J. Math. Soc. Japan
Vol. 27, No. 1, 1975

Point derivations on commutative Banach algebras and estimates of the $A(X)$-metric norm

By Mikihiro HAYASHI

(Received May 28, 1973)

§ 1. Introduction.

Let A be a commutative Banach algebra with unit 1. We denote the set of all homomorphisms of A onto $\boldsymbol{C}$ by $\mathcal{M}(A)$, which is called the maximal ideal space of A. For $\phi\in\mathcal{M}(A)$, a point derivation on A at ϕ is an (algebraic) linear functional D on A with the property that $D(fg)=\phi(f)D(g)+\phi(g)D(f)$ for all $f, g\in A$. In this paper we consider the point derivations which are defined as follows. Let $\hat{f}(\phi)=\phi(f)$ be the Gelfand transform and let $\{\phi_{r}, t_{r}\}$ be a pair of nets in $\mathcal{M}(A)\times\boldsymbol{C}\backslash\{0\}$ with the following properties:

(1.1) ϕ_{r} converges to ϕ in $\mathcal{M}(A)$ with the weak*-topology,

(1.2) t_{r} converges to 0 in $\boldsymbol{C}$,

(1.3) $\dfrac{\hat{f}(\phi_{r})-\hat{f}(\phi)}{t_{r}}$ converges for any $f\in A$.

Then the limit $D(f)=\lim_{r}\dfrac{\hat{f}(\phi_{r})-\hat{f}(\phi)}{t_{r}}$ defines a point derivation at ϕ.

In section 2 considering this kind of point derivation we shall give an another proof of Browder's theorem; there exists a nonzero point derivation at ϕ if ϕ is not isolated in $\mathcal{M}(A)$ with the metric topology. Also we shall prove that there exists a nonzero continuous point derivation at ϕ if ϕ is not isolated in $\mathcal{M}(A)$ with the metric topology and the norm $\|\psi-\phi\|$ of the metric topology is equivalent to a semi-metric $|\psi(w_{1})-\phi(w_{1})|+\cdots+|\psi(w_{n})-\phi(w_{n})|$ of the weak* topology in some metric neighborhood of ϕ in $\mathcal{M}(A)$, where $w_{1}, \cdots, w_{n}\in A$.

In the remaining sections we shall consider the function algebra $A(X)$ on a compact plane set X. In this case we obtain more exact results. As is well known, these results are translated for the case $R(X)$ and the proofs for the case $R(X)$ are performed similarly. We state here the corresponding results for $R(X)$. Let $R_{0}(X)$ be the set of all rational functions with poles off X. $R(X)$ is the uniform closure of $R_{0}(X)$ on X. The maximal ideal space of $R(X)$ is identified with X. It is known that each of the following conditions is equivalent to the existence of a nonzero continuous point derivation on $R(X)$ at $x\in X$:

(r.1) There exists a constant k such that $|f'(x)|\leqq k\|f\|$ for all $f\in R_0(X)$.

(r.2) (Wilken [5].) There exists a complex representing measure μ for x such that $\int\frac{d|\mu|(z)}{|z-x|}<\infty$.

(r.3) (Hallstrom [4].) $\sum_{n=0}^{\infty}\frac{4^n}{r^2}\gamma(E_n(x;r)\backslash X)<\infty$;

where $E_n(x;r)=\left\{z:\frac{r}{2^{n+1}}<|z-x|<\frac{r}{2^n}\right\}$, and γ is the analytic capacity. For a plane set D, the analytic capacity of D, $\gamma(D)$, is defined by

$$\gamma(D)=\sup\{|f'(\infty)|:f\in\mathcal{A}(D)\},$$

where $\mathcal{A}(D)$ is the set of all functions on the Riemann sphere S^2 such that f is analytic off a compact subset of D, $\|f\|_{S^2}\leqq 1$ and $f(\infty)=0$. In section 3 we shall give two another equivalent conditions:

(r.4) There exists a sequence $x_n\in X$ which converges to x and has the property that $\frac{f(x_n)-f(x)}{x_n-x}$ converges for any $f\in R(X)$.

(r.5) $\varlimsup_{\substack{z\to x\\ z\in X}}\frac{\|z-x\|^R}{|z-x|}<\infty$;

where $\|.\|^R$ denotes the $R(X)$-metric norm, which is defined by $\|x-y\|^R=\sup\{|f(x)-f(y)|:f\in R(X),\ \|f\|\leqq 1\}$.

In section 4 we aim to estimates the $A(X)$-metric norm by the continuous analytic capacity. Although the estimates for $A(X)$ are given in section 4 precisely, we write here the main corresponding three estimates for $R(X)$. Let x, $y\in X$ and let $\Delta(x;r)=\{z:|z-x|<r\}$. We denote the distance between a point z and a set T by $d(z,T)$.

$$\text{(R-2)}\qquad \|x-y\|^R\geqq\frac{1}{5}\frac{4\sum_{n=0}^{\infty}\frac{2^n}{r}\gamma(E_n(x;r)\backslash X)}{3+4\sum_{n=0}^{\infty}\frac{2^n}{r}\gamma(E_n(x;r)\backslash X)}-\frac{\gamma(\Delta(x;r)\backslash X)}{d(y,\Delta(x;r)\backslash X)}.$$

Let $0<\sigma<1$ and C be a universal constant. If $|x-y|<\frac{r}{4}$, then

$$\text{(R-4)}\qquad \|x-y\|^R\leqq C\Big[\frac{|x-y|}{r-2|x-y|}+\frac{4\sqrt{|x-y|}}{\sqrt{r}-2\sqrt{|x-y|}}\sum_{n=0}^{\infty}\frac{2^n}{r}\gamma(E_n(x;r)\backslash X)$$
$$+\frac{4}{\sigma}\sum_{n=\left[\frac{1}{2}\log_2\frac{r}{|x-y|}\right]}^{\infty}\frac{2^n}{r}\gamma(E_n(x;r)\backslash X)+\frac{4}{1-\sigma}\sum_{n=0}^{\infty}\frac{2^n}{\sigma|x-y|}\gamma(E_n(y;\sigma|x-y|)\backslash X)\Big],$$

where $\left[\frac{1}{2}\log_2\frac{r}{|x-y|}\right]$ denotes the maximum integer which does not exceed

$\frac{1}{2}\log_2\frac{r}{|x-y|}$. If $|x-y|<\frac{r}{8}$, then

(R-5) $$\|x-y\|^R \leqq C|x-y|\Big[\frac{1}{r-2|x-y|}+\frac{48}{\sigma}\sum_{n=0}^{\infty}\frac{4^n}{r^2}\gamma(E_n(x;r)\backslash X) + \frac{1}{1-\sigma}\frac{4}{|x-y|}\sum_{n=0}^{\infty}\frac{2^n}{\sigma|x-y|}\gamma(E_n(y;\sigma|x-y|)\backslash X)\Big].$$

In section 5 we shall prove the following results by the application of the above estimates. Let $x_n \in X$ be a sequence which converges to x. Then x_n converges to x in the $R(X)$-metric topology if and only if

$$\lim_{n\to\infty}\sum_{k=0}^{\infty}\frac{2^k}{\sigma|x-x_n|}\gamma(E_k(x_n;\sigma|x-x_n|)\backslash X)=0$$

for any fixed $0<\sigma<1$. And x_n has the property (r.4) if and only if

$$\overline{\lim_{n\to\infty}}\frac{1}{|x-x_n|}\sum_{k=0}^{\infty}\frac{2^k}{\sigma|x-x_n|}\gamma(E_k(x_n;\sigma|x-x_n|)\backslash X)<\infty$$

for any fixed $0<\sigma<1$.

The notations $\Delta(x;r)$, $E_n(x;r)$, and $d(z, T)$ remain valid throughout the paper.

ACKNOWLEDGEMENTS. The author has been studying the theory of function algebras with Professors S. Koshi and T. Ando at Hokkaido University. He thanks them for all their encouragement and advice during his research. The argument in section 3 is an analogue of Browder [**1**] and the estimates (A-1) and (A-2) in section 4 are essentially due to Curtis [**2**].

§2. Sequential derivations.

Let A be a commutative Banach algebra and $\mathcal{M}(A)$ the maximal ideal space. For $\phi\in\mathcal{M}(A)$, $D_\phi(A)(T_\phi(A))$ denotes the set of all (continuous) point derivations on A at ϕ, and A_ϕ denotes the kernel of ϕ. It is easy to see that a linear functional D on A is a point derivation at ϕ if and only if $D(fg)=0$ for all $f, g\in A_\phi$ and $D(1)=0$. Thus $D_\phi(A)$ is identified with the algebraic dual space of A_ϕ/A_ϕ^2, where $A_\phi^2=\{f_1g_1+\cdots+f_kg_k : f_i, g_i\in A_\phi\}$.

2.1. DEFINITION. Let $\phi\in\mathcal{M}(A)$ and let $\{\phi_r, t_r\}$ be a pair of nets in $\mathcal{M}(A)\times \boldsymbol{C}\backslash\{0\}$ with the properties (1.1), (1.2) and (1.3). Then we say that $\{\phi_r, t_r\}$ is a sequential derivation at ϕ for A and that $D(f)=\lim_{r}\frac{\hat{f}(\phi_r)-\hat{f}(\phi)}{t_r}$ is the point derivation defined by $\{\phi_r, t_r\}$.

2.2. THEOREM. *If $\{\phi_n, t_n\}_{n=1}^{\infty}$ is a sequential derivation at ϕ, then the point derivation D defined by $\{\phi_n, t_n\}$ is continuous; more precisely, it follows that*

$$\|D\| \leqq \varliminf_{n\to\infty} \frac{\|\phi_n-\phi\|}{|t_n|} \leqq \varlimsup_{n\to\infty} \frac{\|\phi_n-\phi\|}{|t_n|} < \infty .$$

In particular, ϕ_n must converge to ϕ in the metric topology.

PROOF. Regarding ϕ_n and ϕ as bounded linear functionals on the Banach space A, we have $\left(\frac{\phi_n-\phi}{t_n}\right)(f)=\frac{\hat{f}(\phi_n)-\hat{f}(\phi)}{t_n}$ for $f\in A$. Hence, the theorem follows from the uniform boundedness theorem.

2.3. COROLLARY (Browder [1]). *If ϕ is not isolated in $\mathcal{M}(A)$ with the metric topology, then there exists a nonzero point derivation at ϕ.*

PROOF. By the hypothesis we can take a sequence ϕ_n in $\mathcal{M}(A)$ such that $\phi_n\neq\phi$ and ϕ_n converges to ϕ in the metric topology. If $D_\phi(A)=\{0\}$, then $A_\phi=A_\phi^2$. Therefore any element f of A can be represented in the form $f-\hat{f}(\phi)=g_1h_1+\cdots+g_kh_k$ for some $g_i, h_i\in A_\phi$. Hence, we have

$$\varlimsup_{n\to\infty} \frac{|\hat{f}(\phi_n)-\hat{f}(\phi)|}{\|\phi_n-\phi\|} \leqq \varlimsup_{n} \frac{\sum_{i=1}^{k} |\phi_n(g_i)-\phi(g_i)|\,|\phi_n(h_i)|}{\|\phi_n-\phi\|}$$

$$\leqq \varlimsup_{n} \sum_{i=1}^{k} \frac{\|\phi_n-\phi\|\,\|g_i\|}{\|\phi_n-\phi\|}\,|\phi_n(h_i)|=0 .$$

Since this holds for all $f\in A$, we have

$$\varlimsup_{n} \frac{|\hat{f}(\phi_n)-\hat{f}(\phi)|}{\|\phi_n-\phi\|^2} \leqq \varlimsup_{n} \sum_{i=1}^{k} \frac{|\phi_n(g_i)-\phi(g_i)|}{\|\phi_n-\phi\|}\,\frac{|\phi_n(h_i)-\phi(h_i)|}{\|\phi_n-\phi\|}=0 .$$

Therefore $\{\phi_n, \|\phi_n-\phi\|^2\}$ must be a sequential derivation. Since $\varlimsup_{n} \frac{\|\phi_n-\phi\|}{\|\phi_n-\phi\|^2}=\infty$, we have a contradiction.

Now we shall consider a pair $\{\phi_\gamma, t_\gamma\}$ of nets under a slight weak condition.

2.4. LEMMA. *Let $\{\phi_\gamma, t_\gamma\}$ be a pair of nets in $\mathcal{M}(A)\times \boldsymbol{C}\backslash\{0\}$ with the properties* (1.1), (1.2) *and*

$$p(f)=\varlimsup_{\gamma}\left|\frac{\hat{f}(\phi_\gamma)-\hat{f}(\phi)}{t_\gamma}\right|<\infty \qquad \text{for all } f\in A . \tag{2.1}$$

Then p is a semi-norm on A. If a linear functional D on A satisfies

$$|D(f)|\leqq p(f) \qquad \text{for all } f\in A , \tag{2.2}$$

then D is a point derivation at ϕ. And if $\varlimsup_{\gamma} \frac{\|\phi_\gamma-\phi\|}{t_\gamma}<\infty$, then the point derivation D is continuous.

PROOF. It is clear that p is a semi-norm. If D is a linear functional on A with the property (2.2), then for any $f, g\in A_\phi$ it holds that

$$|D(fg)|\leqq p(fg)=\varlimsup_{\gamma}\left|\frac{\hat{f}(\phi_\gamma)-\hat{f}(\phi)}{t_\gamma}\hat{g}(\phi_\gamma)+\frac{\hat{g}(\phi_\gamma)-\hat{g}(\phi)}{t_\gamma}\hat{f}(\phi)\right|$$

$$\leqq p(f)|\phi(g)|+p(g)|\phi(f)|=0 ,$$

and $|D(1)| \leqq p(1)=0$. Hence D is a point derivation at ϕ. Since $p(f) \leqq \overline{\lim_{\gamma}} \frac{\|\phi_\gamma-\phi\|}{|t_\gamma|}\|f\|$, the last statement holds.

As an application of the Hahn-Banach extension theorem, we have the following corollary.

2.5. COROLLARY. *If there is an element w of A such that $p(w) \neq 0$, then there exists a nonzero point derivation D at ϕ such that $D(w)=p(w)$ and $|D(f)| \leqq p(f)$ for all $f \in A$.*

2.6. THEOREM. *Let $w_1, \cdots, w_n \in A_\phi$. Let $\{\phi_\gamma\}$ be a net in $\mathcal{M}(A)$ which converges to ϕ and has the property*

$$\overline{\lim_{\gamma}} \frac{\|\phi_\gamma-\phi\|}{|\phi_\gamma(w_1)|+\cdots+|\phi_\gamma(w_n)|}<\infty . \tag{2.3}$$

Then there exists a nonzero continuous point derivation at ϕ. Moreover, if (2.3) *diverges for any lack of elements $w_1, \cdots, w_n$, then* $\dim T_\phi(A) \geqq n$.

PROOF. Removing elements out of $w_1, \cdots, w_n$ within the property (2.3) as we can, it suffices to verify the last statement. Hence we assume (2.3) and that for each $k=1, \cdots, n$,

$$\overline{\lim_{\gamma}} \frac{\|\phi_\gamma-\phi\|}{\sum_{i \neq k}|\phi_\gamma(w_i)|}=\infty .$$

Then there exists a subnet ϕ_{γ_k} for each k such that

$$\lim_{\gamma_k} \frac{\|\phi_{\gamma_k}-\phi\|}{\sum_{i \neq k}|\phi_{\gamma_k}(w_i)|}=\infty ,$$

and this yields

$$\begin{aligned} \overline{\lim_{\gamma_k}} \frac{\|\phi_{\gamma_k}-\phi\|}{|\phi_{\gamma_k}(w_k)|} &= \overline{\lim_{\gamma_k}} \frac{1}{\dfrac{\sum_{i=1}^{n}|\phi_{\gamma_k}(w_i)|}{\|\phi_{\gamma_k}-\phi\|}-\dfrac{\sum_{i \neq k}|\phi_{\gamma_k}(w_i)|}{\|\phi_{\gamma_k}-\phi\|}} \\ &= \overline{\lim_{\gamma_k}} \frac{\|\phi_{\gamma_k}-\phi\|}{\sum_{i=1}^{n}|\phi_{\gamma_k}(w_i)|}<\infty . \end{aligned}$$

Therefore we can define continuous semi-norms p_k on A by

$$p_k(f)=\overline{\lim_{\gamma_k}} \frac{|\hat{f}(\phi_{\gamma_k})-\hat{f}(\phi)|}{|\phi_{\gamma_k}(w_k)|} \quad \text{for } f \in A .$$

Since $p_k(w_k)=1$, there exist continuous point derivations D_k at ϕ by Corollary 2.5 such that

$$D_k(w_k)=1 \quad \text{and} \quad |D_k(f)| \leqq p_k(f) \quad \text{for } f \in A .$$

Finally, $D_1, \cdots, D_n$ are linearly independent, for

$$p_k(w_j)=\overline{\lim_{r_k}}\frac{|\hat{w}_j(\phi_{r_k})|}{|\phi_{r_k}(w_k)|}\leqq\overline{\lim_{r_k}}\frac{\sum_{i\neq k}|\phi_{r_k}(w_i)|}{\|\phi_{r_k}-\phi\|}\frac{\|\phi_{r_k}-\phi\|}{|\phi_{r_k}(w_k)|}=0 \quad \text{if} \quad k\neq j\,.$$

This completes the proof.

2.7. COROLLARY. *Let ϕ be a non-isolated point of $\mathcal{M}(A)$ in the metric topology. If there exist $w_1, \cdots, w_n\in A_\phi$ with the property;*

$$\underline{\lim}_{\psi\to\phi}\frac{\|\psi-\phi\|}{|\psi(w_1)|+\cdots+|\psi(w_n)|}<\infty,$$

then there exists a nonzero continuous point derivation at ϕ.

For $w_1, \cdots, w_n\in A$, $(w_1, \cdots, w_n)(\psi, \phi)=\sum_{i=1}^{n}|\psi(w_i)-\phi(w_i)|$ is a semi-metric on $\mathcal{M}(A)$ for the weak* topology. The semi-metric $(w_1, \cdots, w_n)(\psi, \phi)$ and the metric of the norm $\|\psi-\phi\|$ are said to be equivalent on a subset M of $\mathcal{M}(A)$ if and only if for ψ, $\phi\in M$

$$K\|\psi-\phi\|\leqq(w_1, \cdots, w_n)(\psi, \phi)\leqq\max_{1\leqq i\leqq n}\|w_i\|\cdot\|\psi-\phi\|\,;$$

where K is some constant and the last inequality holds always.

2.8. COROLLARY. *Let $w_1, \cdots, w_n\in A$. If the semi-metric $(w_1, \cdots, w_n)(\psi, \phi)$ and the metric of the norm $\|\psi-\phi\|$ are equivalent on a metric open set U of $\mathcal{M}(A)$, then there exists a nonzero continuous point derivation at any non-isolated point ϕ of U.*

The following lemma is for the next section.

2.9. LEMMA. *Let $\phi\in\mathcal{M}(A)$. Let $\{\phi_r, t_r\}$ be a pair of nets in $\mathcal{M}(A)\times\boldsymbol{C}\backslash\{0\}$ with the properties* (1.1), (1.2) *and* $\overline{\lim_r}\frac{\|\phi_r-\phi\|}{|t_r|}<\infty$. *Suppose there exists a dense subset A_0 of A such that*

$$\lim_r\frac{\hat{f}(\phi_r)-\hat{f}(\phi)}{t_r} \text{ exist for all } f\in A_0\,.$$

Then $\{\phi_r, t_r\}$ is a sequential derivation at ϕ for A.

The proof is formal and will be omitted.

§ 3. Point derivations for $A(X)$.

From now on X denotes a compact subset of the complex plane $\boldsymbol{C}$, and $A(X)$ denotes the uniform closed algebra of all continuous functions on X which is analytic in the interior of X. The interior of X will be denoted by X^0. Let $x\in X$. Let $A(X; x)$ be the set of all functions of $A(X)$ which admit analytic continuation to some neighborhood of x. Then $A(X; x)$ is a uniformly dense subalgebra of $A(X)$, and this implies that the maximal ideal space of $A(X)$ is X ([3], Chap. II, Th. 1.8 and Cor. 1.10). When the functional $f\mapsto f'(x)$ is continuous on $A(X; x)$, the unique continuous extension on $A(X)$ of this functional is a continuous point derivation on $A(X)$ at x, and so we may use the notation $f'(x)$ also for all $f\in A(X)$. We can easily verify that any continuous point derivation on $A(X)$ at x is a constant multiple of $f\mapsto f'(x)$ if there exists a nonzero continuous point derivation on $A(X)$ at x.

Now we prove the equivalence of the following conditions which were stated for $R(X)$ in section 1:

(a.0) There exists a nonzero continuous point derivation on $A(X)$ at x.

(a.1) There exists a constant k such that $|f'(x)|\leqq k\|f\|$ for all $f\in A(X; x)$.

(a.2) There exists a complex representing measure μ such that $\int\frac{d|\mu|(z)}{|z-x|}<\infty$.

(a.3) $\sum_{n=0}^{\infty}\frac{4^n}{r^2}\alpha(E_n(x; r)\backslash X^0)<\infty$; where α denotes the continuous analytic capacity (see section 4).

(a.4) There exists a sequential derivation of the form $\{x_n, x_n-x\}$ at x for $A(X)$.

(a.5) $\varlimsup_{\substack{z\to x\\ z\in X}}\frac{\|z-x\|^A}{|z-x|}<\infty$;

where $\|z-x\|^A$ denotes the metric norm for $A(X)$, i.e.,

$$\|z-x\|^A=\sup\{|f(z)-f(x)| : f\in A(X),\ \|f\|\leqq 1\}.$$

The equivalence of (a.0) and (a.1) follows from the comments at the beginning of this section. Hence the equivalence of the conditions (a.0)$\sim$(a.3) is a formal modification of Wilken's [5] and Hallstrom's [3]. By Theorem 2.2, (a.4) implies (a.1). Furthermore, Corollary 2.7 and Lemma 2.9 imply:

3.1. THEOREM. *$\{x_n, x_n-x\}$ is a sequential derivation at x if and only if* $\varlimsup_{n\to\infty}\frac{\|x_n-x\|^A}{|x_n-x|}<\infty$.

This shows the equivalence of (a.4) and (a.5). Hence, to complete the equivalence, it suffices to show that (a.2) implies (a.4).

Let m be the two-dimensional Lebesgue measure on the complex plane $\boldsymbol{C}$. Let μ be a (regular Borel) measure on X. For each $x\in\boldsymbol{C}$, we put

$$\tilde{\mu}(z)=\int\frac{d|\mu|(w)}{|w-z|}$$

$$\hat{\mu}(z)=\int\frac{d\mu(w)}{w-z}\qquad\text{when}\quad\tilde{\mu}(z)<\infty.$$

An application of Fibini's theorem shows that $\tilde{\mu}$ is locally integrable with respect to m, in particular, $\tilde{\mu}(z)<\infty$ a.e. (m). If $\tilde{\mu}(x)<\infty$, then $\frac{\mu(w)}{w-x}$ is a measure on X. Hence, for each $z\in \boldsymbol{C}$, we also put

$$\tilde{\mu}(z, x)=\Big(\frac{\mu(w)}{w-x}\Big)^{\sim}(z)=\int\frac{d|\mu|(w)}{|w-z||w-x|} \tag{3.1}$$

$$\hat{\mu}(z, x)=\Big(\frac{\mu(w)}{w-x}\Big)^{\wedge}(z)=\int\frac{d\mu(w)}{(w-z)(w-x)} \quad \text{when} \quad \tilde{\mu}(z, x)<\infty. \tag{3.2}$$

Let μ be a complex representing measure for x, i.e., $\int f\,d\mu=f(x)$ for all $f\in A(X)$. When $\tilde{\mu}(z)<\infty$, we put

$$c=\int\frac{w-x}{w-z}d\mu(w)=1+(z-x)\hat{\mu}(z). \tag{3.3}$$

If $c\neq 0$, then we can easily see that the point z belongs to X and

$$d\nu(w)=\frac{w-x}{c(w-z)}\,d\mu(w) \tag{3.4}$$

is a complex representing measure for z.

3.2. LEMMA (Browder [1]). *Let μ be a measure on X, and let $x\in \boldsymbol{C}$. For each positive integer n, let $\Delta_n=\Big\{z:|z-x|\leqq\frac{1}{n}\Big\}$. Then*

$$\frac{1}{m(\Delta_n)}\int_{\Delta_n}|z-x|\tilde{\mu}(z)\,dm(z)\longrightarrow|\mu|(\{x\}) \qquad \textit{as} \quad n\to\infty.$$

3.3. LEMMA. *Let $x\in X$, let μ be a complex representing measure with the property* (a.2). *Let $\varepsilon>0$ and $\delta=\varepsilon/(\varepsilon+2\|\mu\|+2)$. For each $z\in X$,*

$$|z-x|\tilde{\mu}(z, x)\leqq\delta \quad \textit{and} \quad |z-x|\tilde{\mu}(z)\leqq\delta\min(1, 1/\tilde{\mu}(x))$$

imply

$$\Big|\frac{f(z)-f(x)}{z-x}-f'(x)\Big|\leqq\varepsilon\|f\| \qquad \textit{for all} \quad f\in A(X).$$

PROOF. Our assumption is $\tilde{\mu}(x)<\infty$. Thus we can define a measure μ' by

$$d\mu'(w)=\Big(\frac{1}{w-x}-\hat{\mu}(x)\Big)d\mu(w).$$

Then we have $\int f\,d\mu'(w)=f'(x)$ for all $f\in A(X)$; indeed, it suffices to show this only for $f\in A(X; x)$, but this will follow easily. Take $z\in X$ in our assumption. Since $\delta<1$, $c=1+(z-x)\hat{\mu}(z)\neq 0$. Thus we can define a complex representing measure ν for z by (3.4). Now

$$\begin{aligned}\frac{d\nu(w)-d\mu(w)}{z-x}&=\frac{1}{z-x}\Big(\frac{w-x}{c(w-z)}-1\Big)d\mu(w)\\&=\frac{1}{z-x}\,\frac{w-x-(1+(z-x)\hat{\mu}(z))(w-z)}{c(w-z)}\,d\mu(w)\end{aligned}$$

$$=\Big(\frac{1}{c(w-z)}-\frac{\hat{\mu}(z)}{c}\Big)d\mu(w)\,.$$

Moreover,

$$\frac{d\nu(w)-d\mu(w)}{z-x}-d\mu'(w)$$

$$=\Big[\frac{1}{c(w-z)}-\frac{1}{w-x}-\Big(\frac{\hat{\mu}(z)}{c}-\hat{\mu}(x)\Big)\Big]d\mu(w)$$

$$=\Big[\frac{w-x-(1+(z-x)\hat{\mu}(z))(w-z)}{c(w-z)(w-x)}-\int\Big(\frac{1}{c(\zeta-z)}-\frac{1}{\zeta-x}\Big)d\mu(\zeta)\Big]d\mu(w)$$

$$=\frac{z-x}{c}\Big[\frac{1}{(w-z)(w-x)}-\frac{\hat{\mu}(z)}{w-x}-\int\Big(\frac{1}{(\zeta-z)(\zeta-x)}-\frac{\hat{\mu}(z)}{\zeta-x}\Big)d\mu(\zeta)\Big]d\mu(w)$$

$$=\frac{z-x}{c}\Big[\frac{1}{(w-z)(w-x)}-\frac{\hat{\mu}(z)}{w-x}-(\hat{\mu}(z,\,x)-\hat{\mu}(z)\hat{\mu}(x))\Big]d\mu(w)\,.$$

Thus, for $f\in A(X)$,

$$\Big|\frac{f(z)-f(x)}{z-x}-f'(x)\Big|=\Big|\int f(w)\Big(\frac{d\nu(w)-d\mu(w)}{z-x}-d\mu'(w)\Big)\Big|$$

$$\leqq\|f\|\frac{|z-x|}{|c|}\Big[\tilde{\mu}(z,\,x)+|\hat{\mu}(z)|\tilde{\mu}(x)+(|\hat{\mu}(z,\,x)|+|\hat{\mu}(z)\hat{\mu}(x)|)\|\mu\|\Big]$$

$$\leqq\|f\|\frac{|z-x|}{1-|z-x|\tilde{\mu}(z)}(\tilde{\mu}(z,\,x)+\tilde{\mu}(z)\tilde{\mu}(x))(1+\|\mu\|)$$

$$\leqq\|f\|\frac{2\delta}{1-\delta}(1+\|\mu\|)=\varepsilon\|f\|\,.$$

This proves the lemma.

3.4. THEOREM. *Suppose there exists a nonzero continuous point derivation on $A(X)$ at x. For any $\varepsilon>0$, we put*

$$D_\varepsilon(x)=\Big\{z\in X:\Big|\frac{f(z)-f(x)}{z-x}-f'(x)\Big|\leqq\varepsilon\|f\|\quad\text{for all } f\in A(X)\Big\},$$

and

$$\Delta_n=\Big\{z\in \boldsymbol{C}:\ |z-x|\leqq\frac{1}{n}\Big\}.$$

Then

$$\lim_{n\to\infty}\frac{m(D_\varepsilon(x)\cap\Delta_n)}{m(\Delta_n)}=1\,.$$

PROOF. Let μ be a complex representing measure for x with the property (a.2), i.e., $\tilde{\mu}(x)<\infty$. Clearly, $|\mu|(\{x\})=0$, and $\Big|\frac{\mu(w)}{w-x}\Big|(\{x\})=0$. Let $b=\min(1,\,1/\tilde{\mu}(x))$. We put

$$K_n=\{z\in\Delta_n:\ |z-x|\tilde{\mu}(z,\,x)\leqq\delta\}$$

$$L_n = \{z \in \Delta_n : |z-x|\tilde{\mu}(z) \leqq b\delta\} .$$

Then, by Lemma 3.3,

$$m(D_\varepsilon(x) \cap \Delta_n) \geqq m(L_n \cap K_n)$$
$$\geqq m(\Delta_n) - \left(\frac{1}{\delta}\int_{\Delta_n} |z-x|\tilde{\mu}(z,\, x)\, dm(z)\right.$$
$$\left. + \frac{1}{b\delta}\int_{\Delta_n} |z-x|\tilde{\mu}(z)\, dm(z)\right).$$

Hence we obtain

$$1 \geqq \frac{m(D_\varepsilon(x) \cap \Delta_n)}{m(\Delta_n)} \geqq 1 - \frac{1}{\delta}\frac{1}{m(\Delta_n)}\int_{\Delta_n} |z-x|\left(\frac{\mu(w)}{w-x}\right)^{\sim}(z)\, dm(z)$$
$$- \frac{1}{b\delta}\frac{1}{m(\Delta_n)}\int_{\Delta_n} |z-x|\tilde{\mu}(z)\, dm(z) .$$

Now the theorem follows from Lemma 3.2.

3.5. COROLLARY. *If there exists a nonzero continuous point derivation at* $x \in X$, *then there exists a sequence* x_n *in* X *such that* x_n *converges to* x *and, as linear functionals on* $A(X)$,

$$\frac{f(x_n)-f(x)}{x_n - x} \longrightarrow f'(x) \quad (\textit{uniformly}) .$$

That completes the equivalence of the conditions (a.0)~(a.5).

3.6. COROLLARY. *Let* $D_x : f \mapsto f'(x)$ *be the point derivation at* x. *Then*

$$\|D_x\| = \varliminf_{z \to x} \frac{\|z-x\|^A}{|z-x|} .$$

PROOF. Let $M = \varliminf_{z \to x} \frac{\|z-x\|^A}{|z-x|}$. There is a sequence x_n in X with $\lim_n \frac{\|x_n - x\|^A}{|x_n - x|} = M$. Since the linear functional $f \mapsto \frac{f(x_n)-f(x)}{x_n - x}$ strongly converges to D_x by Lemma 2.9, $\|D_x\| \leqq M$. The reverse inequality follows from the above Corollary.

§ 4. Estimates of the $A(X)$-metric norm.

Let D be a plane set. The continuous analytic capacity of D, $\alpha(D)$, is defined by

$$\alpha(D) = \sup \{ |f'(\infty)| : f \in \mathcal{AC}(D)\} ,$$

where $\mathcal{AC}(D)$ is the set of all continuous functions f on the Riemann sphere S^2 such that f are analytic off a compact subset of D, $\|f\|_{S^2} \leqq 1$ and $f(\infty) = 0$.

For integers t, N, M($N < M$, and it may be $M = \infty$) and a positive number r, we employ the notation $\alpha_t^{N,M}(x\,;\, r)$ which is defined by

$$\alpha_t^{N,M}(x\,;r)=2^{t+1}\sum_{n=N}^{M}\frac{2^{nt}}{r^t}\alpha(E_n(x\,;r)\backslash X^0)\,;$$

as it were, this means continuous analytic capacity at x of dgree t with radius r and ratio 1/2 from N to M. We shall use the notation $\alpha_t(x\,;r)$ instead of $\alpha_t^{0,\infty}(x\,;r)$. Let us attend to the following facts;

(4.1) $x\in X$ is a peak point for $A(X)$ if and only if $\alpha_1(x\,;r)=\infty$ (Melnikov),

(4.2) there exists a nonzero continuous point derivation on $A(X)$ at x if and only if $\alpha_2(x\,;r)<\infty$ (Hallstrom, cf. (a.3)).

To obtain one side estimates, we repeat the argument in Curtis [2] which is based only on the above definition of α and the following lemma.

4.1. LEMMA (see [2], [3]). *Let K be a compact plane set and f a continuous function on S^2 which is analytic off K and vanishes at ∞. Then*

$$|f(z)|\leqq\frac{\alpha(K)}{d(z,K)}\|f\|_{S^2}\,;$$

where we admit the right hand to attain the value ∞ when $z\in K$.

4.2. ESTIMATE (essentially due to Curtis [2]). For x, $y\in X$, it holds

(A-1) $$\|x-y\|^A\geqq\frac{\alpha(\Delta(x\,;r)\backslash X^0)}{r+\alpha(\Delta(x\,;r)\backslash X^0)}-\frac{\alpha(\Delta(x\,;r)\backslash X^0)}{d(y,\Delta(x\,;r)\backslash X^0)},$$

(A-2) $$\|x-y\|^A\geqq\frac{1}{5}\frac{\alpha_1(x\,;r)}{3+\alpha_1(x\,;r)}-\frac{\alpha(\Delta(x\,;r)\backslash X^0)}{d(y,\Delta(x\,;r)\backslash X^0)}.$$

PROOF. We shall only prove (A-2). Let $\varepsilon>0$ and let M be a positive integer, and we set $E_n=E_n(x\,;r)$. Then there exist compact sets $K_n\subset E_n\backslash X^0$ and functions $f_n\in\mathcal{AC}(K_n)$ for $0\leqq n\leqq M$ such that

$$\alpha(E_n\backslash X^0)-\frac{\varepsilon}{M+1}\frac{r}{2^n}\leqq f_n'(\infty)\leqq\alpha(E_n\backslash X^0)\,.$$

We define a function g on S^2 by

$$g(z)=\sum_{n=0}^{M}\frac{2^n}{r}(f_n'(\infty)-(z-x)f_n(z))$$

$$=\sum_{n=0}^{M}\frac{2^n}{r}(f_n'(\infty)-\sum_{n=0}^{M}\frac{2^n}{r}(z-x)f_n(z)\,.$$

Since $g(\infty)=0$, g is continuous on S^2 and analytic off $K=\bigcup_{n=0}^{M}K_n$. Hence $g\in A(X)$, and $g(x)=\sum_{n=0}^{M}\frac{2^n}{r}f_n'(\infty)$. Since $(z-x)f_n(z)$ is analytic off K_n, by the maximum modulus principle,

$$\left\|\frac{2^n}{r}(z-x)f_n(z)\right\|=\left\|\frac{2^n(z-x)}{r}f_n(z)\right\|_{E_n}\leqq1\,.$$

On the other hand, applying Lemma 4.1 to f_n,

$$|f_n(z)|\leqq\frac{\alpha(K_n)}{d(z,K_n)}\leqq\frac{\alpha(E_n\backslash X^0)}{d(z,E_n)}.$$

Hence, if $\frac{r}{2^{j+1}}\leqq|z-x|<\frac{r}{2^j}$, then the distance from z to E_n is at least $\frac{r}{2^{j+1}}-\frac{r}{2^{j+2}}$ for $n\neq j-1,\ j,\ j+1$, and hence for such n

$$\left|\frac{2^n}{r}(z-x)f_n(z)\right|\leqq\frac{2^n}{r}\frac{r}{2^j}\frac{\alpha(E_n\backslash X^0)}{\frac{r}{2^{j+1}}-\frac{r}{2^{j+2}}}\leqq\frac{2^{n+2}}{r}\alpha(E_n\backslash X^0).$$

Now, for $z\in\bigcup_{n=0}^{M}E_n$, we have

$$\begin{aligned}|g(z)|&\leqq\sum_{n=0}^{M}\frac{2^n}{r}f_n'(\infty)+\sum_{n=0}^{M}\left|\frac{2^n}{r}(z-x)f_n(z)\right|\\&\leqq\sum_{n=0}^{M}\frac{2^n}{r}\alpha(E_n\backslash X^0)+4\sum_{n=0}^{M}\frac{2^n}{r}\alpha(E_n\backslash X^0)+3\\&=\frac{5}{4}\alpha_1^{0,M}(x;r)+3.\end{aligned}$$

Again, by the maximum modulus principle, this estimate holds for all $z\in S^2$, and applying Lemma 4.1 to g, we obtain

$$|g(z)|<\frac{\alpha(K)}{d(z,K)}\Big(\frac{5}{4}\alpha_1^{0,M}(x;r)+3\Big).$$

Since $\|g\|_X\leqq\frac{5}{4}\alpha_1^{0,M}(x;r)+3$, we have for $y\in X$

$$\begin{aligned}\|x-y\|^A&\geqq\frac{\sum_{n=0}^{M}\frac{2^n}{r}f_n'(\infty)}{\frac{5}{4}\alpha_1^{0,M}(x;r)+3}-\frac{\alpha(K)}{d(y,K)}\\&\geqq\frac{1}{5}\frac{\alpha_1^{0,M}(x;r)-4\varepsilon}{3+\alpha_1^{0,M}(x;r)}-\frac{\alpha(\Delta(x;r)\backslash X^0)}{d(y,\Delta(x;r)\backslash X^0)}.\end{aligned}$$

Now let $\varepsilon\downarrow 0$ and $M\to\infty$, we have the estimate (A-2). One can also prove the estimate (A-1) by the similar modification of [**2**], Theorem 3.2.

To obtain the opposite estimate, we need the following theorem ([**3**], Chap. VIII, Th. 12.6).

(Melnikov's Estimate) Let J be an open anulus of conformal radius r, and let K be a compact subset of $\bar{J}$ and f a continuous function on $\bar{J}$ which is analytic in $J\backslash K$. Then

$$\left|\int_{bJ}f(z)dz\right|\leqq\frac{c}{1-r}\|f\|_J\alpha(K\cap J),$$

where c is a universal constant and bJ denotes the boundary of J.

The estimate is not so simple, that we first make some calculations. Let x, y be distinct points in X. We denote by $A(X; x, y)$ the set of all functions of $A(X)$ which admit analytic continuation to some neighborhood of x and y. Let $f \in A(X; x, y)$. We extend f to a continuous function on S^2 such that f is analytic in some neighborhood of x and y, and the norm $\|f\|_{S^2}$ is sufficiently near to $\|f\|_X$. Now let r_1 be a positive number such that f is analytic in $\Delta(x; r_1)$. Our aim is to estimate $|f(x)-f(y)|$. Let $\Gamma=\{z: |z-x| =r_1\}$ and

$$g(z)=\frac{f(z)-f(y)}{z-y}.$$

Then g is continuous on S^2 and analytic wherever f is. Thus it follows from Cauchy's integral formula

$$f(z)-f(y)=\frac{x-y}{2\pi i}\int_\Gamma \frac{g(\zeta)}{\zeta-x}d\zeta. \tag{4.3}$$

Now we fix a number δ such that $0<\delta<|x-y|$, and take a continuously differentiable function h on S^2 such that h is supported on $\bar{\Delta}(y; \delta)$, $\left\|\frac{\partial h}{\partial \bar{z}}\right\| \leqq \frac{4}{\delta}$, $0 \leqq h(z) \leqq 1$ on S^2 and $h(z)=1$ when $z \in \Delta\left(y; \frac{\delta}{2}\right)$. In order to estimate (4.3) we consider the integral of the form

$$\begin{aligned} G(w)&=\frac{1}{\pi}\iint \frac{g(z)-g(w)}{z-w}\frac{\partial h}{\partial \bar{z}}(z)\,d\xi d\eta \\ &=g(w)h(w)+\frac{1}{\pi}\iint g(z)\frac{\partial h}{\partial \bar{z}}(z)\frac{1}{z-w}\,d\xi d\eta\,; \end{aligned} \tag{4.4}$$

where $z=\xi+i\eta$. From [3], Chap. II, Lemma 1.7, G is analytic wherever g is and analytic in $S^2\backslash\bar{\Delta}(y; \delta)$, and $G-g$ is analytic wherever g is and analytic in $\Delta\left(y; \frac{\delta}{2}\right)$. Moreover, a crude estimate yields

$$|G(w)| \leqq |g(w)|+\frac{1}{\pi}\|g\|_{\Delta(y; \frac{\delta}{2})^c}\left\|\frac{\partial h}{\partial \bar{z}}\right\|\iint_{\Delta(y; \delta)}\frac{d\xi d\eta}{|z-w|}.$$

Since the last integral attains the maximum $2\pi\delta$ when $w=y$, the definition of g yields

$$|G(w)| \leqq |g(w)|+\frac{1}{\pi}\frac{2\|f\|}{\frac{\delta}{2}}\frac{4}{\delta}\cdot 2\pi\delta=|g(w)|+\frac{32}{\delta}\|f\|. \tag{4.5}$$

And, by the maximum modulus principle,

$$\|G-g\|_{S^2}=\|G-g\|_{\Delta(y; \frac{\delta}{2})^c} \leqq 2\|g\|_{\Delta(y; \frac{\delta}{2})^c}+\frac{32}{\delta}\|f\| \tag{4.6}$$

$$\leqq 2\cdot\frac{2\|f\|}{\frac{\delta}{2}}+\frac{32}{\delta}\|f\|=\frac{40}{\delta}\|f\|.$$

If $w\notin\bar{\Delta}(y;\delta)$, attending to that $h(w)=0$ in (4.4), we have

$$|G(w)-g(w)|\leqq|g(w)|+\frac{1}{\pi}\|g\|_{\Delta(y;\frac{\delta}{2})^c}\left\|\frac{\partial h}{\partial\bar{z}}\right\|\frac{\delta^2}{|w-y|-\delta}$$

$$\leqq\frac{2\|f\|}{|w-y|-\delta}+\frac{2\|f\|}{\frac{\delta}{2}}\frac{4}{\delta}\frac{\delta^2}{|w-y|-\delta}.$$

Therefore,

$$(4.7)\qquad |G(w)-g(w)|\leqq\frac{18\|f\|}{|w-y|-\delta}\qquad\text{for}\quad |w-y|>\delta.$$

Now return to (4.3),

$$\int_\Gamma\frac{g(\zeta)}{\zeta-x}d\zeta=\int_\Gamma\frac{(g-G)(\zeta)}{\zeta-x}d\zeta+\int_\Gamma\frac{G(\zeta)}{\zeta-x}d\zeta.$$

If necessary, we shrink r_1 sufficiently so that $\bar{\Delta}(x;r_1)\cap\bar{\Delta}(y;\delta)=\emptyset$. Since $G(z)$ is analytic in $S^2\backslash\bar{\Delta}(y;\delta)$, an application of Cauchy's integral formula yields

$$\int_\Gamma\frac{g(\zeta)}{\zeta-x}d\zeta=\int_\Gamma\frac{(g-G)(\zeta)}{\zeta-x}d\zeta+\int_{|\zeta-y|=\delta}\frac{G(\zeta)}{\zeta-x}d\zeta.$$

We compute the integrals separately.

(The first integral): Let $r>|x-y|+\delta$, and we fix a integer $k>0$ such that $\frac{r}{2^{k-1}}>|x-y|+\delta$. For sufficiently large integer $M\geqq k$, we may assume $\Gamma=\left\{z;\ |z-x|=\frac{r}{2^{M+1}}\right\}$. Then

$$\int_\Gamma\frac{(g-G)(\zeta)}{\zeta-x}d\zeta=\int_{|\zeta-x|=r}\frac{(g-G)(\zeta)}{\zeta-x}d\zeta+\sum_{n=0}^{M}\int_{bE_n(x;r)}\frac{(g-G)(\zeta)}{\zeta-x}d\zeta.$$

Divide the second term into two parts at k, apply (4.6) for $i<k$ and the first term, and apply (4.7) for $i\geqq k$. Then Melnikov's estimate yields

$$\left|\int_\Gamma\frac{(g-G)(\zeta)}{\zeta-x}d\zeta\right|\leqq\frac{2\pi r\cdot 18\|f\|}{r(r-|x-y|-\delta)}$$

$$+2c\sum_{n=0}^{k-1}\frac{18\|f\|}{\frac{r}{2^{n+1}}-|x-y|-\delta}\frac{\alpha(E_n(x;r)\backslash X^0)}{\frac{r}{2^{n+1}}}$$

$$+2c\sum_{n=k}^{M}\frac{40}{\delta}\|f\|\frac{\alpha(E_n(x;r)\backslash X^0)}{\frac{r}{2^{n+1}}}.$$

(The second integral): This time we can write as follows;

$$\int_{|\zeta-y|=\delta}\frac{G(\zeta)}{\zeta-x}d\zeta=\sum_{n=0}^{\infty}\int_{bE_n(y;\delta)}\frac{G(\zeta)}{\zeta-x}d\zeta,$$

since the terms of the right hand are vanishing for sufficiently large n; because $\frac{G(\zeta)}{\zeta-x}$ is analytic in some neighborhood of y. Thus, by (4.5) and the definition of g, Melnikov's estimate yields

$$\left|\int_{|\zeta-y|=\delta}\frac{G(\zeta)}{\zeta-x}d\zeta\right|\leqq\frac{1}{|x-y|-\delta}\cdot 2c\sum_{n=0}^{\infty}\left(\frac{2\|f\|}{\frac{\delta}{2^{n+1}}}+\frac{32}{\delta}\|f\|\right)\alpha(E_n(y;\delta)\backslash X^0)$$

$$\leqq\frac{2c}{|x-y|-\delta}\sum_{n=0}^{\infty}18\|f\|\frac{2^{n+1}}{\delta}\alpha(E_n(y;\delta)\backslash X^0).$$

We put these together, and let $M\to\infty$. Since $A(X;x,y)$ is uniformly dense in $A(X)$, we have the following estimate.

4.3. ESTIMATE. Let $x, y\in X$. Let $r>0$, $\delta>0$ such that $r>|x-y|+\delta$, and $0<\delta<|x-y|$. Let k be a positive integer such that $\frac{r}{2^{k-1}}>|x-y|+\delta$, then

$$\text{(A-3)}\qquad \|x-y\|^A\leqq C|x-y|\Big[\frac{1}{r-|x-y|-\delta}+4\sum_{n=0}^{k-1}\frac{1}{\frac{r}{2^{n+1}}-|x-y|-\delta}\frac{2^n}{r}\alpha(E_n(x;r)\backslash X^0)+\frac{1}{\delta}\alpha_1^{k,\infty}(x;r)+\frac{1}{|x-y|-\delta}\alpha_1(y;\delta)\Big],$$

where C is a universal constant, for instance, we take $C=36\pi c$.

Now we shall derive two versions of (A-3) which have meaning in the cases (4.1), (4.2) respectively.

4.4. ESTIMATE. Let $x, y\in X$, $0<\sigma<1$ and $r>0$. If $|x-y|<\frac{r}{4}$, then

$$\text{(A-4)}\quad \|x-y\|^A\leqq C\Big[\frac{|x-y|}{r-2|x-y|}+\frac{\sqrt{|x-y|}}{\sqrt{r}-2\sqrt{|x-y|}}\alpha_1(x;r)+\frac{1}{\sigma}\alpha_1^{k,\infty}(x;r)+\frac{1}{1-\sigma}\alpha_1(y;\sigma|x-y|)\Big],$$

where $k=\left[\frac{1}{2}\log_2\frac{r}{|x-y|}\right]$; that is, k is the maximum integer with $\frac{1}{2^k}\geqq\sqrt{\frac{|x-y|}{r}}$.

PROOF. Let $\delta=\sigma|x-y|$ in (A-3). Then we have only to consider the second term in (A-3). This is converted as follows;

$$|x-y|\cdot 4\sum_{n=0}^{k-1}\frac{1}{\frac{r}{2^{n+1}}-|x-y|-\delta}\frac{2^n}{r}\alpha(E_n(x;r)\backslash X^0)$$

$$\leqq \frac{4|x-y|}{\sqrt{r|x-y|}-2|x-y|}\sum_{n=0}^{k-1}\frac{2^n}{r}\alpha(E_n(x;r)\backslash X^0)$$

$$\leqq \frac{\sqrt{|x-y|}}{\sqrt{r}-2\sqrt{|x-y|}}\alpha_1(x;r).$$

4.5. ESTIMATE. Let $x, y\in X$, $0<\sigma<1$ and $r>0$. If $|x-y|<\frac{r}{8}$, then

$$\text{(A-5)}\qquad \|x-y\|^A \leqq C|x-y|\Big[\frac{1}{r-2|x-y|}+\frac{6}{\sigma}\alpha_2(x;r)+\frac{1}{1-\sigma}\frac{\alpha_1(y;\sigma|x-y|)}{|x-y|}\Big].$$

PROOF. Let k be the maximum integer such that $\frac{r}{2^k}\geqq 4|x-y|$. We must be concerned with the second and the third terms in (A-3). Since

$$\frac{r}{2^{n+1}}-|x-y|-\delta \geqq \frac{r}{2^{n+1}}-2\cdot\frac{r}{4\cdot 2^k}\geqq\frac{1}{2}\cdot\frac{r}{2^{n+1}}$$

for $n<k$, the second term is converted as follows;

$$4\sum_{n=0}^{k-1}\frac{1}{\frac{r}{2^{n+1}}-|x-y|-\delta}\frac{2^n}{r}\alpha(E_n(x;r)\backslash X^0)$$

$$\leqq 4\sum_{n=0}^{k-1}\frac{2^{n+2}}{r}\frac{2^n}{r}\alpha(E_n(x;r)\backslash X^0)\leqq\frac{2}{\sigma}\alpha_2(x;r).$$

Also, the third term is converted as follows; let $\delta=\sigma|x-y|$,

$$\frac{1}{\delta}\cdot 4\sum_{n=k}^{\infty}\frac{2^n}{r}\alpha(E_n(x;r)\backslash X^0)=\frac{4}{\delta}\sum_{n=k}^{\infty}\frac{(2^n)^2}{r^2}\alpha(E_n(x;r)\backslash X^0)\cdot\frac{r}{2^n}$$

$$\leqq\frac{4|x-y|}{\delta}\cdot 8\sum_{n=k}^{\infty}\frac{2^{2n}}{r^2}\alpha(E_n(x;r)\backslash X^0)\leqq\frac{4}{\sigma}\alpha_2(x;r).$$

COMMENT. Our estimates are unintelligible. However, if we consider when x is fixed and σ is a constant, then the first three terms in (A-4) and the first two terms in (A-5) are determined by the usual metric $|x-y|$, and hence we have only to warry about the last term containing $\alpha_1(y;\sigma|x-y|)$ for (A-4) and (A-5).

§ 5. Applications of the estimates.

We use the following simple facts:

$$\text{(5.1)}\qquad \alpha(\Delta(x;r))=r.$$

$$\text{(5.2)}\qquad \text{If } D_1\subset D_2,\text{ then } \alpha(D_1)\leqq\alpha(D_2).$$

The first application is to show the following well known theorem (see [2], [3]).

5.1. THEOREM. *Let* $x\in X$.

(a) *If* $\overline{\lim_{r\to 0}}\dfrac{\alpha(\varDelta(x;r)\backslash X^0)}{r}>0$, *then x is a peak point for* $A(X)$.

(b) *If* $\alpha_1(x;r)=\infty$ *for some* $r>0$, *then x is a peak point for* $A(X)$.

PROOF. (a) By the assumption, there exist $r_n\downarrow 0$ such that

$$\lim_{n\to\infty}\frac{\alpha(\varDelta(x;r_n)\backslash X^0)}{r_n}>0\,.$$

Let $y\in X$ be a distinct point from x, and set $C_n=\alpha(\varDelta(x;r_n)\backslash X^0)$. Then, by (A-1),

$$\|x-y\|^{A}\geqq\lim_{n\to\infty}\left(\frac{\dfrac{C_n}{r_n}}{1+\dfrac{C_n}{r_n}}-\frac{C_n}{d(y,\varDelta(x;r_n)\backslash X^0)}\right)=\frac{\lim_n\dfrac{C_n}{r_n}}{1+\lim_n\dfrac{C_n}{r_n}}>0\,,$$

since $d(y,\varDelta(x;r_n)\backslash X^0)\to|x-y|$ and $C_n=\alpha(\varDelta(x;r_n)\backslash X^0)\leqq r_n\to 0$. This shows that x is isolated in X with the $A(X)$-metric topology. Therefore x must be a peak point for $A(X)$ by a Corollary of Theorem 2 in Browder's [**1**]. (b) will follow from the similar argument with the use of (A-2) instead of (A-1).

5.2. THEOREM. *Let* $x\in X$ *and let* x_n $(\neq x)$ *be a sequence in* X.

(a) *If* x_n *converges to x in the* $A(X)$-*metric topology, then for any* $\sigma>0$

$$\lim_{n\to\infty}\frac{\alpha(\varDelta(x_n;\sigma|x_n-x|)\backslash X^0)}{|x_n-x|}=0\,.$$

(b) *If* $\{x_n, x_n-x\}$ *is a sequential derivation for* $A(X)$ *at x, then for any* $\sigma>0$

$$\lim_{n\to\infty}\frac{\alpha(\varDelta(x_n;\sigma|x_n-x|)\backslash X^0)}{|x_n-x|^2}=0\,.$$

PROOF. (a) By the assumption x is not a peak point for $A(X)$. Hence $\lim_{r\to 0}\dfrac{\alpha(\varDelta(x;r)\backslash X^0)}{r}=0$ by Theorem 5.1 (a), so the conclusion yields to the inequality

$$\frac{\alpha(\varDelta(x_n;\sigma|x_n-x|)\backslash X^0)}{|x_n-x|}\leqq(1+\sigma)\frac{\alpha(\varDelta(x;(1+\sigma)|x_n-x|)\backslash X^0)}{(1+\sigma)|x_n-x|}\,.$$

(b) In this case there is a nonzero bounded point derivation on $A(X)$ at x. Hence $\lim_{r\to 0}\dfrac{\alpha(\varDelta(x;r)\backslash X^0)}{r^2}=0$ ([**4**], Th. 2). The remains are similar.

5.3. THEOREM. *Let* $x\in X$. *Let* x_n *be a sequence in X which converges to x in the natural topology of* ***C***.

(a) *Suppose x is not a peak point for* $A(X)$. x_n *converges to x in the* $A(X)$-*metric topology if and only if*

$$\lim_{n\to\infty}\alpha_1(x_n;\sigma|x_n-x|)=0\,;$$

where σ is any fixed number $0<\sigma<1$.

(b) *Suppose there exists a nonzero continuous point derivation on* $A(X)$ *at* x. *Then* $\{x_n, x_n-x\}$ *is a sequential derivation for* $A(X)$ *at* x *if and only if*

$$\varlimsup_{n\to\infty} \frac{\alpha_1(x_n\,;\,\sigma|x_n-x|)}{|x_n-x|}<\infty\,;$$

where σ is any fixed number $0<\sigma<1$.

PROOF. (a) We shall prove "if only" part. From (A-2),

$$\|x_n-x\|^A \geqq \frac{1}{5}\,\frac{\alpha_1(x_n\,;\,\sigma|x_n-x|)}{3+\alpha_1(x_n\,;\,\sigma|x_n-x|)} - \frac{\alpha(\Delta(x_n\,;\,\sigma|x_n-x|)\backslash X^0)}{|x_n-x|-\sigma|x_n-x|}\,.$$

Since we have seen $\lim_{n\to\infty}\frac{\alpha(\Delta(x_n\,;\,\sigma|x_n-x|)\backslash X^0)}{|x_n-x|}=0$ in Theorem 5.2 (a), the second term in the right hand tends to 0 as $n\to\infty$, and the assumption is $\|x_n-x\|^A\to 0$, thus $\alpha_1(x_n\,;\,\sigma|x_n-x|)$ must converge to 0. "if" part is an easy consequence of the estimate (A-4) and the last comment in the previous section.

(b) We shall prove "if only" part. From (A-2),

$$\frac{\|x_n-x\|^A}{|x_n-x|} \geqq \frac{1}{5}\,\frac{\dfrac{\alpha_1(x_n\,;\,\sigma|x_n-x|)}{|x_n-x|}}{3+\alpha_1(x_n\,;\,\sigma|x_n-x|)} - \frac{\alpha(\Delta(x_n\,;\,\sigma|x_n-x|)\backslash X^0)}{|x_n-x|(|x_n-x|-\sigma|x_n-x|)}\,.$$

We have seen in Theorem 5.2 (b) that the second term in the right hand tends to zero. And since x_n converges to x in the metric topology, $\alpha_1(x_n\,;\,\sigma|x_n-x|)$ converges to 0 by (a). Thus Theorem 3.1 implies the first half. The latter half follows from (A-5) and the last comment in the previous section.

Addendum. After this paper was submitted for publication, James Liming Wang sent to the author his paper "An approximate Taylor's theorem for $R(X)$" (Aarhus Univ. Preprint Series, 1972/73, No. 59). With another remarkable facts he showed independently in it that the arguments in section 3 are valid for t-th order point derivation.

Suggested by his paper, the author obtained the estimates in the case of t-th order point derivation. In particular, it was proved that the linear functional $\frac{f(x_n)-f(x)}{x_n-x}$ on $A(X)$ converges uniformly to $f'(x)$ if and only if $\lim_n \frac{\alpha_1(x_n\,;\,\sigma|x_n-x|)}{|x_n-x|}=0$, for any fixed $0<\sigma<1$.

References

[1] A. Browder, Point derivations on function algebras, J. Functional Anal., **1** (1967), 22–27.
[2] P.C. Curtis, Peak points for algebras of analytic functions, J. Functional Anal., **3** (1969), 35–47.
[3] T.W. Gamelin, Uniform Algebras, Prentice-Hall, New York, 1969.
[4] A.P. Hallstrom, On bounded point derivations and analytic capacity, J. Functional Anal., **4** (1969), 153–165.
[5] D.R. Wilken, Bounded point derivations and representing measures on $R(X)$, Proc. Amer. Math. Soc., **24** (1970), 371–373.

Mikihiro HAYASHI
Department of Mathematics
Hokkaido University

Present address:
Department of Mathematics
Faculty of Science
Ibaraki University
Bunkyo, Mito, Ibaraki
Japan