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On the riemannian structure all of whose geodesics are closed and of the same length

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§1. Introduction.

Let (M, h_0) be an n-dimensional riemannian manifold all of whose geodesics are closed and of the same length $2\pi L$. We call such a riemannian structure a C_L-structure. In the present paper we shall give a characterization of C_L-structure (M, h_0) among all the riemannian structures on M.

Let $\mathrm{Geod}(M, h_0)$ be the set of all oriented closed geodesics of (M, h_0). Then $\mathrm{Geod}(M, h_0)$ has a structure of compact $2(n-1)$-dimensional differentiable manifold. Moreover on $\mathrm{Geod}(M, h_0)$ there is the natural symplectic form Ω by which we may define the volume element

$$\omega := \{(-1)^{n(n-1)/2}/(n-1)!\}\,\Omega^{n-1}$$

on $\mathrm{Geod}(M, h_0)$ (see §2). $\mathcal{V}$ will denote the volume of $\mathrm{Geod}(M, h_0)$ with respect to ω. Let $\mathfrak{M}_M$ be the space of all riemannian structures g on M. Now we shall define the function f over $\mathfrak{M}_M$ as follows:

$$f(g) := \mathrm{vol}(M, g)/[c(M, g)]^n\,,$$

where $\mathrm{vol}(M, g)$ is the volume of M with respect to the canonical measure ν_g derived from g and $c(M, g)$ is an average of the length of $c \in \mathrm{Geod}(M, h_0)$ with respect to g, that is,

$$c(M, g) := (1/\mathcal{V})\int_{\mathrm{Geod}(M,h_0)}\left\{\int_0^{2\pi L}\|\dot{c}(s)\|_g ds\right\}\omega\,.$$

In the above definition all geodesics $c \in \mathrm{Geod}(M, h_0)$ are parametrized by the arc length relative to h_0, and $\|\dot{c}(s)\|_g$ denotes the norm of the velocity vector $\dot{c}(s)$ of $c(s)$ with respect to g. Then f is a "smooth" function on $\mathfrak{M}_M$, i.e., for any differentiable one parameter family $g(t)$ of riemannian structures on M, $f(g(t))$ depends differentiably on t. Indeed, a critical point of f is a riemannian structure g on M such that $d/dt(f(g(t)))_{|t=0} = 0$ does hold for every differentiable one parameter family $g(t)$ with $g(0) = g$.

In the present paper we shall give a characterization of C_L-structure in terms of a critical point of the function f.

MAIN THEOREM. *Let (M, h_0) be a riemannian manifold with C_L-structure*

h_0 and f be the function on $\mathfrak{M}_M$ defined as above by this C_L-structure. Then $g \in \mathfrak{M}_M$ is a critical point of f if and only if g is equal to h_0 up to the homothety.

In a previous paper ([5]), the author has shown that this theorem holds for the real projectve spaces with the riemannian structure of constant curvature. On the other hand, the most typical examples of C_L-manifolds are the compact symmetric spaces of rank one with their canonical riemannian structures. So we have a characterization of these standard riemannian structures (i.e., spheres S^n, projective spaces $P^n(K)$ for $K=\boldsymbol{R}$, $\boldsymbol{C}$, or $\boldsymbol{H}$, and the Cayley projective plane $P^2(\Gamma)$, with their canonical metrics). Among other examples of C_L-manifolds, there are so-called Zoll's surfaces which are surfaces of revolution diffeomorphic to S^2 but not isometric to (S^2, g_0) of constant curvature. Of course, for Zoll's surface (S^2, g_Z), Geod(S^2, g_Z) is different from the space of oriented great circles of S^2, and consequently the function f defined by (S^2, g_Z) is different from that defined by (S^2, g_0).

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§ 2. The contact structure on the unit tangent bundle.

Let (M, g) be a riemannian manifold. Let TM be the tangent bundle to M, and $U(M, g) := \{X \in TM \mid g(X, X)=1\}$ be the differentiable manifold of unit tangent vectors to M. Then there is a natural riemannian structure G on TM (resp. $U(M, g)$). See S. Sasaki [5], or M. Berger [1]. It is defined by

$$G(\tilde{X}, \tilde{Y}) := g(\pi_* \tilde{X}, \pi_* \tilde{Y}) + g(K\tilde{X}, K\tilde{Y}) \qquad \text{for} \quad \tilde{X}, \tilde{Y} \in TTM, \tag{2.1}$$

where $\pi : TM$ (resp. $U(M, g)) \to M$ denotes the projection mapping and $K: TTM \to TM$ denotes the connection mapping of the Levi-Civita connection of g (see D. Gromoll, W. Klingenberg and W. Meyer [3]). Or equivalently by the expression using local coordinates (x^i, a^i) ($X := (x^i, a^i)$ means $X = \sum a^i \partial/\partial x^i$), G takes the form

$$G := \begin{pmatrix} g_{ij} + \Gamma^m_{ia} \Gamma^n_{jb} g_{mn} a^a a^b & \Gamma^k_{il} g_{jk} a^l \\ g_{ik} \Gamma^k_{ij} a^l & g_{ij} \end{pmatrix}, \tag{2.2}$$

where Γ^i_{jk} is the Christoffel's symbol. It is obvious from (2.1) that $\pi : (TM, G)$ (resp. $(U(M, g), G)) \to (M, g)$ is a riemannian submersion. Note that fibre $(U_m(M, g), G_{|U_m(M,g)})$ over $m \in M$ is a standard sphere (S^{n-1}, g_0) of constant curvature 1. Now, on TM there is a horizontal (with respect to the riemannian submersion π) vector field ξ which is called the geodesic spray of (M, g). For $X \in TM$, ξ_X is defined as the tangent vector to the orbit of the geodesic flow $t \to \varphi_t X$ at $t=0$. In terms of local coordinates, ξ has the form

(2.3) $$\xi_{(x^i,a^i)}:=(x^i,\ a^i,\ a^i,\ -\Gamma^i_{jk}a^ja^k)\,.$$

Since $\xi_X, X\in U(M, g)$ is tangent to $U(M, g)$, we may consider ξ as a vector field on $U(M, g)$. Let η be a one form on $U(M, g)$ which is a dual of ξ relative to G, i.e., $\eta(\tilde{X})=G(\tilde{X}, \xi)$ for any $\tilde{X}\in TU(M, g)$. In local coordinates expression, η takes the form

(2.4) $$\eta_{(x^i,a^i)}:=(x^i,\ a^i,\ g_{ij}a^j,\ 0)=\sum g_{ij}a^j dx^i\,.$$

Now we recall the notion of contact structure. Let M^{2n-1} be a $(2n-1)$-dimensional differentiable manifold. Then differentiable one form η on M^{2n-1} is called a contact form if $\eta\wedge(d\eta)^{n-1}\neq 0$ holds everywhere. By the condition $\eta(\xi)=1$ and $i(\xi)d\eta=0$, the unique vector field ξ is determined and is called the characteristic vector field of η. The contact structure is called regular, if ξ is regular in the sense of one dimensional distribution. That is, for every point $m\in M$, there exists an adapted coordinate system $\{(x^1, \cdots, x^{2n-1}), U:=\{|x^i|<a\}\}$ around $m=(0, \cdots, 0)$ such that slices $x^i=c^i$ (constant), $i=1, \cdots, 2n-2$, are integral curves of ξ and the intersection of an integral curve of ξ and U consists of at most one such slice.

Now we return to $U(M, g)$. The following lemma is due to S. Sasaki ([**5**]). We shall give a simple proof.

LEMMA 2.1. *η defined by* (2.4) *is a contact form on $U(M, g)$ and ξ defined by* (2.3) *is the characteristic vector field of η.*

PROOF. Take a normal coordinates (x^i) around $m\in M$. Then at X, $\boldsymbol{\pi}_*(X)=m$, η and $d\eta$ takes the form

$$\eta=\sum a^i dx^i\,, \quad \text{and} \quad d\eta=-\sum dx^i\wedge da^i\,.$$

So we have at m,

(2.5) $$\begin{aligned}\eta\wedge(d\eta)^{n-1}&=(-1)^{n(n-1)/2}(n-1)!\,dx^1\wedge\cdots\wedge dx^n\\&\qquad\wedge\left(\sum(-1)^{j-1}a^j da^1\wedge\cdots\widehat{da^j}\cdots\wedge da^n\right)\\&=(-1)^{n(n-1)/2}(n-1)!\ \text{volume element of } (U(M, g), G)\,,\end{aligned}$$

because $\boldsymbol{\pi}:(U(M, g), G)\to(M, g)$ is a riemannian submersion with fibre (S^{n-1}, g_0). Next, $\eta(\xi)=1$ is trivial and

$$\begin{aligned}i(\xi)d\eta&=\sum(\xi\cdot a^i)dx^i-\sum(\xi\cdot x^i)da^i\\&=-\sum a^i da^i=0\,,\end{aligned}$$

since $\sum(a^i)^2=1$ holds. q. e. d.

Now we have,

PROPOSITION 2.2. *Let M be a compact differentiable manifold. Then (M, h_0) is a C_L-manifold for some positive L if ond only if $(U(M, h_0), \eta)$ is a regular contact structure, where η denotes the contact structure on $U(M, h_0)$ defined by* (2.4).

PROOF. This follows from the following observations.

(i) Compact contact manifold is regular if and only if all trajectories of the characteristic vector field are closed and of the same period.

(ii) Trajectory through $X\in U(M, g)$ of the geodesic spray ξ on $U(M, g)$ is given by the orbit of the geodesic flow $\{\varphi_t X\}_{t\geqq 0}$.

(iii) (M, h_0) is a C_L-manifold if and only if all the orbits of the geodesic flow are periodic with the least period $2\pi L$. q. e. d.

Let (M, h_0) be a C_L-structure. Then from the general theory of regular contact structure we have the following (see Boothby-Wang [2], Sasaki [6]).

(a) $U(M, h_0)$ has a structure of principal circle bundle over the base manifold $B^{2(n-1)}$ of dimension $2(n-1)$. p will denote the projection.

(b) η defines a connection over this bundle.

(c) $B^{2(n-1)}$ is a symplectic manifold and its fundamental 2-form Ω is the curvature form of this connection, i. e.,

$$d\eta = p^*\Omega \tag{2.6}$$

is the equation of the structure of the connection.

(d) $\Omega/2\pi L$ determines an integral cocycle on $B^{2(n-1)}$.

Since fibres of p are trajectories of ξ, which are the orbits of the geodesic flow, $B^{2(n-1)}$ may be identified with the space Geod(M, h_0) of all oriented closed geodesics of (M, h_0).

REMARK 1. The cohomology class determined by the closed form $-\Omega/2\pi L$ is nothing but the first Chern class of the $U(1)$-bundle $U(M, h_0)\rightarrow B$.

REMARK 2. We have the following geometrical interpretation of Ω. Let $\tilde{X}, \tilde{Y}\in T_c\,\mathrm{Geod}(M, h_0)$. Then $\tilde{X}, \tilde{Y}$ may be identified with periodic Jacobi fields along the closed geodesic c such that $\langle\tilde{X}(s), \dot{c}(s)\rangle=\langle\tilde{Y}(s), \dot{c}(s)\rangle=0$. Then a direct calculation shows

$$-2\Omega(\tilde{X}, \tilde{Y})=\langle\tilde{X}(s), \nabla Y(s)\rangle-\langle\nabla\tilde{X}(s), \tilde{Y}(s)\rangle\,.$$

Note that the right-hand side of the above equation is independent of s.

Now we have the following lemma which plays an essential role in the proof of the theorem.

LEMMA 2.3. *Let (M, h_0) be a C_L-manifold and φ be a continuous function on $U(M, h_0)$. Let dS^{n-1} be the canonical measure of the sphere S^{n-1} of constant curvature* 1. *Then we have*

$$\int_{c\in\mathrm{Geod}(M,h_0)}\left\{\int_0^{2\pi L}\varphi(\dot{c}(s))ds\right\}\omega=\int_M\left\{\int_{U_m(M,h_0)}\varphi(x)dS^{n-1}\right\}\nu_{h_0}\,. \tag{2.6}$$

PROOF. The left-hand side of (2.6)

$$=(-1)^{n(n-1)/2}/(n-1)!\int_{\mathrm{Geod}}\left\{\int_{p^{-1}(c)}\iota^*(\varphi\eta)\right\}\Omega^{n-1}\ \ (\iota: p^{-1}(c)\rightarrow U(M, h_0),\ \text{inclusion})$$

$$=(-1)^{n(n-1)/2}/(n-1)!\int_{U(M,h_0)}\varphi\eta\wedge(p^*\Omega)^{n-1} \quad \text{(see M. Berger [1], p. 14)}$$

$$=(-1)^{n(n-1)/2}/(n-1)!\int_{U(M,h_0)}\varphi\eta\wedge(d\eta)^{n-1}=\int_{U(M,h_0)}\varphi\nu_G$$

$$=\int_M\left\{\int_{U_m(M,h_0)}\varphi(x)dS^{n-1}\right\}\nu_{h_0}.$$

The last equality holds since $\pi:(U(M, h_0), G)\rightarrow(M, h_0)$ is a riemannian submersion with fibre (S^{n-1}, h_0) of constant curvature 1. q. e. d.

REMARK 3. The fibring p of $U(M, h_0)$ has been considered in A. Weinstein [7] without using contact structure.

§ 3. Proof of the theorem.

1°. First we shall compute the first variation formula. Let $g(t)$ be a differentiable one parameter family of riemannian structures on a C_L-manifold (M, h_0). We put $k=g'(0)$ and $\mathrm{trace}_g k$ will denote the trace of symmetric tensor k relative to g, i. e., $\mathrm{trace}_g k=\sum g^{ij}k_{ij}$. Then by the well-known formula

$$\{\mathrm{vol}\,(M, g(t))\}'=1/2\int_M \mathrm{trace}_{g(t)}g'(t)\nu_{g(t)},$$

we get

(3.1) (First Variation Formula). $d/dt(f(g(t)))=\{[c(M, g(t))]^{n-1}/2cV\}\times$

$$\left[\int_M \mathrm{trace}_{g(t)}g'(t)\nu_{g(t)}\cdot\int_{\mathrm{Geod}}\left\{\int_0^{2\pi L}\|\dot{c}(s)\|_{g(t)}ds\right\}\omega-\right.$$

$$\left.n\,\mathrm{vol}\,(M, g(t))\int_{\mathrm{Geod}}\left\{\int_0^{2\pi L}g'(t)(\dot{c}(s), \dot{c}(s))/\|\dot{c}(s)\|_{g(t)}ds\right\}\omega\right].$$

Now we shall assume $g(0)=\alpha^2h_0$, where α is a positive constant, and show that $d/dt(f(g(t)))_{|t=0}=0$ does hold for all such $g(t)$. Let ω_{n-1} be the volume of (S^{n-1}, g_0) of constant curvature 1. After the homothetic deformation $h_0\rightarrow\alpha^2h_0$, we have the following.

(3.2) $$\int_M \mathrm{trace}_{\alpha^2h_0}k\nu_{\alpha^2h_0}=\alpha^{n-2}\int_M \mathrm{trace}_{h_0}k\nu_{h_0},$$

$$\int_{\mathrm{Geod}}\left\{\int_0^{2\pi L}\|\dot{c}(s)\|_{\alpha^2h_0}ds\right\}\omega=\alpha\int_{\mathrm{Geod}}\left\{\int_0^{2\pi L}ds\right\}\omega=\mathrm{vol}\,(U(M, h_0), G),$$

$$\mathrm{vol}\,(M, \alpha^2h_0)=\alpha^n\,\mathrm{vol}\,(M, h_0),$$

$$\int_{c\in\mathrm{Geod}}\left\{\int_0^{2\pi L}k(\dot{c}(s), \dot{c}(s))/\|\dot{c}(s)\|_{\alpha^2h_0}ds\right\}\omega$$

$$=1/\alpha\int_{U(M,h_0)}k(x, x)\nu_G=\omega_{n-1}/(n\alpha)\int_M \mathrm{trace}_{h_0}k\nu_{h_0}.$$

The last identity follows from $\int_{U_m(M,h_0)} k(x, x)dS^{n-1} = \omega_{n-1}/n \operatorname{trace}_{h_0} k(m)$. Since $\operatorname{vol}(U(M, h_0), G) = \omega_{n-1} \operatorname{vol}(M, h_0)$ via Lemma 2.3, our assertion is clear from these formulas.

2°. To begin with, we shall prepare the following lemma.

LEMMA. *Let $a_1, \cdots, a_n$ be positive numbers. We put*

$$A_i := \int_{x_1^2+\cdots+x_n^2=1} a_i x_i^2 / \sqrt{a_1 x_1^2 + \cdots + a_n x_n^2}\, dS^{n-1}$$

(not summed up with respect to i). If $A_1 = \cdots = A_n$ holds, then we have $a_1 = \cdots = a_n$.

PROOF OF LEMMA. By an elementary calculus we get for $i \neq j$,

$$A_i - A_j = (a_i - a_j)\int_{S^{n-1}} (\text{positive function on } S^{n-1}) dS^{n-1}.$$

In fact, put $\alpha_1 := \sqrt{a_1 x_1^2 + \cdots + a_n x_n^2}$, $\alpha_2 := \sqrt{a_1 x_2^2 + a_2 x_1^2 + a_3 x_3^2 + \cdots + a_n x_n^2}$. Then we have by the formula for transformation of integral,

$$\begin{aligned} A_1 - A_2 &= \int_{S^{n-1}} (a_1 x_1^2)/\alpha_1 dS^{n-1} - \int_{S^{n-1}} (a_2 x_2^2)/\alpha_1 dS^{n-1} \\ &= \int_{S^{n-1}} (a_1 x_1^2/\alpha_1 - a_2 x_1^2/\alpha_2) dS^{n-1} \\ &= \int_{S^{n-1}} x_1^2 \{a_1 a_2 (a_1 - a_2) x_1^2 + (a_1^3 - a_2^3) x_2^2 + a_3 (a_1^2 - a_2^2) x_3^2 + \cdots\} \\ &\qquad\qquad / \alpha_1 \alpha_2 (a_1 \alpha_2 + a_2 \alpha_1) dS^{n-1} \\ &= (a_1 - a_2) \int_{S^{n-1}} (\text{positive function on } S^{n-1}) dS^{n-1}. \end{aligned}$$

q. e. d.

Now assume that g is a critical point of f. Then for any differentiable one parameter family $g(t)$ of riemannian structures with $g(0) = g$, we have $d/dt(f(g(t)))_{|t=0} = 0$, i. e.,

$$\int_{\text{Geod}} \left\{\int_0^{2\pi L} \|\dot{c}(s)\|_g ds\right\} \omega \cdot \int_M \operatorname{trace}_g k \nu_g - n \operatorname{vol}(M, g) \int_{\text{Geod}} \left\{\int_0^{2\pi L} k(\dot{c}(s), \dot{c}(s))/\|\dot{c}(s)\|_g ds\right\} \omega = 0, \tag{3.3}$$

where we have put $k = g'(0)$. Let $S^2(M)$ be the set of all symmetric tensor fields on M of type $(0, 2)$. Then g is a critical point of f if and only if

$$\int_{\text{Geod}} \left\{\int_0^{2\pi L} k(\dot{c}(s), \dot{c}(s))/\|\dot{c}(s)\|_g ds\right\} \omega \Big/ \int_M \operatorname{trace}_g k \nu_g = \text{constant} \tag{3.4}$$

does hold for any $k \in S^2(M)$. On the other hand we have via Lemma 2.3

$$\int_{\text{Geod}} \left\{\int_0^{2\pi L} k(\dot{c}(s), \dot{c}(s))/\|\dot{c}(s)\|_g ds\right\} \omega = \int_M \left\{\int_{U_m(M,h_0)} k(x, x)/\|x\|_g\right\} \nu_{h_0},$$

and

$$\int_M \mathrm{trace}_g k\nu_g = n/\omega_{n-1}\cdot\int_M\Big\{\int_{U_m(M,g)} k(x,x)dS^{n-1}\Big\}\nu_g\,.$$

So (3.4) is turned into

$$\int_M\Big\{\int_{U_m(M,h_0)} k(x,x)/\|x\|_g dS^{n-1}\Big\}\nu_{h_0}\Big/\int_M\Big\{\int_{U_m(M,g)} k(x,x)dS^{n-1}\Big\}\nu_g$$
$$=\text{constant}$$

for any $k\in S^2(M)$. On the other hand we have

$$\nu_g(m)=\sqrt{\det(g_{ij}(m))/(\det((h_0)_{ij}(m))}\nu_{h_0}(m)\,.$$

So if g is a critical point of f, then there exists a C^∞-function $\beta(m)$ on M such that

$$\int_M\Big\{\int_{U_m(M,h_0)} k(x,x)/\|x\|_g dS^{n-1}-\beta(m)\int_{U_m(M,g)} k(x,x)dS^{n-1}\Big\}\nu_{h_0}=0$$

does hold for any $k\in S^2(M)$. Then at every point $m\in M$, we have

$$\text{(3.6)}\qquad \alpha_k(m):=\int_{U_m(M,h_0)} k(x,x)/\|x\|_g dS^{n-1}-\beta(m)\int_{U_m(M,g)} k(x,x)dS^{n-1}=0$$

for any $k\in S^2(M)$. In fact, assume that (3.6) is not satisfied for some $m\in M$, and $k\in S^2(M)$. Then we may assume that $m\to\alpha_k(m)$ is positive on some neighbourhood U of m. Choose a non-negative C^∞-function φ on M such that $\varphi=1$ on some V $(\subset U)$ and $\varphi=0$ outside U. Then we have $\int_M\varphi\alpha_k\nu_{h_0}$ $=\int_M\alpha_{\varphi k}\nu_{h_0}>0$. But this contradicts to the equation above (3.6). So (3.6) does hold for any $m\in M$ and $k\in S^2(M)$.

Now take an orthonormal frame relative to h_0 in T_mM such that g takes the form

$$g(x,x)=a_1x_1^2+\cdots+a_nx_n^2\qquad(a_1,\cdots,a_n>0)\,,$$

where x_i's are the components of $x\in T_mM$ with respect to this orthonormal basis. In (3.6), take especially $k(x,x)=x_i^2$. Then we get

$$\int_{x_1^2+\cdots+x_n^2=1} x_i^2/\sqrt{a_1x_1^2+\cdots+a_nx_n^2}\,dS^{n-1}=\beta(m)\int_{y_1^2+\cdots+y_n^2=1} y_i^2/a_i dS^{n-1}\,,$$

where we have put $y_i^2=a_ix_i^2$ (not summed up). That is,

$$A_i:=\int_{x_1^2+\cdots+x_n^2=1} a_ix_i^2/\sqrt{a_1x_1^2+\cdots+a_nx_n^2}\,dS^{n-1}=\beta(m)\omega_{n-1}/n$$

holds for $i=1,\cdots,n$, so by the lemma above we have $a_1=\cdots=a_n$. Consequently critical point g $(=a^2h_0)$ is conformally related to the C_L-structure h_0 on M. Finally we must show that this positive C^∞-function a on M reduces to a constant.

Let $k=\varphi h_0$, where φ is any C^∞-function, then from (3.3) we have easily

$$\int_M\left\{\left(\int_M a^n \nu_{h_0}\right)1/a-\left(\int_M a\nu_{h_0}\right)a^{n-2}\right\}\varphi\nu_{h_0}=0$$

and consequently a must be a constant. This completes the proof of the theorem.

REMARK 4. The function f on $\mathfrak{M}_M$ for a C_L-manifold (M, h_0) takes neither maximum nor minimum. It suffices to show that at h_0 f takes neither maximum value nor minimum value. For that purpose we shall give the second variation of f at the critical point h_0. We omit the calculation.

$$\begin{aligned}(3.7)\qquad & 2c(M, h_0)^{n+1}\mathcal{V} d^2/dt^2\{f(g(t))\}_{|t=0} \\ &= -(n+1)/(n+2)\operatorname{vol}(U(M, h_0), G)\int_M \langle k, k\rangle \nu_{h_0} \\ &\quad +(n+3)/2(n+2)\operatorname{vol}(U(M, h_0), G)\int_M (\operatorname{trace}_{h_0}k)^2\nu_{h_0} \\ &\quad -(n-1)/2n\omega_{n-1}\left(\int_M \operatorname{trace}_{h_0}k\nu_{h_0}\right)^2,\end{aligned}$$

where $k=g'(0)$, and $\langle k, k\rangle$ denotes the square of the norm (derived from h_0) of k.

First take a normal coordinate system (x, U) around $m\in M$. Choose a non-zero symmetric tensor k_0 of type $(0, 2)$ with vanishing trace at m. By parallel translation of k_0 along the radial geodesics, we have a non-zero symmetric tensor field k_U with vanishing trace on U. Let φ be a non-negative smooth function such that $\varphi=1$ on V $(V\subset U)$ and $\varphi=0$ outside U. If we define $k=\varphi k_U$ on U and $k=0$ outside U, then we have a smooth symmetric tensor field k on M such that $\operatorname{trace}_{h_0}k=0$, but $\int_M \langle k, k\rangle\nu_{h_0}>0$. Now put $g(t)=h_0+kt$, then we have $d^2/dt^2\{f(g(t))\}_{|t=0}<0$ by (3.7), and we see that f can not take a minimum value at h_0.

Secondly put $g'(0)=k=\varphi h_0$, where φ is any C^∞-function. Then

$$\begin{aligned}&2(c(M, h_0))^{n+1}\mathcal{V} d^2/dt^2\{f(g(t))\}_{|t=0}=n(n-1)/2\operatorname{vol}(U(M, h_0), G) \\ &\quad\times\left\{\int_M \varphi^2\nu_{h_0}-1/\operatorname{vol}(M, h_0)\left(\int_M \varphi\nu_{h_0}\right)^2\right\}\geqq 0\end{aligned}$$

does hold by virtue of Schwarz's inequality, where the equality holds if and only if φ is a constant function. Then if we take any $g(t)$ with $g'(0)=\varphi h_0$ where φ is a non-constant smooth function on M, then we have $f(g(t))>f(h_0)$ for every sufficiently small $t\neq 0$. So $t\to f(g(t))$ takes a strictly relative minimum at $t=0$, and f can not take a maximum value at the critical points of f.

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