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Pseudo-differential operators of multiple symbol and the Calderón-Vaillancourt theorem

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Introduction.

In the present paper we shall define a class $S_{\lambda,\rho,\delta}^{\tilde{m}_\nu;\tilde{m}'_\nu}$ of multiple symbol as an extension of the class $S_{\rho,\delta}^{m,m'}$ of double symbol in our previous paper [**6**], where $\tilde{m}_\nu=(m_1, \cdots, m_\nu)$ and $\tilde{m}'_\nu=(m'_0, m'_1, \cdots, m'_\nu)$ are real vectors and $m'_j \geqq 0$, $j=0, 1, \cdots, \nu$. The multiple symbol has the form $p(x^0, \tilde{\xi}^\nu, \tilde{x}^\nu)=p(x^0, \xi^1, x^1, \cdots, \xi^\nu, x^\nu)$ and the associated pseudo-differential operator $P=p(X^0, D_{\tilde{x}^\nu}, \tilde{X}^\nu)$ is defined as the map $P: \mathcal{B}\rightarrow\mathcal{B}$ by using oscillatory integrals developed in Kumano-go [**7**] and Kumano-go-Taniguchi [**8**], where $\mathcal{B}$ denotes the set of C^∞-functions with bounded derivatives of any order in R^n. Then, the (single) symbol $\sigma(P)(x, \xi)$ is given by $\sigma(P)(x, \xi)=e^{-ix\cdot\xi}P(e^{ix\cdot\xi})$.

We shall give a theorem which represents $\sigma(P)(x, \xi)$ by the oscillatory integral of the multiple symbol $p(x^0, \tilde{\xi}^\nu, \tilde{x}^\nu)$ and the asymptotic expansion formula for $\sigma(P)(x, \xi)$ will be given. As an application we shall prove the Calderón-Vaillancourt theorem in [**3**] (see also [**11**]) for the L^2-continuity of pseudo-differential operators of class $S_{\lambda,\delta,\delta}^0$ $(0\leqq\delta<1)$ only by symbol calculus. Another application is found in Tsutsumi [**10**], where our theorem is used to construct the fundamental solution $U(t)$ for a degenerate parabolic pseudo-differential operator in the class $S_{\rho,\delta}^0$ with a parameter t.

We believe that our theorem will be useful when we try to solve operator-valued integral equations with pseudo-differential operators as their kernels.

§ 1. Oscillatory integrals.

DEFINITION 1.1. We say that a C^∞-function $p(\eta, y)$ in $R_{\eta,y}^{2n}$ belongs to a class $\mathcal{A}_{\delta,\tilde{\tau}}^m$ for $-\infty<m<\infty$, $\delta<1$ and a sequence $\tilde{\tau}$; $0\leqq\tau_1\leqq\tau_2\leqq\cdots\leqq\tau_l\leqq\cdots$, when for any multi-index α, β we have

$$(1.1)\qquad |p_{(\beta)}^{(\alpha)}(\eta, y)|\leqq C_{\alpha,\beta}\langle\eta\rangle^{m+\delta|\beta|}\langle y\rangle^{\tau_{|\beta|}}$$

for a constant $C_{\alpha,\beta}$ and set $\mathcal{A}_\delta=\bigcup_{-\infty<m<\infty}\bigcup_{\tilde{\tau}}\mathcal{A}_{\delta,\tilde{\tau}}^m$, where $p_{(\beta)}^{(\alpha)}=\partial_\eta^\alpha D_y^\beta p$, $D_{y_j}=-i\partial/\partial y_j$, $\partial_{\eta_j}=\partial/\partial\eta_j$, $j=1, \cdots, n$, $\langle y\rangle=\sqrt{1+|y|^2}$, $\langle\eta\rangle=\sqrt{1+|\eta|^2}$ (cf. [**7**], [**8**]).

DEFINITION 1.2. For a $p(\eta, y)\in \mathcal{A}^m_{\delta,\tau}$ we define the oscillatory integral by

$$O_s[e^{-iy\cdot\eta}p(\eta, y)] \equiv O_s-\iint e^{-iy\cdot\eta}p(\eta, y)dyd\!\!^-\eta \tag{1.2}$$
$$=\lim_{\varepsilon\to 0}\iint e^{-iy\cdot\eta}\chi_\varepsilon(\eta, y)p(\eta, y)dyd\!\!^-\eta\,,$$

where $d\!\!^-\eta=(2\pi)^{-n}d\eta$, $y\cdot\eta=y_1\eta_1+\cdots+y_n\eta_n$ and $\chi_\varepsilon(\eta, y)=\chi(\varepsilon\eta, \varepsilon y)$, $0<\varepsilon<1$, for a $\chi\in\mathcal{S}$ (the class of rapidly decreasing functions) in $R^{2n}_{\eta,y}$ such that $\chi(0)=1$ (cf. [4] for the general form).

Then, the following propositions are found in [7], [8].

PROPOSITION 1.1. *For a $p\in\mathcal{A}^m_{\delta,\tau}$ we choose positive integers l, l' such that $-2l(1-\delta)+m<-n$, $-2l'+\tau_{2l}<-n$. Then, we can write $O_s[e^{-iy\cdot\eta}p(\eta, y)]$ as*

$$O_s[e^{-iy\cdot\eta}p]=\iint e^{-iy\cdot\eta}\langle y\rangle^{-2l'}\langle D_\eta\rangle^{2l'}\{\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l}p(\eta, y)\}dyd\!\!^-\eta\,, \tag{1.3}$$

and for $l_0=2(l+l')$ and a constant C we have $|O_s[e^{-iy\cdot\eta}p]|\leqq C|p|^{(m)}_{l_0}$, where $|p|^{(m)}_{l_0}$ is a semi-norm defined by $|p|^{(m)}_{l_0}=\max_{|\alpha+\beta|\leqq l_0}\inf\{C_{\alpha,\beta}$ of (1.1)$\}$.

PROPOSITION 1.2. *Let $\{p_\varepsilon\}_{0<\varepsilon<1}$ be a bounded set of $\mathcal{A}^m_{\delta,\tau}$ in the sense: $\sup_\varepsilon\{|p_\varepsilon|^{(m)}_{l_0}\}\leqq M_{l_0}$, $l_0=0, 1, 2, \cdots$, for constants M_{l_0}.*

Suppose that there exists a $p_0\in\mathcal{A}^m_{\delta,\tau}$ such that $p_\varepsilon(\eta, y)\to p_0(\eta, y)$ as $\varepsilon\to 0$ uniformly on any compact set of $R^{2n}_{\eta,y}$. Then we have $\lim_{\varepsilon\to 0}O_s[e^{-iy\cdot\eta}p_\varepsilon]=O_s[e^{-iy\cdot\eta}p_0]$.

PROPOSITION 1.3. *For $p\in\mathcal{A}_\delta$ we have $O_s[e^{-iy\cdot\eta}y^\alpha p]=O_s[e^{-iy\cdot\eta}D^\alpha_\eta p]$ and $O_s[e^{-iy\cdot\eta}\eta^\beta p]=O_s[e^{-iy\cdot\eta}D^\beta_y p]$.*

§2. Multiple symbols and theorems.

Let $\lambda(\xi)$ is a C^∞-function in R^n_ξ such that for constants A_0 and A_α

$$1\leqq\lambda(\xi)\leqq A_0\langle\xi\rangle \quad\text{and}\quad |\lambda^{(\alpha)}(\xi)|\leqq A_0\lambda(\xi)^{1-|\alpha|} \qquad \text{(cf. [5])}. \tag{2.1}$$

Let $(x^0, \tilde{x}^\nu)=(x^0, x^1, \cdots, x^\nu)$ and $\tilde{\xi}^\nu=(\xi^1, \cdots, \xi^\nu)$ be a $(\nu+1)$-tuple of points $x^0, x^1, \cdots, x^\nu\in R^n_x$ and a ν-tuple of $\xi^1, \cdots, \xi^\nu\in R^n_\xi$ respectively. By $(\beta^0, \tilde{\beta}^\nu)=(\beta^0, \beta^1, \cdots, \beta^\nu)$ and $\tilde{\alpha}^\nu=(\alpha^1, \cdots, \alpha^\nu)$ we denote a $(\nu+1)$-tuple of multi-indices $\beta^0, \beta^1, \cdots, \beta^\nu$ in R^n and a ν-tuple of $\alpha^1, \cdots, \alpha^\nu$ of R^n respectively.

DEFINITION 2.1. i) For $0\leqq\delta\leqq\rho\leqq 1$, $\delta<1$, real vectors $\tilde{m}_\nu=(m_1, \cdots, m_\nu)$ and $\tilde{m}'_\nu=(m'_0, \cdots, m'_\nu)$ ($m'_j\geqq 0, j=0, \cdots, \nu$) we say that a C^∞-functions $p(x^0, \tilde{\xi}^\nu, \tilde{x}^\nu)=p(x^0, \xi^1, x^1, \cdots, \xi^\nu, x^\nu)$ in $R^{(2\nu+1)n}$ is a multiple symbol of class $S^{\tilde{m}_\nu;\tilde{m}'_\nu}_{\lambda,\rho,\delta}$, when for any $\tilde{\alpha}^\nu$ and $(\beta^0, \tilde{\beta}^\nu)$ we have for a constant $C=C(\tilde{\alpha}^\nu, \beta^0, \tilde{\beta}^\nu)$

$$|p^{(\tilde{\alpha}^\nu)}_{(\beta^0,\tilde{\beta}^\nu)}|\leqq C\lambda(\xi^1)^{m'_0+\delta|\beta^0|}\prod_{j=1}^{\nu}\lambda(\xi^j)^{m_j-\rho|\alpha^j|}(\lambda(\xi^j)+\lambda(\xi^{j+1}))^{m'_j+\delta|\beta^j|} \tag{2.2}$$
$$(\xi^{\nu+1}=0)\,,$$

where $p^{(\tilde{\alpha}^\nu)}_{(\beta^0,\tilde{\beta}^\nu)}=p^{(\alpha^1,\cdots,\alpha^\nu)}_{(\beta^0,\beta^1,\cdots,\beta^\nu)}=\partial^{\tilde{\alpha}^\nu}_{\tilde{\xi}^\nu}D^{\beta^0}_{x^0}D^{\tilde{\beta}^\nu}_{\tilde{x}^\nu}p=\partial^{\alpha^1}_{\xi^1}\cdots\partial^{\alpha^\nu}_{\xi^\nu}D^{\beta^0}_{x^0}\cdots D^{\beta^\nu}_{x^\nu}p$. When $\tilde{m}'_\nu=0$ we denote $S^{\tilde{m}_\nu;0}_{\lambda,\rho,\delta}$ simply by $S^{\tilde{m}_\nu}_{\lambda,\rho,\delta}$ (cf. $S^{M,m}_{\Phi,\varphi}$ of [**1**], [**2**]).

ii) The associated pseudo-differential operator $P=p(X, D_{\tilde{x}^\nu}, \tilde{X}^\nu)=p(X^0, D_{x^1}, X^1, \cdots, D_{x^\nu}, X^\nu)$ with symbol $p(x^0, \tilde{\xi}^\nu, \tilde{x}^\nu)$ is defined by

$$Pu(x)=O_s\text{-}\iint e^{-i\tilde{y}^\nu\cdot\tilde{\xi}^\nu}p(x, \xi^1, x+\bar{y}^1, \cdots, \xi^\nu, x+\bar{y}^\nu)u(x+\bar{y}^\nu)d\tilde{y}^\nu d\tilde{\xi}^\nu \quad \text{for } u\in\mathcal{B} \tag{2.3}$$

where $d\tilde{y}^\nu d\tilde{\xi}^\nu=dy^1 d\xi^1 \cdots dy^\nu d\xi^\nu$, $\tilde{y}^\nu\cdot\tilde{\xi}^\nu=y^1\cdot\xi^1+\cdots+y^\nu\cdot\xi^\nu$, $\bar{y}^1=y^1, \cdots, \bar{y}^\nu=y^1+\cdots+y^\nu$. In case $\nu=1$ we often write $P=p(X, D_x, X')$ as in [**6**].

REMARK 1°. For $p\in S^{\tilde{m}_\nu;\tilde{m}'_\nu}_{\lambda,\rho,\delta}$ we define semi-norms $|p|^{(\tilde{m}_\nu;\tilde{m}'_\nu)}_{l,l'}$, $l, l'=0, 1, \cdots,$ by

$$|p|^{(\tilde{m}_\nu;\tilde{m}'_\nu)}_{l,l'}=\max_{|\alpha^j|\leq l, |\beta^j|\leq l'} \inf\{C=C(\tilde{\alpha}^\nu, \beta^0, \tilde{\beta}^\nu) \text{ of } (2.2)\}\,. \tag{2.4}$$

Then $S^{\tilde{m}_\nu;\tilde{m}'_\nu}_{\lambda,\rho,\delta}$ makes a Fréchet space.

2°. For $P_j=p_j(X, D_x, X')\in S^{m_j}_{\lambda,\rho,\delta}$, $j=1, \cdots, \nu$, set $p(x^0, \tilde{\xi}^\nu, \tilde{x}^\nu)=p_1(x^0, \xi^1, x^1) \cdots p_\nu(x^{\nu-1}, \xi^\nu, x^\nu)$. Then $p(x^0, \tilde{\xi}^\nu, \tilde{x}^\nu)\in S^{\tilde{m}_\nu}_{\lambda,\rho,\delta}$ for $\tilde{m}_\nu=(m_1, \cdots, m_\nu)$ and we have

$$P_1\cdots P_\nu u(x)=p(X^0, D_{\tilde{x}^\nu}, \tilde{X}^\nu)u(x)\,. \tag{2.5}$$

In fact, take $\chi_\varepsilon(\tilde{\xi}^\nu, \tilde{y}^\nu)=\chi(\varepsilon\xi^1, \varepsilon y^1)\cdots\chi(\varepsilon\xi^\nu, \varepsilon y^\nu)$ for $\chi(\xi, x)\in\mathcal{S}$ in $R^{2n}_{\xi,x}$ such that $\chi(0, 0)=1$, and set

$$p_{j,\varepsilon}(x, \xi, x')=p_j(x, \xi, x')\chi(\varepsilon\xi, \varepsilon(x'-x))\,, \qquad j=1, \cdots, \nu\,.$$

Then by the change of variables $x+\bar{y}^1=z^1, \cdots, x+\bar{y}^\nu=z^\nu$ in (2.3) we have

$$p(x^0, D_{\tilde{x}^\nu}, \tilde{x}^\nu)u(x)=\lim_{\varepsilon\to\infty}\int\cdots\int\prod_{j=1}^{\nu}(e^{i(z^{j-1}-z^j)\cdot\xi^j}p_{j,\varepsilon}(z^{j-1}, \xi^j, z^j))u(z^\nu)d\tilde{z}^\nu d\tilde{\xi}^\nu \qquad (z^0=x)\,, \tag{2.6}$$

and have

$$\iint e^{i(z^{j-1}-z^j)\cdot\xi^j}p_{j,\varepsilon}(z^{j-1}, \xi^j, z^j)v(z^j)dz^j d\xi^j \longrightarrow (p_j(X, D_x, X')v)(z^{j-1})$$

in $\mathcal{S}$ for $v\in\mathcal{S}$. So we have (2.5).

3°. For $P=p(X, D_x)\in S^m_{\lambda,\rho,\delta}$ we have

$$p(x, \xi)=e^{-ix\cdot\xi}P(e^{ix\cdot\xi})\,, \tag{2.7}$$

since

$$e^{-ix\cdot\xi}P(e^{ix\cdot\xi})=e^{-ix\cdot\xi}O_s[e^{-iy\cdot\eta}p(x, \eta)e^{i(x+y)\cdot\xi}]$$

$$=O_s[e^{-iy(\eta-\xi)}p(x, \eta)]=O_s\text{-}\iint e^{-iy\cdot\eta}p(x, \xi+\eta)dyd\eta=p(x, \xi)\,.$$

The following lemma is fundamental in the present paper.

LEMMA A. *For a symbol* $p(x^0, \tilde{\xi}^\nu, \tilde{x}^\nu)\in S^{\tilde{m}_\nu;\tilde{m}'_\nu}_{\lambda,\rho,\delta}$, *define a single symbol* $q_\theta(x, \xi, x')$, $|\theta|\leq 1$, *by*

$$(2.8)\qquad q_\theta(x,\xi,x')=O_s-\iint e^{-i\tilde{y}^{\nu-1}\cdot\tilde{\eta}^{\nu-1}}p(x,\xi+\theta\eta^1,x+\bar{y}^1,\cdots,\xi+\theta\eta^{\nu-1},x+\bar{y}^{\nu-1},\xi,x')\cdot d\tilde{y}^{\nu-1}d\tilde{\eta}^{\nu-1}.$$

Then there exists a constant $C>0$, *depending on* $M=\sum_{j=1}^{\nu-1}(|m_j|+|m'_j|)+m'_0$ *but independent of* ν, *such that*

$$(2.9)\qquad |q_\theta(x,\xi,x')|\leqq C^{\nu+1}|p|_{l,l'}^{(\bar{m}_\nu;\bar{m}_\nu')}\lambda(\xi)^{\bar{m}_\nu+\bar{m}'_\nu}\qquad(|\theta|\leqq 1),$$

where $\bar{m}_\nu=m_1+\cdots+m_\nu$, $\bar{m}'_\nu=m'_0+m'_1+\cdots+m'_\nu$, *and*

$$(2.10)\qquad l=2[n/2+1],\qquad l'=2[(n+M)/(2(1-\delta))+1].$$

Proof will be given in § 3.

THEOREM 2.1. *For* $P=p(X^0,D_{\tilde{x}^\nu},\tilde{X}^\nu)\in S_{\lambda,\rho,\delta}^{\bar{m}_\nu;\bar{m}'_\nu}$, *set*

$$(2.11)\qquad \begin{aligned} &q(x,\xi,x')\ (=q_1(x,\xi,x')\ \text{ of } (2.8))\\ &=O_s-\iint e^{-i\tilde{y}^{\nu-1}\cdot\tilde{\eta}^{\nu-1}}\cdot p(x,\xi+\eta^1,x+\bar{y},\cdots,\xi+\eta^{\nu-1},x+\bar{y}^{\nu-1},\xi,x')d\tilde{y}^{\nu-1}d\tilde{\eta}^{\nu-1}.\end{aligned}$$

Then we have

$$(2.12)\qquad P=q(X,D_x,X')\qquad\text{and}\qquad q(x,\xi,x')\in S_{\lambda,\rho,\delta}^{\bar{m}_\nu+\bar{m}'_\nu}.$$

Furthermore, for any l, l' *there exists a constant* C *such that*

$$(2.13)\qquad |q|_{l,l'}^{(\bar{m}_\nu+\bar{m}'_\nu)}\leqq C^{\nu+1}|p|_{l_0,l_0'}^{(\bar{m}_\nu;\bar{m}'_\nu)},$$

where

$$(2.14)\qquad \begin{cases} l_0=l+2[n/2+1],\\ l'_0=l'+2[(n+\sum_{j=1}^{\nu-1}(|m_j|+|m'_j|)+m'_0+\rho l+\delta l')/(2(1-\delta))+1].\end{cases}$$

PROOF. As we got (2.6), by the change of variables $x'=x+\bar{y}^\nu=x+\bar{y}^{\nu-1}+y^\nu$, $\xi=\xi^\nu$, $\eta^j=\xi^j-\xi^\nu$, $j=1,\cdots,\nu-1$, we have

$$Pu(x)=\lim_{\varepsilon\to\infty}\iint e^{i(x-x')\cdot\xi}\Big\{\iint e^{-i\tilde{y}^{\nu-1}\cdot\tilde{\eta}^{\nu-1}}\chi_\varepsilon(\xi+\eta^1,x+\bar{y}^1,\cdots,\xi+\eta^{\nu-1},x+\bar{y}^{\nu-1},\xi,x'-x-\bar{y}^{\nu-1})P(x,\xi+\eta^1,\cdots,x+\bar{y}^{\nu-1},\xi,x')d\tilde{y}^{\nu-1}d\tilde{\eta}^{\nu-1}\Big\}u(x')dx'd\xi,$$

and using Proposition 1.2 we get $P=q(X,D_x,X')$. Since

$$\begin{aligned} &q_{(\beta,\beta')}^{(\alpha)}(x,\xi,x')\\ &=O_s-\iint e^{-i\tilde{y}^{\nu-1}\cdot\tilde{\eta}^{\nu-1}}\cdot\partial_\xi^\alpha D_x^\beta D_{x'}^{\beta'}p(x,\xi+\eta^1,\cdots,x+\bar{y}^{\nu-1},\xi,x')d\tilde{y}^{\nu-1}d\tilde{\eta}^{\nu-1},\end{aligned}$$

we have by Lemma A (2.13) and $q\in S_{\lambda,\rho,\delta}^{\bar{m}_\nu+\bar{m}'_\nu}$. Q. E. D.

THEOREM 2.2. *Let* $P_j=p_j(X,D_x,X')\in S_{\lambda,\rho,\delta}^{m_j}$, $j=1,\cdots,\nu$. *Then*, $Q=P_1\cdots P_\nu\in S_{\lambda,\rho,\delta}^{\bar{m}_\nu}$ *and for any* l, l' *there exists a constant* $C=C(l,l')$ *such that*

(2.15) $$|\sigma(Q)|_{l,l'}^{(\bar{m}_\nu)} \leqq C^{\nu+1} \prod_{j=1}^{\nu} |p|_{l_0,l_0'}^{(m_j)},$$

where l_0, l_0' *is defined by* (2.14).

Proof is clear from Theorem 2.1.

THEOREM 2.3 (Calderón-Vaillancourt [3]). *Let* $P=p(X, D_x, X')\in S^0_{\lambda,\rho,\delta}$ $(0\leqq\delta<1)$. *Then there exists a constant* C *such that*

(2.16) $$\|Pu\|_{L^2}\leqq C|p|_{l_1,l_2}^{(0)}\|u\|_{L^2} \quad \textit{for} \quad u\in\mathcal{S},$$

where $l_1=2[n/2+1]$, $l_2=2[n/(2(1-\delta))+1]$.

PROOF. I) For $P_0=p_0(X, D_x, X')\in S^0_{\lambda,\rho,\delta}$ we first assume that $p_0(x, \xi, x')=0$ for $|x|+|\xi|+|x'|\geqq R$ (>0). Then setting $K_0(x, x')=\int e^{i(x-x')\cdot\xi}p_0(x, \xi, x')đ\xi$ we have $P_0u(x)=\int K_0(x, x')u(x')dx'$, and $|K_0(x, x')|\leqq C_R|p_0|_{0,0}^{(0)}$ for $C_R=$the volume of $\{|\xi|\leqq R\}$. So, noting $K_0(x, x')=0$ for $|x|+|x'|\geqq R$, we have

(2.17) $$\|P_0u\|_{L^2}\leqq C_R^2|p_0|_{0,0}^{(0)}\|u\|_{L^2} \quad \text{for} \quad u\in\mathcal{S}.$$

II) Consider $Q_\nu=\overbrace{P^*P\cdots P^*P}^{\nu}$ for $\nu=2^l$, $l=1, 2, \cdots$. Then we have $\|P\|^\nu\leqq\|Q_\nu\|$. Note that $\sigma(Q_\nu)(x, \xi, x')=0$ for $|x|+|\xi|+|x'|\geqq R$ if $p(x, \xi, x')=0$ for $|x|+|\xi|+|x'|\geqq R$, and that $\sigma(P^*)(x, \xi, x')=\overline{p(x', \xi, x)}$. Then we get by (2.17) and Theorem 2.2

$$\|P\|^\nu\leqq\|Q_\nu\|\leqq C_R^2|\sigma(Q_\nu)|_{0,0}^{(0)}\leqq C_R^2C^{\nu+1}(|p|_{l_1,l_2}^{(0)})^\nu,$$

and by letting $\nu\to\infty$ we get (2.16). For the general $P=p(X, D_x, X')\in S^0_{\lambda,\rho,\delta}$ we have $Pu=\lim_{\varepsilon\to0}P_\varepsilon u$ in L^2 for $u\in\mathcal{S}$, if we set $\sigma(P_\varepsilon)(x, \xi, x')=\chi(\varepsilon x, \varepsilon\xi, \varepsilon x')\cdot p(x, \xi, x')$ for $\chi(x, \xi, x')\in C_0^\infty$ in $\{|x|+|\xi|+|x'|<1\}$ such that $\chi(0)=1$. Hence, noting $\sigma(P_\varepsilon)=0$ for $|x|+|\xi|+|x'|\geqq\varepsilon^{-1}$ we get (2.16) for the general case.

Q. E. D.

THEOREM 2.4. *For* $P=p(X^0, D_{\tilde{x}^\nu}, \tilde{X}^\nu)\in S^{\tilde{m}_\nu;\bar{m}'_\nu}_{\lambda,\rho,\delta}$ *set*

(2.18) $$p_{\tilde{\alpha}^{\nu-1}}(x, \xi, x')=p^{(\alpha^1,\cdots,\alpha^{\nu-1},0)}_{(0,\alpha_1^1,\alpha_2^1+\alpha_1^2,\cdots,\alpha_{\nu-1}^1+\cdots+\alpha_1^{\nu-1},0)}(x, \xi, \cdots, x, \xi, x'),$$

which belongs to $S^{\bar{m}_\nu+\bar{m}_{\nu'}-(\rho-\delta)|\tilde{\alpha}^{\nu-1}|}_{\lambda,\rho,\delta}$, *where*

(2.19) $$\begin{cases} \tilde{\alpha}^{\nu-1}=(\alpha^1, \cdots, \alpha^{\nu-1}), & |\tilde{\alpha}^{\nu-1}|=|\alpha^1|+\cdots+|\alpha^{\nu-1}|, \\ \alpha^j=\alpha_1^j+\cdots+\alpha_{\nu-j}^j & (j=1, \cdots, \nu-1). \end{cases}$$

Then for any N *there exists* $r_N(x, \xi, x')\in S^{\bar{m}_\nu+\bar{m}_{\nu'}-(\rho-\delta)N}_{\lambda,\rho,\delta}$ *such that*

(2.20) $$\sigma(P)(x, \xi, x')=\sum_{|\tilde{\alpha}^{\nu-1}|<N}\frac{1}{\prod_{j=1}^{\nu-1}(\alpha_1^j!\cdots\alpha_{\nu-j}^j!)}p_{\tilde{\alpha}^{\nu-1}}(x, \xi, x') + r_N(x, \xi, x') \quad (\textit{cf. the expansion form in } [\mathbf{9}]).$$

PROOF. We take Taylor's expansion for $p(x, \xi+\eta^1, \cdots, x+\bar{y}^{\nu-1}, \xi, x')$ with respect to η. Then by Proposition 1.3 and Lemma A we get easily (2.20).

§ 3. Proof of Lemma A.

For $n_0=[n/2+1]$ we first write by integration by parts

$$(3.1)\qquad q_\theta(x, \xi, x')=O_s-\iint e^{-i\tilde{y}^{\nu-1}\cdot\tilde{\eta}^{\nu-1}}\prod_{j=1}^{\nu-1}(1+(-\Delta_{\eta^j})^{n_0}(\lambda(\xi+\theta\eta^j)^{2n_0\delta}\cdot)$$
$$\times\prod_{j=1}^{\nu-1}(1+\lambda(\xi+\theta\eta^j)^{2n_0\delta}|y^j|^{2n_0})^{-1}p(x, \xi+\theta\eta^1, \cdots, x+\bar{y}^{\nu-1}, \xi, x')d\tilde{y}^{\nu-1}d\tilde{\eta}^{\nu-1}.$$

Then by the change of variables: $(y^1, \cdots, y^{\nu-1})\to(\bar{y}^1, \cdots, \bar{y}^{\nu-1})$ such that $\bar{y}^1=y^1, \cdots, \bar{y}^{\nu-1}=y^1+\cdots+y^{\nu-1}$, and by integration by parts we have for $0\leqq k_j=k$, $(\eta^j, \eta^{j+1})\leqq l'/2$

$$(3.2)\qquad q_\theta(x, \xi, x')=\iint e^{-i\sum_{j=1}^{\nu-1}\bar{y}^j\cdot(\eta^j-\eta^{j+1})}\Big(\prod_{j=1}^{\nu-1}|\eta^j-\eta^{j+1}|^{-2k_j}\Big)$$
$$\times\prod_{j=1}^{\nu-1}(-\Delta_{\bar{y}^j})^{k_j}r_\theta(x, \xi, x'; \tilde{\eta}^{\nu-1}, \bar{y}^{\nu-1})d\tilde{\bar{y}}^{\nu-1}d\tilde{\eta}^{\nu-1}\qquad(\eta^\nu=0),$$

where

$$(3.3)\qquad r_\theta(x, \xi, x'; \tilde{\eta}^{\nu-1}, \tilde{\bar{y}}^{\nu-1})=\prod_{j=1}^{\nu-1}(1+(-\Delta_{\eta^j})^{n_0}(\lambda(\xi+\theta\eta^j)^{2n_0\delta}\cdot)$$
$$\times\prod_{j=1}^{\nu-1}(1+\lambda(\xi+\theta\eta^j)^{2n_0\delta}|\bar{y}^j-\bar{y}^{j-1}|^{2n_0})^{-1}p(x, \xi+\theta\eta^1, \cdots, x+\bar{y}^{\nu-1}, \xi, x')$$
$$(\bar{y}^0=0).$$

Noting that

$$\int(1+\lambda(\xi+\theta\eta^j)^{2n_0\delta}|\bar{y}^j-\bar{y}^{j-1}|^{2n_0})^{-1}d\bar{y}^j\leqq C_1\lambda(\xi+\theta\eta^j)^{-n\delta},$$

we have for a constant $C=C(l, l')$

$$(3.4)\qquad |q_\theta(x, \xi, x')|\leqq C^{\nu+1}|p|_{l,l'}^{(\tilde{m}_\nu;\tilde{m}'_\nu)}\lambda(\xi)^{m_\nu+m_{\nu'}}\int\lambda(\xi+\theta\eta^1)^{m_0'}$$
$$\times\prod_{j=1}^{\nu-1}(|\eta^j-\eta^{j+1}|^{-2k_j}\lambda(\xi+\theta\eta^j)^{m_j-n\delta}(\lambda(\xi+\theta\eta^j)+\lambda(\xi+\theta\eta^{j+1}))^{m_j'+2k_j\delta})d\tilde{\eta}^{\nu-1}$$
$$(\eta^\nu=0).$$

We have from (2.1)

$$(3.5)\qquad \lambda(\xi+\theta\eta^{j+1})/2\leqq\lambda(\xi+\theta\eta^j)\leqq 2\lambda(\xi+\theta\eta^{j+1})$$
$$\text{for}\quad |\eta^j-\eta^{j+1}|\leqq c_0\lambda(\xi+\theta\eta^{j+1}),$$

and

$$(3.6)\qquad \lambda(\xi+\theta\eta^j)\leqq C_2|\eta^j-\eta^{j+1}|\qquad\text{for}\quad |\eta^j-\eta^{j+1}|\geqq c_0\lambda(\xi+\theta\eta^{j+1})$$

for a small constant $c_0>0$ and a large constant $C_2>0$.

Set $\Omega_{j,1}=\{\eta^j;|\eta^j-\eta^{j+1}|\leqq c_0\lambda(\xi+\theta\eta^{j+1})^{\delta}\}$, $\Omega_{j,2}=\{\eta^j;\ c_0\lambda(\xi+\theta\eta^{j+1})^{\delta}\leqq|\eta^j-\eta^{j+1}|\leqq c_0\lambda(\xi+\theta\eta^{j+1})\}$ and $\Omega_{j,3}=\{\eta^j;|\eta^j-\eta^{j+1}|\geqq c_0\lambda(\xi+\theta\eta^{j+1})\}$, and set $k_j=0$ for $\eta^j\in\Omega_{j,1}$ and $=l'/2$ for $\eta^j\in\Omega_{j,2}\cup\Omega_{j,3}$. Then we can prove by induction

$$(3.7)\qquad A_{j_0}\equiv\int\lambda(\xi+\theta\eta')^{m_0'}\prod_{j=1}^{j_0}\{|\eta^j-\eta^{j+1}|^{-2k_j}\lambda(\xi+\theta\eta^j)^{m_j}$$

$$\times(\lambda(\xi+\theta\eta^j)+\lambda(\xi+\theta\eta^{j+1}))^{m_j'+2k_j\delta}\}d\eta^j$$

$$\leqq C_3^{j_0+1}\lambda(\xi+\theta\eta^{j_0+1})^{\bar{m}_{j_0}+\bar{m}_{j_0}'},\qquad (j_0=1,2,\cdots,\nu-1)$$

for a large constant C_3 independent of j_0 and ν. To get (3.7) we have only to prove that

$$\int\{|\eta^{j_0}-\eta^{j_0+1}|^{-2k_{j_0}}\lambda(\xi+\theta\eta^{j_0})^{\bar{m}_{j_0}+\bar{m}_{j_0-1}'-n\delta}$$

$$\times(\lambda(\xi+\theta\eta^{j_0})+\lambda(\xi+\theta\eta^{j_0+1}))^{m'_{j_0}+2k_{j_0}\delta}\}d\eta^{j_0}\leqq C_4\lambda(\xi+\theta\eta^{j_0+1})^{\bar{m}_{j_0}+\bar{m}'_{j_0}},$$

which can be done by dividing the integrand into three parts $\Omega_{j_0,1}$, $\Omega_{j_0,2}$ and $\Omega_{j_0,3}$ and using (3.5) and (3.6). Here we use the condition (2.10) to obtain

$$\lambda(\xi+\theta\eta^{j_0})^{\bar{m}_{j_0}+\bar{m}'_{j_0-1}-n\delta}\leqq C_4\lambda(\xi+\theta\eta^{j_0+1})^{(\bar{m}_{j_0})_++\bar{m}'_{j_0-1}},$$

$$-l'(1-\delta)+n+(\bar{m}_{j_0})_+\leqq\bar{m}_{j_0},\ j_0=1,\cdots,\nu-1,\ (\bar{m}_{j_0})_+=\max(0,\bar{m}_{j_0}).$$

Finally setting $j_0=\nu-1$ in (3.7) we get (2.9) from (3.4). Q. E. D.

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