J. Math. Soc. Japan
Vol. 27, No. 2, 1975

A remark on G. Glauberman's theorem

By Makoto HAYASHI

(Received Jan. 10, 1974)
(Revised May 13, 1974)

In his paper [2], Glauberman proved an interesting theorem for p-stability. The purpose of this note is to prove an immediate corollary of Glauberman's theorem. We introduce notation which will be used in this note. For a finite group H; H_p denotes a Sylow p-subgroup of H and $\pi(H:p)$ is the set of primes $\bigcup_P \pi(N_H(P)/C_H(P))$, where P ranges over all p-subgroups of H. Other notations are standard (see [3]).

THEOREM*. *Let G be a finite group. Then the following* (i), (ii), (iii) *are equivalent.*

(i) $G \triangleright G_2$.

(ii) $N_G(ZJ(G_p)) \triangleright N_G(ZJ(G_p))_2$, *for every odd prime* $p \in \pi(G)$.

(iii) $2 \notin \pi(N_G(ZJ(G_p)):p)$, *for every odd prime* $p \in \pi(G)$.

COROLLARY. *Let G be a finite group admitting a fixed-point-free automorphism ϕ of a prime power order p^n. Suppose that the order of $C_G(\phi^{p^{n-1}})$ is odd. Then G is solvable.*

PROOF OF THE THEOREM. (i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(iii) follow immediately from definition. To prove (iii)$\Rightarrow$(i), we assume by way of contradiction that G_2 is not normal in G. Then we have $2 \in \pi(G:p)$ for some odd prime in $\pi(G)$ by Baer's theorem (see [3] page 105). So the order of $N_G(W)/C_G(W)$ is divisible by 2 for some p-subgroup W of G. But the functor ZJ controls strong fusion in G_p with respect to G, because, otherwise, Corollary 2 of Theorem A of [2] would imply that $2 \in \pi(N_G(ZJ(G_p):p))$, contrary to our assumption (iii). Then for $g \in N_G(W)$, we have $g = cn$, $c \in N_G(W)$, $n \in N_G(ZJ(G_p))$. Then we have $n \in N_{N_G(ZJ(G_p))}(W)$, and n has the same action on W as g. So we have $2 \in \pi(N_G(ZJ(G_p)):p))$. This contradicts (iii), q.e.d.

PROOF OF THE COROLLARY. Let G be a minimal counter example group of the corollary. Then all ϕ-invariant proper subgroups of G are solvable.

*) Originally, to prove this theorem, I did not use Glauberman's theorem, but Thompson's N-group classification theorem. I would like to thank the referee for his several suggestions, including improvement in the proof of the theorem.

Let $N=N_G(ZJ(G_p))$ be a ϕ-invariant subgroup of G, p odd. Suppose $G=N$, then we have $G/O_p(G)$ is solvable by induction. Hence G is solvable, which is a contradiction. Thus N is a ϕ-invariant proper subgroup of G. Since N is solvable, we have $N=F(N)C_N(\phi^{p^{n-1}})$ by F. Gross [4]. Hence we have $N \triangleright N_2$. Since p is an arbitrary odd prime in $\pi(G)$, we have $G \triangleright G_2$ by the Theorem. Since G/G_2 is solvable by the fundamental theorem of W. Feit and J. G. Thompson [1], we have G is solvable. It's a contradiction. Thus the proof is complete.

References

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Makoto HAYASHI
Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo, Japan