)$. This completes the proof.

If we construct a metrizable cone $C(K_1)$ over the subset K_1 of $M(X)$, then the union $M(X)\cup C(K_1)$ is an AR. Hence we have

COROLLARY 1. *For every finite dimensional metric space X there is an* AR *$M(X)$ satisfying the conditions* (1)-(4) *in Theorem* 1.

Let Y be a discrete space consisting of uncountable points. Then there does not exist a locally compact AR containing Y. Hence we can not strengthen Corollary 1 by replacing (1)-(4) by (1)-(5).

COROLLARY 2. *Let τ be an infinite cardinal number. For each $n=0, 1, 2, \cdots$, there is an* AR *$A(\tau, n)$ with $w(A(\tau, n))=\tau$ and* $\dim A(\tau, n)=n+1$ *such that if X is a metric space with $w(X)\leqq\tau$ and* $\dim X\leqq n$ *then X is embedded into $A(\tau, n)$.*

This is a consequence of Nagata [**15**] and Corollary 1.

§ 3. Fundamental skeletons and Δ-spaces.

Let X be a space and $n=0, 1, 2, \cdots$. K. Borsuk [**1**] introduced the concept of homological and homotopical n-skeletons of X. As a shape theoretical modification of it we introduce the following concept (see [**10**, p. 44]).

DEFINITION 1. Let X be a space and $n=0, 1, 2, \cdots$. By a *fundamental n-skeleton X^n* of X we mean a subset of X satisfying the following conditions:

(i) X^n is a closed subset of X with $\dim X^n\leqq n$.

(ii) If $x_0\in X^n$ and $i:(X^n, x_0)\to(X, x_0)$ is the inclusion map, then the induced homomorphisms i_* of $\check{H}^c_k(X^n : G)$ into $\check{H}^c_k(X : G)$ and of $\underline{\pi}_k(X^n, x_0)$ into $\underline{\pi}_k(X, x_0)$ are isomorphisms for $0 \leqq k < n$ and epimorphisms for $k = n$ respectively. Here $\check{H}^c_*$ is the Čech homology group with compact carriers, G is an arbitrary abelian group and $\underline{\pi}_*$ is the fundamental group defined in [**3**, § 32].

The n-skeleton of a simplicial complex is a fundamental n-skeleton of it. For every continuum X, every closed 0-dimensional subset of X is its fundamental 0-skeleton. If X is totally disconnected and $\dim X > 0$, then there is no fundamental 0-skeleton of X.

EXAMPLE 1. Let Y be a solenoid of Van Dantzig. Then Y has a fundamental 0-skeleton Y^0 such that Y^0 is homeomorphic to a Cantor discontinuum and the quotient space Y/Y^0 is arcwise connected.

EXAMPLE 2. It is known that every compact ANR has a fundamental k-skeleton for $k=0, 1$. However there is a compact ANR which has no fundamental k-skeleton for each $k=2, 3, \cdots$. Such an ANR X is given by a modification of the example constructed by Borsuk [**1**]. Consider a 2-sphere S^2. Let A be an arc in S^2. Take a map f from A onto the Hilbert cube Q. Let X be the adjunction space obtained by S^2, Q and f. Obviously X does not have any fundamental k-skeletons for $k \geqq 2$.

EXAMPLE 3. Consider the continua M_R, M_{R_p}, M_{Z_p} and M_{Q_p} constructed in [**14**, Appendix, pp. 228-230]. Each of them does not have any fundamental 1-skeleton, because any open set in it contains a 1-sphere which represents non homologous cycle.

DEFINITION 2. A metric space X is said to be a *Δ-space* if there is an inverse sequence $\{K_n, \pi_{n\,n+1}\}$ consisting of simplicial complexes K_n with metric topology and simplicial maps $\pi_{n\,n+1} : K_{n+1} \to K_n$ whose limit space is homeomorphic to X (cf. [**10**] and [**12**]).

Every polytope is a Δ-space. As known in [**12**, Theorem 1] there is a 1-dimensional compact AR which is not a Δ-space. For examples given above, it is known that any solenoid of Van Dantzig is a Δ-space but each of continua M_R, M_{R_p}, M_{Z_p} and M_{Q_p} and the AR in Example 2 are not.

THEOREM 2. *Let X be a finite dimensional locally compact metric space. Then there is a locally compact Δ-space Y such that $Sh_W(X) = Sh_W(Y)$ and* $\dim X = \dim Y$.

Here $Sh_W(X)$ is the weak shape of X defined by K. Borsuk (see for definition [**2**, p. 79] and [**4**, § 5]). In case X is compact, the theorem has been proved in [**12**] by using Mardešić-Segal approach to shape [**13**]. In this approach by means of ANR sequences, note that the compactness of a space is essential.

PROOF OF THEOREM 2. We shall make use of an AR $M(X)$ constructed in the proofs of Theorem 1 and Corollary 1. Let $\mathfrak{U}_n$, $n=1, 2, \cdots$, be a sequence of locally finite open covers of X such that each member of $\mathfrak{U}_n$ has a compact closure, order of $\mathfrak{U}_n \leqq \dim X+1$, $\mathfrak{U}_{n+1} \overset{*}{>} \mathfrak{U}_n$ for each n and mesh of $\mathfrak{U}_n \to 0$ $(n \to \infty)$. Let K_n be the nerve of $\mathfrak{U}_n$ and let $\pi_{n\,n+1}$ be a simplicial map of K_{n+1} into the barycentric subdivision of K_n such that $\{K_n, \pi_{n\,n+1}\}$ forms an inverse sequence whose limit is X (see the proof of Theorem 1). Then $M(X)$ is a union of X, metrizable mapping cylinders $M(K_{n+1}, K_n, \pi_{n\,n+1})$, $n=1, 2, \cdots$, and a metrizable cone $C(K_1)$ over K_1. Construct an inverse sequence $\{K'_n, \mu_{n\,n+1}\}$ as follows. Put $K'_n = K_n$, $n=1, 2, \cdots$. Let $\mu_{n\,n+1}: K_{n+1} \to K_n$ be a natural simplicial projection, that is, a vertex v of K_{n+1} corresponding to an element $V \in \mathfrak{U}_{n+1}$ is mapped by $\mu_{n\,n+1}$ to a vertex w corresponding to an element $W \in \mathfrak{U}_n$ containing V. Then two maps $\pi_{n\,n+1}$ and $\mu_{n\,n+1}$ of $K_{n+1} = K'_{n+1}$ into $K_n = K'_n$ are contiguous. Consider an inverse sequence $\{K'_n, \mu_{n\,n+1}\}$ and put $Y = \varprojlim \{K'_n\}$. It is easy to know that Y is a locally compact Δ-space and $\dim Y = \dim X$. By the same argument as in the construction of $M(X)$ we can construct an AR $M(Y)$ which is a union of Y, metrizable mapping cylinders $M(K'_{n+1}, K'_n, \mu_{n\,n+1})$, $n=1, 2, \cdots$, and a metrizable cone $C(K'_1)$. For each $n=1, 2, \cdots$, let $M^X_n = \bigcup_{m=1}^{n-1} M(K_{m+1}, K_m, \pi_{m\,m+1}) \cup C(K_1)$ and $M^Y_n = \bigcup_{m=1}^{n-1} M(K'_{m+1}, K'_m, \mu_{m\,m+1}) \cup C(K'_1)$. By $\nu^X_n: M(X) \to M^X_n$ and $\nu^Y_n: M(Y) \to M^Y_n$ denote the strong deformation retractions constructed in the proof of Theorem 1. By local compactness of X and Y each of ν^X_n and ν^Y_n is a perfect map. We define maps $f_n: M(X) \to M(Y)$ and $g_n: M(Y) \to M(X)$, $n=1, 2, \cdots$, as follows. Let $f'_n: C(K_1) \cup \bigcup_{k=2}^{n} K_2 \to C(K'_1) \cup \bigcup_{k=2}^{n} K'_k \subset M(Y)$ be the identity map. For $k=1, 2, \cdots, n-1$, let us extend f'_n over $M(K_{k+1}, K_k, \pi_{k\,k+1})$. Since maps $\pi_{k\,k+1}$ and $\mu_{k\,k+1}$ are contiguous, there is a homotopy $H: K_{k+1} \times I \to K'_k$ defined by $H(x, t) = t \cdot \pi_{k\,k+1}(x) + (1-t) \cdot \mu_{k\,k+1}(x)$ for $(x, t) \in K_{k+1} \times I$. Define f'_n on $M(K_{k+1}, K_k, \pi_{k\,k+1})$, $k=1, \cdots, n-1$, by

$$f'_n(x, t) = (x, 2t), \quad x \in K_{k+1} \text{ and } 0 \leqq t \leqq 1/2,$$

$$f'_n(x, t) = H(x, 2t-1), \quad x \in K_{k+1} \text{ and } 1/2 \leqq t \leqq 1.$$

We obtain a continuous map $f'_n: M^X_n \to M(Y)$. Let $f_n = f'_n \nu^X_n: M(X) \to M(Y)$. Similarly let us define $g'_n: M^Y_n \to M(X)$ by

$$g'_n | C(K'_1) \cup \bigcup_{k=2}^{n} K'_k = \text{the identity},$$

$$g'_n(x, t) = (x, 2t), \quad x \in K'_{n+1}, \quad 0 \leqq t \leqq 1/2, \ k=1, \cdots, n-1,$$

$$g'_n(x, t) = (2t-1) \cdot \mu_{k\,k+1}(x) + (2-2t) \cdot \pi_{k\,k+1}(x), \quad x \in K'_{k+1},$$

$$1/2 \leqq t \leqq 1, \ k=1, \cdots, n-1,$$

and let $g_n = g'_n \nu^Y_n: M(Y) \to M(X)$. We obtain sequences of maps $\underline{f} = \{f_n\}: M(X)$

$\rightarrow M(Y)$ and $\underset{\sim}{g}=\{g_n\}: M(Y)\rightarrow M(X)$. Let F be a compact set of X. Let H_n be a finite subcomplex of K_n consisting of all closed simplexes intersecting $\nu_n^X(F)$ and put $F'=Y\cap(\nu_n^Y)^{-1}f_n(H_n)$, where n is any positive integer. Then it is easy to see that F' is a compact set of Y which is $\underset{\sim}{f}$-assigned to F (see [**4**, p. 142]). Similarly, for a compact set F' of Y, if H_n' is a finite subcomplex of K_n' consisting of closed simplexes intersecting $\nu_n^Y(F')$ and we put $F=X\cap(\nu_n^X)^{-1}g_n(H_n')$, then F is a compact set of X being $\underset{\sim}{g}$-assigned to F'. By the definitions of $\underset{\sim}{f}$ and $\underset{\sim}{g}$, since $g_nf_n\cong\nu_n^X$ and $f_ng_n\cong\nu_n^Y$ for each n, it is easy to see that $\underset{\sim}{g}\cdot\underset{\sim}{f}\cong\underset{\sim}{i}_{X,M(X)}$ and $\underset{\sim}{f}\cdot\underset{\sim}{g}\cong\underset{\sim}{i}_{Y,M(Y)}$, where $\underset{\sim}{i}_{X,M(X)}$ and $i_{Y,M(Y)}$ are the identity W-sequences for X in $M(X)$ and for Y in $M(Y)$ respectively (see [**4**, § 2]). This completes the proof.

REMARK. If we use the same argument as in the proof of [**11**, Theorem 2], then it is known that $Sh(X)=Sh(Y)$ for the Δ-space Y constructed in Theorem 2. Here $Sh(X)$ means the shape of X defined by Fox [**7**].

COROLLARY 3. *Let X be an n-dimensional locally compact metric space. For every $m<n$, there is an n-dimensional Δ-space Z such that $\check{H}_k^c(X:G)\cong\check{H}_k^c(Z:G)$ for $k>m+1$ and $\check{H}_k^c(Z:G)=0$ for $k\leqq m$, where G is an arbitrary abelian group and $\check{H}_*^c$ is the reduced Čech homology group with compact carriers.*

PROOF. Let Y be an n-dimensional Δ-space such that $Sh_W(X)=Sh_W(Y)$. Let $\{K_k, \pi_{k\,k+1}\}$ be an inverse sequence consisting of simplicial complexes and simplicial bonding maps whose limit space is Y. Let $m<n$. Consider the inverse sequence $\{K_k^m, \pi_{k\,k+1}|K_{k+1}^m\}$, where K_k^m is the m-skeleton of K_k, and put $Y^m=\underset{\longleftarrow}{\lim}\{K_k^m\}$. Then it is easy to see that Y^m is a fundamental k-skeleton of Y. Let N_k be the union of K_k and a metrizable cone over K_k^m. Extend $\pi_{k\,k+1}: K_{k+1}\rightarrow K_k$ naturally to a simplicial map $\mu_{k\,k+1}: N_{k+1}\rightarrow N_k$. Consider the inverse sequence $\{N_k, \mu_{k\,k+1}\}$ and put $Z=\underset{\longleftarrow}{\lim}\{N_k\}$. Obviously Z is an n-dimensional Δ-space satisfying the conditions of the corollary.

EXAMPLE 4. We can not remove the local compactness of X in Theorem 2. Let X be the set of all rational numbers in a real line. If Y is a 0-dimensional metric space such that $Sh_W(Y)=Sh_W(X)$, then X and Y are homeomorphic by [**11**, Theorem 1]. Therefore such a space Y is not completely metrizable. Since every finite dimensional Δ-space is completely metrizable, Y is not a Δ-space.

Finally, we shall give a partial answer to a problem [**4**, (8.8)] raised by K. Borsuk.

THEOREM 3. *Let X and Y be finite dimensional metric spaces. Suppose that there exist sequences of compact sets $\{A_k\}$ and $\{B_k\}$ of X and Y and a sequence of onto homeomorphisms $\{f_k\}$, $f_k: X\rightarrow Y$, satisfying the conditions;*

(1) $A_{k+1}\subset \operatorname{Int} A_k$ *and* $B_{k+1}\subset\operatorname{Int} B_k$, $k=1, 2, \cdots,$

(2) $f_k(A_k)=B_k$, $k=1, 2, \cdots$,

(3) $f_k|A_k \cong f_{k'}|A_k$ rel. Bd A_k *in* B_k, *for every* $k \leqq k'$,

(4) $f_k|(X\backslash A_k)=f_{k'}|(X\backslash A_k)$ *for every* $k \leqq k'$,

where Int A *is the interior of* A *and* Bd A *is the boundary of* A. *Then* $\operatorname{Pos}(X, \bigcap_{k=1}^{\infty} A_k)=\operatorname{Pos}(Y, \bigcap_{k=1}^{\infty} B_k)$.

For the proof we need the following lemma.

LEMMA 4. *Let* X, Y *be finite dimensional metric spaces and let* $\{\mathfrak{U}_n^X\}$ *and* $\{\mathfrak{U}_n^Y\}$ *be sequences consisting of locally finite open covers of* X *and* Y *respectively. By* K_n^X *and* K_n^Y *denote the nerves of* $\mathfrak{U}_n^X$ *and* $\mathfrak{U}_n^Y$. *Let* $\pi_{n\,n+1}^X: K_{n+1}^X \to K_n^X$ *and* $\pi_{n\,n+1}^Y: K_{n+1}^Y \to K_n^Y$ *be piecewise linear maps constructed in the proof of Theorem* 1 *for* $n=1, 2, \cdots$. *Denote by* $M(X)$ *and* $M(Y)$ *ANR's constructed for the inverse sequences* $\{K_n^X, \pi_{n\,n+1}^X\}$ *and* $\{K_n^Y, \pi_{n\,n+1}^Y\}$ *and put* $X_n=X\cup\bigcup_{k=n}^{\infty} M(K_{k+1}^X, K_k^X, \pi_{k\,k+1}^X)$ *and* $Y_n=Y\cup\bigcup_{k=n}^{\infty} M(K_{k+1}^Y, K_k^Y, \pi_{k\,k+1}^Y)$, *where* $X_1=M(X)$ *and* $Y_1=M(Y)$. *Let* $f: X\to Y$ *be a map such that* $\mathfrak{U}_n^X > f^{-1}\mathfrak{U}_n^Y$, $n=1, 2, \cdots$. *Then* f *has an extension* $\tilde{f}: M(X)\to M(Y)$ *such that* $\tilde{f}(X_n)\to Y_n$ *for each* n. *Let* f *and* g *be homotopic maps and let* $\xi: X\times I\to Y$ *be a homotopy connecting* f *and* g. *Suppose that for each* n *there is an open cover* $\mathfrak{W}_n$ *of* I *such that* $\mathfrak{U}_n^X\times\mathfrak{W}_n > \xi^{-1}\mathfrak{U}_n^Y$. *Then there is a homotopy* $\tilde{\xi}: M(X)\times I\to M(Y)$ *such that* $\tilde{\xi}(x, 0)=\tilde{f}(x)$ *and* $\tilde{\xi}(x, 1)=\tilde{g}(x)$ *for* $x\in M(X)$ *and* $\tilde{\xi}(X_n\times I)\subset Y_n$ *for each* n.

PROOF. Since $\mathfrak{U}_n^X > f^{-1}\mathfrak{U}_n^Y$ for each n, there is a natural simplicial projection $\varphi_n: K_n^X\to K_n^Y$. Note that $\pi_{n\,n+1}^Y\varphi_{n+1}$ and $\varphi_n\pi_{n\,n+1}^X$ are contiguous. Hence we can define the map $\phi_n: M(K_{n+1}^X, K_n^X, \pi_{n\,n+1}^X)\to M(K_{n+1}^Y, K_n^Y, \pi_{n\,n+1}^Y)$ as follows:

$$\phi_n(x, t)=(\varphi_{n+1}(x), 2t), \quad x\in K_{n+1}^X \quad \text{and} \quad 0\leqq t\leqq 1/2,$$
$$\phi_n(x, t)=(2t-1)\cdot\varphi_n\pi_{n\,n+1}^X(x)+(2-2t)\cdot\pi_{n\,n+1}^Y\varphi_{n+1}(x), \quad x\in K_{n+1}^X \quad \text{and} \quad 1/2\leqq t\leqq 1,$$
$$\phi_n(x)=\varphi_n(x), \quad x\in K_n^X.$$

Let us define $\tilde{f}: M(X)\to M(Y)$ by $\tilde{f}|X=f$ and $\tilde{f}|M(K_{n+1}^X, K_n^X, \pi_{n\,n+1}^X)=\phi_n$, $n=1, 2, \cdots$. Obviously $\tilde{f}$ is a continuous extension of f and $\tilde{f}(X_n)\subset Y_n$ for each n. The second assertion is proved by the same argument as in the first part and we omit it.

PROOF OF THEOREM 3. By (3) and (4), for $k<k'$ there is a homotopy $\xi_{kk'}^X: X\times I\to Y$ such that

$$\xi_{kk'}^X(x, 0)=f_k(x) \quad \text{and} \quad \xi_{kk'}^X(x, 1)=f_{k'}(x) \qquad \text{for } x\in X,$$
$$\xi_{kk'}^X(x, t)=f_k(x)=f_{k'}(x) \qquad \text{for } x\in X\backslash A_k.$$

Since $f_k^{-1}|B_k\cong f_{k'}^{-1}|B_k$ rel. Bd B_k in A_k, there is a homotopy $\xi_{kk'}^Y: Y\times I\to X$ such that

$$\xi^Y_{kk'}(y, 0)=f_k^{-1}(y) \quad \text{and} \quad \xi^Y_{kk'}(y, 1)=f_{k'}^{-1}(y) \qquad \text{for} \quad y\in Y,$$

$$\xi^Y_{kk'}(y, t)=f_k^{-1}(y)=f_{k'}^{-1}(y) \qquad \text{for} \quad y\in Y\backslash B_k,\ k<k'.$$

Let $\mathfrak{U}^Y_1$ and $\mathfrak{U}^X_1$ be locally finite open covers of Y and X such that mesh $\mathfrak{U}^Y_1<1$, mesh $\mathfrak{U}^X_1<1$, order of $\mathfrak{U}^Y_1=$order of $\mathfrak{U}^X_1\leqq \dim X+1$ and $\mathfrak{U}^X_1>f_1^{-1}\mathfrak{U}^Y_1$. For $n=2$, choose locally finite open covers $\mathfrak{U}^X_2$ and $\mathfrak{U}^Y_2$ of X and Y as follows;

(5) $$\mathfrak{U}^Y_2\overset{*}{>}\mathfrak{U}^Y_1\wedge f_1\mathfrak{U}^X_1,\quad \mathfrak{U}^X_2\overset{*}{>}\mathfrak{U}^X_1\wedge f_1^{-1}\mathfrak{U}^Y_2\wedge f_2^{-1}\mathfrak{U}^Y_2 \text{ and for some}$$

$$\text{open cover } \mathfrak{W}_2 \text{ of } I\ \mathfrak{U}^X_2\times\mathfrak{W}_2\overset{*}{>}(\xi^X_{12})^{-1}\mathfrak{U}^Y_2.$$

By the compactness of A_1 we can find $\mathfrak{U}^X_2$ and $\mathfrak{W}_2$ in (5). By repeating this process we can find inductively sequences of locally open covers $\{\mathfrak{U}^X_n\}$ and $\{\mathfrak{U}^Y_n\}$ of X and Y satisfying the following conditions for $n=1, 2, \cdots$;

(6) $$\text{order of } \mathfrak{U}^X_n, \text{ order of } \mathfrak{U}^Y_n\leqq \dim X+1,$$

(7) $$\text{mesh } \mathfrak{U}^X_n, \text{ mesh } \mathfrak{U}^Y_n\to 0 \quad (n\to\infty),$$

(8) $$\mathfrak{U}^Y_{n+1}\overset{*}{>}\bigwedge_{i=1}^{n} f_i\mathfrak{U}^X_n\wedge\mathfrak{U}^Y_n \text{ and for some open cover } \mathfrak{W}'_{n+1} \text{ of } I$$

$$\mathfrak{U}^Y_{n+1}\times\mathfrak{W}'_{n+1}\overset{*}{>}\bigwedge_{i=1}^{n-1}(\xi^Y_{i\,i+1})^{-1}\mathfrak{U}^X_n,$$

(9) $$\mathfrak{U}^X_{n+1}\overset{*}{>}\bigwedge_{i=1}^{n+1} f_i^{-1}\mathfrak{U}^Y_{n+1}\wedge\mathfrak{U}^X_n \text{ and for some open cover } \mathfrak{W}_{n+1} \text{ of } I$$

$$\mathfrak{U}^X_{n+1}\times\mathfrak{W}_{n+1}\overset{*}{>}\bigwedge_{i=1}^{n}(\xi^X_{i\,i+1})^{-1}\mathfrak{U}^Y_{n+1}.$$

Let K^X_n and K^Y_n, $n=1, 2, \cdots$, be the nerves of $\mathfrak{U}^X_n$ and $\mathfrak{U}^Y_n$, and let $\pi^X_{n\,n+1}$: $K^X_{n+1}\to K^X_n$ be a piecewise linear map constructed in the proof of Theorem 1. Similarly, define a piecewise linear map $\pi^Y_{n\,n+1}: K^Y_{n+1}\to K^Y_n$. Then $\{K^X_n, \pi^X_{n\,n+1}\}$ and $\{K^Y_n, \pi^Y_{n\,n+1}\}$ are barycentric systems on $\{\mathfrak{U}^X_n\}$ and $\{\mathfrak{U}^Y_n\}$ respectively. As in the proof of Theorem 1, construct AR's $M(X)$ and $M(Y)$ for $\{K^X_n\}$ and $\{K^Y_n\}$, namely, $M(X)=X\cup C(K^X_1)\cup\bigcup_{n=1}^{\infty}M(K^X_{n+1}, K^X_n, \pi^X_{n\,n+1})$ and $M(Y)=Y\cup C(K^Y_1)$ $\cup\bigcup_{n=1}^{\infty}M(K^Y_{n+1}, K^Y_n, \pi^Y_{n\,n+1})$. For each $k=1, 2, \cdots$, let $C^X(n, k)$ and $D^X(n, k)$ be the subcomplexes of K^X_n spanned by vertices corresponding to elements of $\mathfrak{U}^X_n$ intersecting A_k and $X\backslash A_k$ respectively. Similarly let $C^Y(n, k)$ and $D^Y(n, k)$ be the subcomplexes of K^Y_n for B_k and $Y\backslash B_k$. Put $E^X(n, k)=C^X(n, k)\cap D^X(n, k)$ and $E^Y(n, k)=C^Y(n, k)\cap D^Y(n, k)$. Then for each i and k $\pi^X_{i\,i+1}(C^X(i+1, k))\subset C^X(i, k)$ and $\pi^Y_{i\,i+1}(C^Y(i+1, k))\subset C^Y(i, k)$. For each n and k we put

$$F^X(n,k)=A_k\cup\bigcup_{i=n}^{\infty}M(C^X(i+1,k),\ C^X(i,k),\ \pi^X_{i\,i+1}),$$

$$G^X(n,k)=\overline{X\backslash A_k}\cup\bigcup_{i=n}^{\infty}M(D^X(i+1,k),\ D^X(i,k),\ \pi^X_{i\,i+1}),$$

$$F^Y(n,k)=B_k\cup\bigcup_{i=n}^{\infty}M(C^Y(i+1,k),\ C^Y(i,k),\ \pi^Y_{i\,i+1}),$$

$$G^Y(n,k)=\overline{Y\backslash B_k}\cup\bigcup_{i=n}^{\infty}M(D^Y(i+1,k),\ D^Y(i,k),\ \pi^Y_{i\,i+1}).$$

Let $X_1=M(X)$, $Y_1=M(Y)$, $X_n=X\cup\bigcup_{i=n}^{\infty}M(K^X_{i+1},K^X_i,\pi^X_{i\,i+1})$ and $Y_n=Y\cup\bigcup_{i=n}^{\infty}M(K^Y_{i+1},K^Y_i,\pi^Y_{i\,i+1})$ for $n>1$. Then $F^X(n,k)$ and $G^X(n,k)$ are closed sets of X_n and $F^Y(n,k)$ and $G^Y(n,k)$ are closed sets of Y_n for each n and k. For $m\geqq n\geqq 1$, since $\mathfrak{U}^X_m>f_n^{-1}\mathfrak{U}^Y_m$ by (9), there is an extension $\varphi^X_n: X_n\to Y_n$ of $f_n: X\to Y$ which is given in the proof of Lemma 4. Also, for $m\geqq n>1$, since $\mathfrak{U}^Y_{m+1}>f_n\mathfrak{U}^X_m$ by (8), there is an extension $\varphi^Y_n: Y_n\to X_{n-1}$ of $f_{n-1}^{-1}: Y_n\to X$. From the definition of φ^X_n and φ^Y_n it follows that

$$\varphi^X_n(K^X_m)\subset K^Y_m,\qquad \varphi^Y_n(K^Y_m)\subset K^X_{m-1},$$

$$\varphi^X_n(M(K^X_{m+1},K^X_m,\pi^X_{m\,m+1}))\subset M(K^Y_{m+1},K^Y_m,\pi^Y_{m\,m+1})\qquad\text{for}\quad m\geqq n,$$

$$\varphi^Y_n(M(K^Y_{m+1},K^Y_m,\pi^Y_{m\,m+1}))\subset M(K^X_m,K^X_{m+1},\pi^X_{m-1\,m}).$$

For each $k<n$ we have

$$\begin{aligned}&\varphi^X_n(F^X(n,k))\subset F^Y(n,k),\\ &\varphi^X_n(G^X(n,k))\subset G^Y(n,k),\\ &\varphi^Y_n(F^Y(n,k))\subset F^X(n-1,k),\\ &\varphi^Y_n(G^Y(n,k))\subset G^X(n-1,k).\end{aligned}\tag{10}$$

From (8), (9) and Lemma 4 it follows that there is a homotopy $\mu^X_{n\,n+1}: X_{n+1}\times I\to Y_{n+1}$ connecting $\varphi^X_n|X_{n+1}$ and φ^X_{n+1} which extends $\xi^X_{n\,n+1}: X\times I\to Y$. Similarly we know that there is a homotopy $\mu^Y_{n\,n+1}: Y_{n+1}\times I\to X_n$ connecting $\varphi^Y_n|Y_{n+1}$ and φ^Y_{n+1} which extends $\xi^Y_{n\,n+1}: X\times I\to Y$ and from their definitions (cf. the proof of Lemma 4) the following relations hold:

$$\begin{aligned}&\mu^X_{n\,n+1}(F^X(n+1,k)\times I)\subset F^Y(n+1,k),\qquad k\leqq n,\\ &\mu^X_{n\,n+1}(G^X(n+1,k)\times I)\subset G^Y(n+1,k),\qquad k\leqq n,\\ &\mu^Y_{n\,n+1}(F^Y(n+1,k)\times I)\subset F^X(n,k),\qquad k<n,\\ &\mu^Y_{n\,n+1}(G^Y(n+1,k)\times I)\subset G^X(n,k),\qquad k<n.\end{aligned}\tag{11}$$

Consider the maps φ^X_{n+1} and $\varphi^X_n | X_{n+1}$ of X_{n+1} into Y_{n+1}, and φ^Y_{n+1} and $\varphi^Y_n | Y_{n+1}$ of Y_{n+1} into X_n. From (4), if necessary, by replacing $\{\mathfrak{U}^X_n\}$ and $\{\mathfrak{U}^Y_n\}$ by refinements, we can assume that

$$\text{(12)}\qquad \begin{aligned} &\varphi^X_{n+1}(x) = \varphi^X_n(x) = \mu^X_{n\,n+1}(x, t), \quad x \in D^X(n+1, n) \quad \text{and} \quad t \in I, \\ &\varphi^Y_{n+1}(y) = \varphi^Y_n(y) = \mu^Y_{n\,n+1}(y, t), \quad y \in D^Y(n+1, n-1) \quad \text{and} \quad t \in I. \end{aligned}$$

Let $A = \bigcap_{k=1}^{\infty} A_k$ and $B = \bigcap_{k=1}^{\infty} B_k$. Now, to prove $\mathrm{Pos}(X, A) = \mathrm{Pos}(Y, B)$, we have to find sequences of maps $\underline{a} = \{a_k, X, Y\}_{M(X), M(Y)}$ and $\underline{b} = \{b_k, Y, X\}_{M(Y), M(X)}$ such that

$$\text{(13)}\qquad \begin{aligned} &\underline{a}' = \{a_k, A, B\}_{M(X), M(Y)}, \qquad \underline{a}'' = \{a_k, X\backslash A, Y\backslash B\}_{M(X), M(Y)}, \\ &\underline{b}' = \{b_k, B, A\}_{M(Y), M(X)}, \qquad \underline{b}'' = \{b_k, Y\backslash B, X\backslash A\}_{M(Y), M(X)} \end{aligned}$$

are W-sequences and

$$\text{(14)}\qquad \begin{aligned} &\underline{b}'\underline{a}' \cong i_{A, M(X)}, \qquad \underline{b}''\underline{a}'' \cong \underline{i}_{(X, A), M(X)}, \\ &a'b' \cong i_{B, M(Y)}, \qquad \underline{a}''\underline{b}'' \cong \underline{i}_{(Y, B), M(Y)}. \end{aligned}$$

(See for notations [**4**, pp. 146, 147].) For $k=1$, let $a_1 = \varphi^X_1$ and let b_1 be an arbitrary map of $M(Y)$ into $M(X)$. For $k=2$, we define a_2 and b_2 as follows. Consider $\varphi^Y_2 : Y_2 \to X_1$. Since $X_1 = M(X)$ is an AR, there is an extension $b_2 : M(Y) \to M(X)$ of φ^Y_2. To construct $a_2 : M(X) \to M(Y)$, put $a_2 = \varphi^X_2$ on the set X_2. Consider the sets $D^X(2, 1) \subset K^X_2$, $D^X(1, 1) \subset K^X_1$ and the mapping cylinder $M(D^X(2, 1), D^X(1, 1), \pi^X_{12}) \subset G^X(1, 1)$. Since $\varphi^X_2(x) = \varphi^X_1(x) = a_1(x)$ for $x \in D^X(2, 1)$ by (12), we can put $a_2 = a_1$ on $M(D^X(2, 1), D^X(1, 1), \pi^X_{12})$. Consider the sets $T = C^X(2, 1) \cup M(E^X(2, 1), E^X(1, 1), \pi^X_{12}) \subset M(C^X(2, 1), C^X(1, 1), \pi^X_{12})$ and $S = M(C^Y(2, 1), C^Y(1, 1), \pi^Y_{12})$. By (12) we know $a_2 | T \cong a_1 | T$ rel. $M(E^X(2, 1), E^X(1, 1), \pi^X_{12})$ in S. Since $a_1 | T$ has an extension a_1 over $M(C^X(2, 1), C^X(1, 1), \pi^X_{12})$ and S is an ANR, by homotopy extension theorem $a_2 | T$ has an extension over $M(C^X(2, 1), C^X(1, 1), \pi^X_{12})$. Finally, since $M(Y)$ is an AR, we can extend a_2 to a map from $M(X)$ into $M(Y)$ which we denote by a_2 again. This completes the definition of a_2. Note that $a_2 | M(C^X(2, 1), C^X(1, 1), \pi^X_{12}) \cong a_1 | M(C^X(2, 1), C^X(1, 1), \pi^X_{12})$ rel. $M(E^X(2, 1), E^X(1, 1), \pi^X_{12})$ in $M(C^Y(2, 1), C^Y(1, 1), \pi^Y_{12})$ and as a consequence

$$\text{(15)}\qquad \begin{aligned} &a_2 | F^X(n, 1) \cong a_1 | F^X(n, 1) \quad \text{in} \quad F^Y(n, 1) \qquad \text{for each } n, \\ &a_2 | G^X(1, 1) = a_1 | G^X(1, 1). \end{aligned}$$

By repeating this process we can construct maps $a_k : M(X) \to M(Y)$ and $b_k : M(Y) \to M(X)$, $k = 3, 4, \cdots$, satisfying the following conditions for $n < k$;

$$\text{(16)}\qquad a_k | X_k = \varphi^X_k \quad \text{and} \quad b_k | Y_k = \varphi^Y_k,$$

$$a_k | G^X(n, n) = a_n | G^X(n, n), \tag{17}$$

$$b_k | G^Y(n, n) = b_n | G^Y(n, n), \tag{18}$$

$$a_k | F^X(n, n) \cong a_n | F^X(n, n) \quad \text{in} \quad F^Y(n, n), \tag{19}$$

$$b_k | F^Y(n, n) \cong b_n | F^Y(n, n) \quad \text{in} \quad F^X(n-1, n-1). \tag{20}$$

(a_k is defined as follows; on the set $X \cup \bigcup_{n=1}^{k} G^X(n, n)$ a_k is defined by (16) and (17), and on the set $\bigcup_{n=1}^{k-1} M(C^X(n+1, n), C^X(n, n), \pi^X_{n\,n+1})$ a_k is obtained from a_{k-1} by homotopy extension theorem; the definition of b_k is similar.)

Now it is immediate that $\underline{a} = \{a_k\}$ and $\underline{b} = \{b_k\}$ satisfy (13) and (14). To show that $\underline{a}'$ is a W-sequence, note that $\{F^X(n, n): n = 1, 2, \cdots\}$ and $\{F^Y(n, n): n = 1, 2, \cdots\}$ form neighborhood bases of A and B in $M(X)$ and in $M(Y)$ respectively. Then (19) shows that $\underline{a}'$ is a W-sequence. Also, that $\underline{a}''$ is a W-sequence follows from (17). Next, let us show that $\underline{a}'\underline{b}' \cong \underline{i}_{B,M(Y)}$. Consider the map $a_{n-1}b_n | F^Y(n, n): F^Y(n, n) \to F^Y(n-1, n-1)$. For $k > n$, note two maps $a_{n-1}b_n | K^Y_k$ and $\pi^Y_{k-1\,k}$ of K^Y_k into K^Y_{k-1} are contiguous. Let us define $\eta: Y_n \to Y_{n-1}$ by $\eta | B_n =$ the identity and $\eta(y, t) = (\pi^Y_{k\,k+1}(y), t)$ for $(y, t) \in M(K^Y_{k+1}, K^Y_k, \pi^Y_{k\,k+1})$, $k = n, n+1, \cdots$. Obviously $\eta | F^Y(n, n) \cong i$ in $F^Y(n-1, n-1)$, where i is the inclusion map of $F^Y(n, n)$ into $F^Y(n-1, n-1)$. Since $\eta | F^Y(n, n) \cong a_{n-1}b_n | F^Y(n, n)$ in $F^Y(n-1, n-1)$ by the contiguity of $\pi^Y_{k\,k+1}$ and $a_{n-1}b_n | K^Y_k$ for $n \leqq k$, we know that $a_{n-1}b_n | F^Y(n, n) \cong i$ in $F^Y(n-1, n-1)$. By this relation, (19) and (20), we can conclude $\underline{a}'\underline{b}' \cong \underline{i}_{B,M(Y)}$. The other assertions in (13) and (14) are proved similarly. This completes the proof.

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