J. Math. Soc. Japan
Vol. 27, No. 2, 1975

Ergodic theorems for contraction semi-groups

By Yoshihiro KUBOKAWA

(Received Oct. 31, 1972)
(Revised Dec. 1, 1973)

§1. Introduction.

The main purpose of this note is to introduce the absolute value of a strongly continuous one-parameter semi-group of contractions on $L_1(X)$, which is again a semi-group and to prove the local ergodic theorem and the ratio ergodic theorem by making use of the introduced semi-group. The absolute value of a bounded linear operator on $L_1(X)$ which is bounded also on $L_\infty(X)$ was introduced by N. Dunford and J. Schwartz [7]. The result was generalized by R. Chacon and U. Krengel [6] as described in Lemma 1 of the present note. But, as Krengel [10] remarked, an essentially same result was obtained much earlier by Kantrovič [8]. We shall introduce the absolute value of a contraction semi-group (Theorem 1). The local ergodic theorem for positive contraction semi-groups on $L_1(X)$ was conjectured by U. Krengel and proved by U. Krengel [9] and D. Ornstein [13] independently. M. Akcoglu and R. Chacon [2] and T. Terrell [14, 15] gave different treatments of the theorem. D. Ornstein [13] gave a proof of the theorem for a contraction semi-group on $L_1(X)$ which is a contraction semi-group also on $L_\infty(X)$. T. Terrell [14] independently proved the theorem for an n-parameter contraction semi-group on $L_1(X)$ which is a contraction semi-group also on $L_\infty(X)$. We shall generalize Ornstein's theorem and prove the local ergodic theorem for a contraction semi-group (T_t) (Theorem 2) by making use of the absolute value of the semi-group (T_t). Further we shall prove a ratio ergodic theorem for a contraction semi-group (Theorem 3). This is a continuous version of Chacon's ratio ergodic theorem for a contraction T and a T-admissible sequence [5].

§2. Definitions and theorems.

Let $(X, \mathfrak{B}, m)$ be a σ-finite measure space and $L_1(X)=L_1(X, \mathfrak{B}, m)$ the Banach space of complex-valued integrable functions on X. Let (T_t) $(t \geqq 0)$ be a strongly continuous one-parameter semi-group of linear contractions on $L_1(X)$. In the sequel we call such a semi-group a contraction semi-group. This means that

(A) T_t is a linear operator on $L_1(X)$ such that $\|T_t\| \leqq 1$ for any $t \geqq 0$

(Contraction property on $L_1(X)$),

(B) $T_{t+s}f=T_t\circ T_sf$ for any $t, s\geqq 0$ and $f\in L_1(X)$, and

(C) $\lim_{t\to 0}\|T_tf-f\|=0$ for any $f\in L_1(X)$ (Strong continuity).

A contraction semi-group (T_t) on $L_1(X)$ is said to be positive if it satisfies **(D)**:

(D) If $f\geqq 0$ and $f\in L_1(X)$, then $T_tf\geqq 0$ for any $t\geqq 0$.

Let T be a contraction on $L_1(X)$. A sequence (P_n) $(n=0, 1, 2, \cdots)$ of non-negative functions in $L_1(X)$ is said to be T-admissible, if $|Tf|\leqq P_{n+1}$ holds whenever f and n satisfy $|f|\leqq P_n$ [**1, 5**]. We shall define a continuous version of a T-admissible sequence. Let (T_t) $(t\geqq 0)$ be a contraction semi-group and let (P_t) $(t\geqq 0)$ be a family of non-negative functions in $L_1(X)$ such that $\lim_{t\to s}\|P_t-P_s\|=0$ for any $s\geqq 0$. The family (P_t) is said to be (T_t)-admissible if $|T_tf|\leqq P_{t+s}$ holds for any $t\geqq 0$ whenever f and s satisfy $|f|\leqq P_s$. There exists a $\mathfrak{L}^+\times\mathfrak{B}$-measurable function $g(t, x)$ such that $g(t, x)=P_t(x)$ a. e. for any fixed t, where $\mathfrak{L}^+$ is the σ-algebra of Lebesgue measurable sets on the half real line. We define the integral $\int_a^b P_t(x)dt$ $(0\leqq a<b<\infty)$ by $\int_a^b g(t, x)dt$. Note that if $g(t, x)$ and $\tilde{g}(t, x)$ are two $\mathfrak{L}^+\times\mathfrak{B}$-measurable versions of $P_t(x)$, then $g(t, x)=\tilde{g}(t, x)$ except on a set of $\lambda\times m$ measure zero ($\lambda=$Lebesgue measure) and there is a m-null set $N\in\mathfrak{B}$ such that for $x\notin N$ $\int_a^b g(t, x)dt=\int_a^b \tilde{g}(t, x)dt$ holds for all a and b. The integral $\int_a^b (T_tf)(x)dt$ is defined analogously.

We shall first construct a positive contraction semi-group $(\tilde{T}_t)$ which dominates (T_t) in the absolute value, that is, we shall prove the following.

THEOREM 1. *Let (T_t) $(t\geqq 0)$ be a contraction semi-group on $L_1(X)$. Then there exists a positive contraction semi-group $(\tilde{T}_t)$ such that for any $t\geqq 0$ and $f\in L_1(X)$*

$$(1.1)\qquad (\tilde{T}_t|f|)(x)\geqq|(T_tf)(x)| \quad a. e.$$

The semi-group $(\tilde{T}_t)$ can be chosen in such a way that if a family (P_t) is (T_t)-admissible, then (P_t) is $(\tilde{T}_t)$-admissible.

Making use of this theorem we shall prove the following.

THEOREM 2 (Local ergodic theorem). *Let (T_t) $(t\geqq 0)$ be a contraction semi-group on $L_1(X)$. Then we have*

$$\lim_{\alpha\to 0}\frac{1}{\alpha}\int_0^\alpha (T_tf)(x)dt=f(x) \quad a. e.$$

D. Ornstein [**13**] proved Theorem 2 for a contraction semi-group on $L_1(X)$ which satisfies **(E)**.

$$\textbf{(E)}\qquad \text{ess.}\sup_x|(T_tf)(x)|\leqq\text{ess.}\sup_x|f(x)| \qquad \text{for any } f\in L_1(X)\cap L_\infty(X).$$

T. Terrell [14] proved a similar theorem for an n-parameter contraction semi-group which satisfies the condition analogous to (**E**). He [15] proved Theorem 2 assuming existence of the absolute value of a contraction semi-group.

COROLLARY 1. *If* (T_t) *is a semi-group which satisfies* (**B**), (**C**) *and the following condition* (**F**), *then the local ergodic theorem for* (T_t) *holds.*

(**F**) *There exists a constant* $\beta>0$ *such that* $\|T_t\|\leqq e^{\beta t}$.

COROLLARY 2. *Under the same condition as in Corollary* 1 *we have for any* $f,\ g\in L_1(X)$

$$\lim_{\alpha\to 0}\frac{\int_0^\alpha (T_t f)(x)dt}{\int_0^\alpha (T_t g)(x)dt}=\frac{f(x)}{g(x)} \qquad a.e.\ on\ \{x: g(x)\neq 0\}\,.$$

Lastly we shall prove the following by reduction to Chacon's ratio ergodic theorem [5], employing Theorem 1.

THEOREM 3. *Let* (T_t) $(t\geqq 0)$ *be a contraction semi-group on* $L_1(X)$ *and let a family* (P_t) $(t\geqq 0)$ *be* (T_t)*-admissible. Then for any* $f\in L_1(X)$ *the limit*

$$\lim_{\alpha\to\infty}\frac{\int_0^\alpha (T_t f)(x)dt}{\int_0^\alpha P_t(x)dt}$$

exists and is finite almost everywhere on the set where $\int_0^\alpha P_t(x)dt>0$ *for some* $\alpha>0$.

If (T_t) is a positive contraction semi-group on $L_1(X)$ and $P_t=T_t g$ $(g\geqq 0)$, then (P_t) is (T_t)-admissible. M. Akcoglu and J. Cunsolo [3] proved Theorem 3 in this case. K. Berk [4] gave different treatments of such (T_t) and (P_t).

§ 3. Proof of Theorem 1.

For the proof of Theorem 1 we need several lemmas. In the sequel the order relation $f\leqq g$ for functions in $L_1(X)$ means $f(x)\leqq g(x)$ a.e.

LEMMA 1 (Chacon-Krengel) [6]. *Let* T *be a bounded linear operator on* $L_1(X)$. *Define*

$$|T|f=\sup_{|g|\leqq f}|Tg|$$

for $f\in L_1(X)$ *such that* $f\geqq 0$. *Then* $|T|$ *is uniquely extended to a positive bounded linear operator on* $L_1(X)$ *and*

(1) $$|T|\,|f|\geqq |Tf| \qquad for\ any\quad f\in L_1(X)\,,$$

(2) $$\|\,|T|\,\|=\|T\|\,.$$

If T *is positive, then* $|T|=T$. *If* T_1 *and* T_2 *are bounded linear operators on*

$L_1(X)$, *then*

(3) $$|T_1T_2|f \leqq |T_1||T_2|f$$

for any $f \in L_1(X)$ *such that* $f \geqq 0$.

Put

$$Q(n)=\left\{\frac{l}{2^n} : l \text{ is a non-negative integer}\right\} \text{ and } Q=\bigcup_{n=1}^{\infty} Q(n).$$

LEMMA 2. *Let* (T_t) *be a contraction semi-group on* $L_1(X)$. *If* $f \in L_1(X)$, *then for any* $r \in Q$ *the limit*

(4) $$\lim_{n\to\infty} |T_{1/2^n}|^{[2^n r]}f(x)$$

exists in the sense of almost everywhere convergence as well as strong convergence.

We define $\tilde{T}_r f$ by (4) for any $r \in Q$ and $f \in L_1(X)$.

PROOF. We can assume $f \geqq 0$. If n is large enough, then $[2^n r]=2^n r$. We have by Lemma 1

$$|T_{1/2^{n+1}}|^{2l} g \geqq |T_{1/2^n}|^l g$$

for any positive integer l and $g \geqq 0$. Hence

$$|T_{1/2^{n+1}}|^{2^{n+1}r} f \geqq |T_{1/2^n}|^{2^n r} f \qquad \text{for large } n.$$

Since $|T_{1/2^n}|$ is a contraction by (2) of Lemma 1, we have $\||T_{1/2^n}|^{[2^n r]}f\| \leqq \|f\|$. Therefore the limit of the sequence $(|T_{1/2^n}|^{[2^n r]}f(x))$ exists almost everywhere and the convergence is strong.

LEMMA 3. *The operator* $\tilde{T}_r$ $(r \in Q)$ *defined in Lemma 2 has the following properties.*

(5) $\tilde{T}_r$ *is a positive linear contraction on* $L_1(X)$ *for any* $r \in Q$.

(6) $\tilde{T}_{r+s}f=\tilde{T}_r \circ \tilde{T}_s f$ *for any* $r, s \in Q$ *and* $f \in L_1(X)$.

(7) *If* $f \geqq 0$ *and* $f \in L_1(X)$, *then we have*

$$\|\tilde{T}_r f - f\| \leqq 2\|T_r f - f\| \qquad \text{for any } r \in Q.$$

PROOF. Since the operator $\tilde{T}_r$ is the strong limit of a sequence of positive contractions by Lemma 2 we have (5). We shall prove (6). We can assume $f \geqq 0$. We have

$$\begin{aligned}\|\tilde{T}_{r+s}f-\tilde{T}_r\circ\tilde{T}_s f\| \leqq {} & \|\tilde{T}_{r+s}f-|T_{1/2^n}|^{2^n(r+s)}f\| \\ & +\||T_{1/2^n}|^{2^n r}(|T_{1/2^n}|^{2^n s}f-\tilde{T}_s f)\| \\ & +\|(|T_{1/2^n}|^{2^n r}-\tilde{T}_r)\tilde{T}_s f\|.\end{aligned}$$

The second term on the right-hand side is bounded by

$$\||T_{1/2^n}|^{2^n s}f-\tilde{T}_s f\|.$$

Letting n tend to infinity, we have (6) by the definition of $(\tilde{T}_r)$. Lastly we shall prove (7). Put

$$T_r f = f + g_r \quad \text{and} \quad \tilde{T}_r f = f + h_r .$$

Let $r = l/2^q$, where q, l are positive integers. Since $\tilde{T}_r f$ is the limit of the increasing sequence $(|T_{1/2^n}|^{[2^n r]} f)$ $(n = q, q+1, \cdots)$ by Lemma 2, we have by (1) and (3) of Lemma 1,

$$\tilde{T}_r f \geqq |T_{1/2^q}|^l f \geqq |T_r| f \geqq |T_r f| . \tag{8}$$

Therefore we have

$$f + h_r \geqq f - |g_r| ,$$

or

$$h_r^- \leqq |g_r| , \qquad \text{where} \quad h_r = h_r^+ - h_r^- .$$

Since

$$\int (f + h_r) dm = \int \tilde{T}_r f dm \leqq \int f dm ,$$

it follows that

$$\int h_r^+ dm \leqq \int h_r^- dm \leqq \int |g_r| dm .$$

This means that

$$\|\tilde{T}_r f - f\| = \|h_r\| = \int h_r^+ dm + \int h_r^- dm \leqq 2\|g_r\| = 2\|T_r f - f\| .$$

LEMMA 4. *The strong limit*

$$\underset{\substack{r \to t \\ r \in Q}}{s\text{-}\lim} \tilde{T}_r f \tag{9}$$

exists for any $t \geqq 0$ *and* $f \in L_1(X)$.

We define $\tilde{T}_t f$ by (9) for any $t \geqq 0$ and $f \in L_1(X)$.

PROOF. We can assume $f \geqq 0$. We have by Lemma 3

$$\|\tilde{T}_r f - \tilde{T}_s f\| \leqq \|\tilde{T}_{|r-s|} f - f\| \leqq 2\|T_{|r-s|} f - f\|$$

for r, $s \in Q$. Hence the assertion follows from the strong continuity of (T_t).

LEMMA 5. $(\tilde{T}_t)$ $(t \geqq 0)$ *is a positive contraction semi-group on* $L_1(X)$.

We call the positive contraction semi-group $(\tilde{T}_t)$ the linear modulus of (T_t).

PROOF. It follows easily from the definition and Lemma 3 that $\tilde{T}_t$ is a positive contraction for any $t \geqq 0$. The semi-group property of $(\tilde{T}_t)$ can be proved by an argument similar to Lemma 3. Since we obtain from (7) of Lemma 3

$$\|\tilde{T}_t f - f\| \leqq 2\|T_t f - f\| \qquad \text{for} \quad f \geqq 0 ,$$

(T_t) is strongly continuous.

PROOF OF THEOREM 1. It follows from (1) and (8) that

$$|T_r f| \leqq |T_r| \, |f| \leqq \tilde{T}_r |f| \qquad (r \in Q)$$

for any $f\in L_1(X)$. Hence we have (1.1) by the strong continuity of (T_t) and $(\tilde{T}_t)$. Suppose that a family (P_t) is (T_t)-admissible. Let $|f|\leqq P_s$. If $|g|\leqq|f|$, then $|T_t g|\leqq P_{s+t}$. Therefore we have

(10) $$|T_t||f|=\sup_{|g|\leqq|f|}|T_t g|\leqq P_{s+t}\,.$$

If T is a positive bounded linear operator, $|Tf|\leqq T|f|$. Hence if $r\in Q$, then

$$||T_{1/2^n}|^{2^n r}f|\leqq|T_{1/2^n}|^{2^n r}|f|\leqq P_{s+r}$$

for large n. We have by making n tend to infinity

$$|\tilde{T}_r f|\leqq\tilde{T}_r|f|\leqq P_{s+r}\qquad\text{for any}\quad r\in Q\,.$$

Therefore

$$|\tilde{T}_t f|\leqq P_{s+t}\qquad\text{for any}\quad t\geqq 0\,.$$

§ 4. Proof of Theorem 2.

In this section we shall prove Theorem 2 and its corollaries. U. Krengel and D. Ornstein proved the following [**9**, **13**].

LEMMA 6 (Local ergodic theorem). *Let (T_t) $(t\geqq 0)$ be a positive contraction semi-group on $L_1(X)$. Then for any $f\in L_1(X)$ we have*

$$\lim_{\alpha\to 0}\frac{1}{\alpha}\int_0^\alpha(T_t f)(x)dt=f(x)\qquad a.\,e.$$

REMARK. The author proved the local ergodic theorem for a one-parameter semi-group of positive bounded linear operators on $L_p(X)$ $(p\geqq 1)$ which are not necessarily contractions [**11**, **12**].

LEMMA 7. *Let (T_t) be a contraction semi-group on $L_1(X)$. If $f\in L_1(X)$, then for almost all $s\geqq 0$ we have*

$$\lim_{\alpha\to 0}\frac{1}{\alpha}\int_0^\alpha(T_{t+s}f)(x)dt=(T_s f)(x)\qquad a.\,e.$$

The proof is found in the proof of Lemma 2 of U. Krengel [**9**], (Lemma 7 is not necessary for a proof of Theorem 2. See below and Y. Kubokawa [**11**].)

PROOF OF THEOREM 2. Let ε be a positive number and let $f\in L_1(X)$. By Lemma 7 and the strong continuity of (T_t), there exists a function g such that

$$\lim_{\alpha\to 0}\frac{1}{\alpha}\int_0^\alpha(T_t g)(x)dt=g(x)\qquad\text{a. e.}$$

and

$$\|g-f\|<\varepsilon^2\,.$$

Indeed choose $g=T_s f$ for a suitable s. $\Big($We can choose $g=\frac{1}{s}\int_0^s T_t f dt$, where s satisfies $\sup_{0\leqq t\leqq s}\|T_t f-f\|<\varepsilon^2$, without employing Lemma 7.$\Big)$ We have

$$\left|\frac{1}{\alpha}\int_0^\alpha (T_t f)(x)dt - f(x)\right|$$

$$\leqq \left|\frac{1}{\alpha}\int_0^\alpha (T_t(f-g)(x)dt\right| + \left|\frac{1}{\alpha}\int_0^\alpha (T_t g)(x)dt - g(x)\right| + |g(x)-f(x)|.$$

Let $(\tilde{T}_t)$ be the linear modulus of (T_t). By Theorem 1 we have (see the remark preceding Theorem 1)

$$\left|\frac{1}{\alpha}\int_0^\alpha T_t(f-g)(x)dt\right| \leqq \frac{1}{\alpha}\int_0^\alpha \tilde{T}_t|f-g|(x)dt,$$

which tends to $|f(x)-g(x)|$ a.e. as $\alpha \to 0$ by Lemma 6 applied to $(\tilde{T}_t)$. Hence

$$\limsup_{\alpha\to 0}\left|\frac{1}{\alpha}\int_0^\alpha (T_t f)(x)dt - f(x)\right| \leqq 2|f(x)-g(x)|.$$

We have $|f(x)-g(x)| < \varepsilon$ for any x except on a set with measure less than ε. Since ε is arbitrary, we have Theorem 2.

PROOF OF COROLLARY 1. From the assumption (**F**) we can define a contraction semi-group (S_t) by $S_t f = e^{-\beta t}T_t f$. We have

$$\frac{1}{\alpha}\int_0^\alpha (T_t f)(x)dt = \frac{1}{\alpha}\int_0^\alpha (e^{\beta t}-1)(S_t f)(x)dt + \frac{1}{\alpha}\int_0^\alpha (S_t f)(x)dt.$$

Let $(\tilde{S}_t)$ be the linear modulus of (S_t). We have by Theorem 1

$$\left|\frac{1}{\alpha}\int_0^\alpha (e^{\beta t}-1)(S_t f)(x)dt\right| \leqq \frac{e^{\beta\alpha}-1}{\alpha}\int_0^\alpha (\tilde{S}_t|f|)(x)dt,$$

which tends to zero by Theorem 2 as $\alpha \to 0$. We get the conclusion by Theorem 2.

Corollary 2 follows from Corollary 1.

§ 5. Proof of Theorem 3.

We define a sequence (Q_n) $(n=0, 1, 2, \cdots)$ of non-negative functions in $L_1(X)$ by $Q_n(x) = \int_n^{n+1} P_t(x)dt$ and a function $f_0(x) = \int_0^1 (T_t f)(x)dt$ for any $f \in L_1(X)$. Then we have the following.

LEMMA 8. *The sequence (Q_n) $(n=0, 1, 2, \cdots)$ is T_1-admissible.*

PROOF. We assume that $|f| \leqq Q_n$ for some n. Then by (1) and the positivity of $|T_1|$,

$$|T_1 f| \leqq |T_1|\,|f| \leqq |T_1| Q_n.$$

Since we have $|T_1| P_t \leqq P_{t+1}$ by (10)

$$|T_1| Q_n = \int_n^{n+1} |T_1| P_t(x)dt \leqq \int_n^{n+1} P_{t+1}(x)dt = Q_{n+1}.$$

We have

$$\int_n^{n+1}(T_tf)(x)dt=T_1^nf_0\,,\quad \int_0^{n+1}(T_tf)(x)dt=\sum_{k=0}^n T_1^kf_0\,,$$

$\int_0^{n+1}P_t(x)dt=\sum_{k=0}^n Q_k$. Since the sequence (Q_n) is T_1-admissible we have the following Lemma 9 and Lemma 10 by applying Lemma 1 of Chacon [5] and Chacon's ratio ergodic theorem [5], respectively.

LEMMA 9. *Let (T_t) $(t\geqq 0)$ be a contraction semi-group on $L_1(X)$. If (P_t) is (T_t)-admissible, then for any $f\in L_1(X)$*

$$\lim_{n\to\infty}\frac{\int_n^{n+1}(T_tf)(x)dt}{\int_0^{n+1}P_t(x)dt}=0\qquad a.\,e.$$

on the set where $\int_0^n P_t(x)dt>0$ for some n.

LEMMA 10. *Assume the same conditions as in Lemma 9. Then for any $f\in L_1(X)$, the limit*

$$\lim_{n\to\infty}\frac{\int_0^{n+1}(T_tf)(x)dt}{\int_0^{n+1}P_t(x)dt}$$

exists and is finite almost everywhere on the set where $\int_0^n P_t(x)dt>0$ for some n.

PROOF OF THEOREM 3. Let r be a positive integer. It is enough to give proof on the set where $\int_0^r P_t(x)dt>0$. Let $\alpha\geqq r$. We choose an integer n with $n\leqq\alpha<n+1$. We have

$$\left|\frac{\int_0^\alpha(T_tf)(x)dt}{\int_0^\alpha P_t(x)dt}-\frac{\int_0^n(T_tf)(x)dt}{\int_0^n P_t(x)dt}\right|$$

$$\leqq\left|\frac{\int_n^\alpha(T_tf)(x)dt}{\int_0^n P_t(x)dt}\right|+\left|\frac{\int_0^n(T_tf)(x)dt}{\int_0^n P_t(x)dt}\cdot\frac{\int_n^\alpha P_t(x)dt}{\int_0^\alpha P_t(x)dt}\right|.$$

Let $(\tilde{T}_t)$ be the linear modulus of (T_t). Then the right-hand side does not exceed

$$\frac{\int_{n-1}^n(\tilde{T}_t\circ\tilde{T}_1|f|)(x)dt}{\int_0^n P_t(x)dt}+\left|\frac{\int_0^n(T_tf)(x)dt}{\int_0^n P_t(x)dt}\right|\cdot\frac{\int_n^\alpha P_t(x)dt}{\int_0^\alpha P_t(x)dt}.$$

Since P_t $(t\geqq 0)$ is $(\tilde{T}_t)$-admissible by Theorem 1, the first term tends to zero a. e. as $\alpha\to\infty$ by Lemma 9. The second term also tends to zero a. e. on the set where $\lim_{n\to\infty}\int_0^n(T_tf)(x)dt/\int_0^n P_t(x)dt=0$. Consider the set where

$$\lim_{n\to\infty}\int_0^n (T_t f)(x)dt / \int_0^n P_t(x)dt \neq 0 .$$

We have

$$\frac{\int_n^\alpha P_t(x)dt}{\int_0^\alpha P_t(x)dt} \leqq \frac{\int_n^{n+1} P_t(x)dt}{\int_0^n P_t(x)dt}$$

$$= \frac{\int_0^{n+1} P_t(x)dt}{\int_0^{n+1}(T_t f)(x)dt} \cdot \frac{\int_0^n (T_t f)(x)dt + \int_n^{n+1}(T_t f)(x)dt}{\int_0^n P_t(x)dt} - 1 ,$$

which tends to zero a. e. on the set by Lemma 9 and Lemma 10. Hence we have

$$\lim_{\alpha\to\infty}\left| \frac{\int_0^\alpha (T_t f)(x)dt}{\int_0^\alpha P_t(x)dt} - \frac{\int_0^n (T_t f)(x)dt}{\int_0^n P_t(x)dt} \right| = 0 \qquad \text{a. e.}$$

By Lemma 10 this completes proof of Theorem 3.

ACKNOWLEDGEMENT: The author wishes to express his hearty thanks to Professor K. Yosida of Gakushuin University for valuable discussions on the subject and to Professor S. Tsurumi of Tokyo Metropolitan University for reading the manuscript and correcting some errors. Proof of Theorem 3 was simplified by Professor S. Tsurumi. He also wishes to express his hearty thanks to the referee for valuable advices.

REMARK. After the author proved Theorem 3, S. Tsurumi generalized it [17].

References

[1] M. Akcoglu, Pointwise ergodic theorems, Trans. Amer. Math. Soc., **125** (1966), 296-309.

[2] M.A. Akcoglu and R.V. Chacon, A local ratio theorem, Canad. J. Math., **22** (1970), 545-552.

[3] M. Akcoglu and J. Cunsolo, An ergodic theorem for semigroups, Proc. Amer. Math. Soc., **24** (1970), 166-170.

[4] K. Berk, Ergodic theory with recurrent weights, Ann. Math. Statist., **39** (1968), 1107-1114.

[5] R.V. Chacon, Convergence of operator averages, Ergodic Theory (Proc. Internat. Sympos., Tulane Univ., New Orleans, La., 1961), 89-120, Academic Press, New York, 1963.

[6] R.V. Chacon and U. Krengel, Linear modulus of a linear operator, Proc. Amer. Math. Soc., **15** (1964), 553-559.

[7] N. Dunford and J. Schwartz, Convergence almost everywhere of operator aver-

ages, J. Rat. Mech. Anal., **5** (1956), 129–178.
[8] L.V. Kantrovič, Linear operators in semi-ordered spaces, I. Rec. Math. [Math. Sbornik] N.S., **49** (1940), 209–284.
[9] U. Krengel, A local ergodic theorem, Invent. Math., **6** (1969), 329–333.
[10] U. Krengel, Über den Absolutbetrag stetiger linearer Operatoren und seine Anwendungen auf ergodische Zerlegungen, Math. Scand., **13** (1963), 151–187.
[11] Y. Kubokawa, A general local ergodic theorem, Proc. Japan Acad., **48** (1972), 461–465.
[12] Y. Kubokawa, A local ergodic theorem for semi-group on L_p, Tôhoku Math. J., **26** (1974), 411–422.
[13] D. Ornstein, The sum of iterates of a positive operator, Advances in Probability and Related Topics (Edited by P. Ney), **2** (1970), 87–115.
[14] T.R. Terrell, Local ergodic theorem for n-parameter semi-group of operators, Lecture Notes in Math., No. 160, 262–278, Springer, 1970.
[15] T.R. Terrell, The local ergodic theorem and semi-groups of nonpositive operators, J. Functional Analysis, **10** (1972), 424–429.
[16] S. Tsurumi, Ergodic theorems (Japanese), Sugaku, **13** (1961/62), 80–88.
[17] S. Tsurumi, An ergodic theorem for a semigroup of linear contractions, Proc. Japan Acad., **49** (1973), 306–309.
[18] K. Yosida, Functional Analysis, Springer, 1965.

Yoshihiro KUBOKAWA
Department of Mathematics
Faculty of Education
Saitama University
Urawa, Japan