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General convergence theorems for the numerical function and the nonstationary stochastic processes

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§ 1. Introduction.

We shall discuss the validity of the limit relation

$$\lim_{n\to\infty} n\int_{-\infty}^{\infty} f(x+u)K(nu)du = f(x)\int_{-\infty}^{\infty} K(u)du .$$

A theorem concerning this relation was given in S. Bochner [1] and is well known. It was generalized by S. Bochner and S. Izumi [2], S. Izumi [3]. The corresponding theorem for the stochastic process was obtained by T. Kawata [4], [5]. In the present paper we shall deal with the generalizations of these theorems.

§ 2. N-functions.

Known definitions and results which we are going to use in this paper are given. An N-function $M(u)$ admits the representation

$$M(u) = \int_0^{|u|} p(t)dt ,$$

where the function $p(t)$ is right-continuous for $t \geqq 0$, positive for $t > 0$, non-decreasing, and satisfies the conditions

$$p(0) = 0 , \qquad p(\infty) = \lim_{t\to\infty} p(t) = \infty .$$

Let $q(s) = \sup_{p(t)\leqq s} t$, $(s \geqq 0)$. Then

$$M(u) = \int_0^{|u|} p(t)dt , \qquad N(v) = \int_0^{|v|} q(s)ds$$

are called mutually complementary N-functions. Let $M(u)$ be an N-function. We shall denote by $L_M(G)$, where G denotes a bounded (unbounded) set in a finite-dimensional Euclidean space, the class of real-valued functions $u(x)$, defined on G, for which

$$\rho(u\,;\,M)=\int_G M(u(x))dx<\infty\,.$$

Suppose $M(u)$ and $N(v)$ are mutually complementary N-functions. We shall denote by $L_M^*(G)$ the totality of $u(x)$ satisfying the condition

$$(u,\,v)=\int_G u(x)v(x)dx<\infty$$

for all $v(x)\in L_N(G)$.

We state known results as lemmas [6].

LEMMA 1. *For any pair of functions* $u(x)\in L_M^*(G)$, $v(x)\in L_N^*(G)$,

$$\left|\int_G u(x)v(x)dx\right|\leqq|u|_M|v|_N\,.$$

LEMMA 2. *For an arbitrary N-function* $M(u)$ *and* $u(x)\in L_M^*(G)$,

$$|u|_M=\inf_{k>0}\frac{1}{k}\Big(1+\int_G M(ku(x))dx\Big)\,.$$

§ 3. A convergence theorem for the numerical function.

We shall prove the following theorem which is a generalization of Bochner-Izumi's theorem.

THEOREM 1. *Suppose* $M(u)$ *and* $N(v)$ *are mutually complementary N-functions. If* $f(u)$ *is a function such that*

$$\int_{-\infty}^{\infty}M(f(u))\frac{du}{1+|u|^{1+\alpha}}<\infty\,,\qquad(\alpha\geqq 0)\,,\tag{3.1}$$

$$f(u)\text{ is continuous at the point } u=x\,,\tag{3.2}$$

and

$$\int_{-\infty}^{\infty}|K(u)|du<\infty\quad\text{and}\quad\int_{-\infty}^{\infty}N(u^{1+\alpha}K(u))\frac{du}{1+|u|^{1+\alpha}}<\infty\,,\tag{3.3}$$

then

$$\lim_{n\to\infty}n\int_{-\infty}^{\infty}f(x+u)K(nu)du=f(x)\int_{-\infty}^{\infty}K(u)du\,.\tag{3.4}$$

When $M(u)=|u|^p/p$ $(p>1)$, Theorem 1 reduces, with some modification, to the following corollary.

COROLLARY 1. *If*

$$\int_{-\infty}^{\infty}\frac{|f(u)|^p}{1+|u|}du<\infty\,,\qquad(p>1,\ \alpha=0)\,,\tag{3.1$'$}$$

$$f(u)\text{ is continuous at the point } u=x\,,\tag{3.2$'$}$$

$$\int_{-\infty}^{\infty}|K(u)|du<\infty\quad\text{and}\quad\int_{-\infty}^{\infty}|u^{q-1}K^q(u)|du<\infty\,,\tag{3.3$'$}$$

where $\frac{1}{p}+\frac{1}{q}=1$, *then the relation* (3.4) *holds.*

We note that Corollary 1 is Bochner-Izumi's theorem [2].

We shall now prove Theorem 1. We may obviously assume that $K(u)=0$ for $u<0$ and that $x=0$; in which case the relation (3.4) reads

$$\lim_{n\to\infty} n\int_0^\infty f(u)K(nu)du=f(+0)\int_0^\infty K(u)du\,. \tag{3.5}$$

Let $g(u)$ be any function having the following properties: $g(u)$ is bounded throughout and vanishes outside a certain interval including x, and the limit $g(+0)$ exists and is equal to $f(+0)$ then, the hypothesis

$$\int_0^\infty |K(u)|\,du=A<\infty$$

implies (S. Bochner [1]. Satz 3, a)

$$\lim_{n\to\infty} n\int_0^\infty g(u)K(nu)du=f(+0)\int_0^\infty K(u)du\,.$$

On the other hand $g(u)$ satisfies all conditions laid down for the function $f(u)$ in Theorem 1. Hence, replacing $f(u)$ by $f(u)-g(u)$, we may add the further assumption $f(+0)=0$ and (3.5) reads now

$$\lim_{n\to\infty} n\int_0^\infty f(u)K(nu)du=0 \tag{3.6}$$

which we are going to show.

To any $\varepsilon>0$ there corresponds an $a>0$ such that $|f(u)|\leqq\frac{\varepsilon}{A}$ if $0\leqq u\leqq a$. Since

$$\left|n\int_0^a f(u)K(nu)du\right|\leqq\frac{\varepsilon}{A}\int_0^a n|K(nu)|\,du\leqq\frac{\varepsilon}{A}\int_0^\infty|K(u)|\,du=\varepsilon$$

and ε may be chosen arbitrary small, the relation (3.6) will follow if the relation

$$\lim_{n\to\infty} n\int_a^\infty f(u)K(nu)du=0 \tag{3.7}$$

holds.

i) The case, $\alpha>0$.

It follows from the Young's inequality that

$$\left|n\int_a^\infty f(u)K(nu)du\right|\leqq\int_a^\infty\left|f(u)n^{1+\alpha}u^{1+\alpha}K(nu)\right|\frac{du}{n^\alpha u^{1+\alpha}}$$

$$\leqq\int_a^\infty M(f(u))\frac{du}{n^\alpha u^{1+\alpha}}+\int_a^\infty N(n^{1+\alpha}u^{1+\alpha}K(nu))\frac{du}{n^\alpha u^{1+\alpha}}$$

$$=\frac{1}{n^\alpha}\int_a^\infty M(f(u))\frac{du}{u^{1+\alpha}}+\int_{an}^\infty N(u^{1+\alpha}K(u))\frac{du}{u^{1+\alpha}}\,.$$

From the hypotheses (3.1) and (3.3),

$$\lim_{n\to\infty}\int_{an}^{\infty}N(u^{1+\alpha}K(u))\frac{du}{u^{1+\alpha}}=0\,,$$

$$\lim_{n\to\infty}\frac{1}{n^{\alpha}}\int_{a}^{\infty}M(f(u))\frac{du}{u^{1+\alpha}}=0\,,\qquad (\alpha>0)$$

which shows (3.7).

ii) The case, $\alpha=0$.

If we put $u=e^{\xi}$, then (3.7) becomes

$$\lim_{n\to\infty} n\int_{\log a}^{\infty} f(e^{\xi})K(ne^{\xi})e^{\xi}d\xi=0\,. \tag{3.8}$$

If we put $f^*(\xi)=f(e^{\xi})$, $K^*(n,\xi)=ne^{\xi}K(ne^{\xi})$, then (3.8) becomes

$$\lim_{n\to\infty}\int_{\log a}^{\infty} f^*(\xi)K^*(n,\xi)d\xi=0\,. \tag{3.9}$$

We are now going to show (3.9).

By Lemmas 1, 2 we have

$$\left|\int_{\log a}^{\infty} f^*(\xi)K^*(n,\xi)d\xi\right|\leqq\left\{\inf_{k>0}\frac{1}{k}\Big(1+\int_{\log a}^{\infty}M(kf^*(\xi))d\xi\Big)\right\} \tag{3.10}$$
$$\times\left\{\inf_{k>0}\frac{1}{k}\Big(1+\int_{\log a}^{\infty}N(kK^*(n,\xi))d\xi\Big)\right\}.$$

Since

$$\int_{\log a}^{\infty}M(kf^*(\xi))d\xi=\int_{a}^{\infty}M(kf(u))\frac{du}{u}\,,$$

$$\int_{\log a}^{\infty}N(kK^*(n,\xi))d\xi=\int_{an}^{\infty}N(kuK(u))\frac{du}{u}\,,$$

(3.10) becomes

$$\left|\int_{\log a}^{\infty} f^*(\xi)K^*(n,\xi)d\xi\right|\leqq\left\{\inf_{k>0}\frac{1}{k}\Big(1+\int_{a}^{\infty}M(kf(u))\frac{du}{u}\Big)\right\} \tag{3.11}$$
$$\times\left\{\inf_{k>0}\frac{1}{k}\Big(1+\int_{an}^{\infty}N(kuK(u))\frac{du}{u}\Big)\right\}.$$

From the hypotheses (3.1) and (3.3), we have

$$\lim_{n\to\infty} n\int_{an}^{\infty}N(kuK(u))\frac{du}{u}=0\,,\qquad k>0\,,$$

$$\inf_{k>0}\frac{1}{k}\Big(1+\int_{a}^{\infty}M(kf(u))\frac{du}{u}\Big)<\infty\,.$$

Thus (3.9) and hence the theorem is proved.

COROLLARY 2. *If*

(3.1) $$\int_{-\infty}^{\infty} M(f(u))\frac{du}{1+|u|^{1+\alpha}} < \infty\,, \qquad \alpha > 0$$

(3.2) $f(u)$ *is continuous at* $u = x$

(3.3)″ *Let there exist a monotone decreasing function* $K_0(u)$ *such that*

$$|K(u)| \leqq K_0(u)\,, \quad \int_{-\infty}^{\infty} K_0(u) < \infty\,,$$

$$K_0(u) = O(|u|^{-(1+\alpha)}) \qquad as \quad |u| \to \infty\,.$$

Then the relation (3.4) *holds.*

In order to prove this, it is sufficient to show that

$$\lim_{n\to\infty} \int_{an}^{\infty} N(u^{1+\alpha}K(u))\frac{du}{u^{1+\alpha}} = 0\,, \qquad \alpha > 0\,.$$

In virtue of the convexity of the N-function $N(v)$

$$N(u^{1+\alpha}K(u)) \leqq N(u^{1+\alpha}K_0(u))\,.$$

The right hand side is bounded, since, by (3.3)″,

$$|u|^{1+\alpha}K_0(u) = O(1)\,, \qquad \text{as} \quad |u| \to \infty\,.$$

Therefore

$$\lim_{n\to\infty} \int_{an}^{\infty} N(u^{1+\alpha}K(u))\frac{du}{u^{1+\alpha}} \leqq \lim_{n\to\infty} \int_{an}^{\infty} O(1)\frac{du}{u^{1+\alpha}} = 0\,.$$

§ 4. A convergence theorem for the nonstationary stochastic process.

We shall suppose that the stochastic process $X(x, \omega)$, $-\infty < x < \infty$, $\omega \in \Omega$, being a probability field, satisfies the conditions:

(i) it is measurable and separable,

(ii) $EX(x, \omega) = 0$ for every x,

(iii) the covariance function

$$\rho(s, t) = EX(s, \omega)\overline{X(t, \omega)}$$

is continuous in $-\infty < s,\ t < \infty$.

We shall prove the following theorem which is a generalization of Kawata's theorem.

THEOREM 2. *Suppose* $M(u)$ *and* $N(v)$ *are mutually complementary N-functions. If*

(4.1) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{M(\rho(s, t))}{(1+|s|^{1+\alpha})(1+|t|^{1+\alpha})}\, dsdt < \infty\,, \qquad \alpha \geqq 0\,,$$

(3.3) $$\int_{-\infty}^{\infty} |K(u)|\, du < \infty \quad and \quad \int_{-\infty}^{\infty} N(u^{1+\alpha}K(u))\frac{du}{1+|u|^{1+\alpha}} < \infty\,,$$

and

(4.2) *the N-function $N(u)$ satisfies the Δ'-condition (i.e. if there exists positive constant c and u_0 can be found such that $N(uv) \leqq cN(u)N(v)$, $u, v \geqq u_0$), then for every x*

$$\lim_{n\to\infty} n\int_{-\infty}^{\infty} X(x+u, \omega)K(nu)du = X(x, \omega)\int_{-\infty}^{\infty} K(u)du\,, \tag{4.3}$$

where $\lim_{n\to\infty}$ is taken in the L^2 norm $E^2|\cdot|$.

The proof will be done in the same way as for the case of a numerical function. We only prove the case $\alpha > 0$.

First of all, we have

$$\begin{aligned} I_1 &= E\left|n\int_{|u|<\delta} X(x+u, \omega)K(nu)du - X(x, \omega)\int_{-\infty}^{\infty} K(u)du\right|^2 \\ &\leqq 2E\left|n\int_{|u|<\delta}[X(x+u, \omega)-X(x, \omega)]K(nu)du\right|^2 \\ &\quad +2E\left|X(x, \omega)\right|^2\left|n\int_{|u|\geqq\delta} K(nu)du\right|^2 \\ &\leqq 2n^2\int_{|u|<\delta}\int_{|v|<\delta}\phi(u, v)K(nu)K(nv)dudv \\ &\quad +2\rho(x, x)\left(\int_{|u|\geqq n\delta}|K(u)|du\right)^2, \end{aligned} \tag{4.4}$$

where $\phi(u, v)$ is $\Delta_{uv}\phi(x, x)$,

$$\Delta_{uv}\phi(x, x) = \rho(x+u, x+v) - \rho(x+u, x) - \rho(x, x+v) + \rho(x, x)\,.$$

Since $\rho(s, t)$ is continuous, we can choose δ so small that $|\phi(u, v)| \leqq \varepsilon_1$, for $|u|, |v| < \delta$, ε_1 being an arbitrarily given small number. If we do this, the first term of the right hand side of (4.4) is not greater than $2\varepsilon_1\left(\int_{-\infty}^{\infty}|K(u)|du\right)^2$. The second term converges to zero as n tends to infinity. Accordingly, we can find δ and n_0 such that for any given ε

$$I_1 < \varepsilon\,, \qquad \text{if} \quad n > n_0\,.$$

Next consider,

$$\begin{aligned} I_2 &= E\left|n\int_{|u|>\delta} X(x+u, \omega)K(nu)du\right|^2 \\ &= n^2\int_{|u|>\delta}\int_{|v|>\delta}\rho(x+u, x+v)K(nu)K(nv)dudv \\ &= \int_{|u|>\delta}\int_{|v|>\delta}\rho(x+u, x+v)n^{2\alpha+2}|uv|^{1+\alpha}K(nu)K(nv)\frac{dudv}{n^{2\alpha}|uv|^{1+\alpha}}\,. \end{aligned}$$

It follows from the Young's inequality that

$$I_2 \leqq \int_{|u|>\delta}\int_{|v|>\delta} \frac{M(\rho(x+u,\ x+v))}{n^{2\alpha}|uv|^{1+\alpha}}\,dudv$$

$$+\int_{|u|>\delta}\int_{|v|>\delta} \frac{N(n^{2\alpha+2}(uv)^{1+\alpha}K(nu)K(nv))}{n^{2\alpha}|uv|^{1+\alpha}}\,dudv$$

$$=\frac{1}{n^{2\alpha}}\int_{|u|>\delta}\int_{|v|>\delta} \frac{M(\rho(x+u,\ x+v))}{|uv|^{1+\alpha}}\,dudv$$

$$+\int_{|u|>n\delta}\int_{|v|>n\delta} \frac{N((uv)^{1+\alpha}K(u)K(v))}{|uv|^{1+\alpha}}\,dudv$$

which is, in view of (4.2),

$$\leqq \frac{1}{n^{2\alpha}}\int_{|u|>\delta}\int_{|v|>\delta} \frac{M(\rho(x+u,\ x+v))}{|uv|^{1+\alpha}}\,dudv$$

$$+c\Big(\int_{|u|>n\delta} \frac{N(u^{1+\alpha}K(u))}{|u|^{1+\alpha}}\,du\Big)^2,$$

where c is a constant. Hence for a given $\varepsilon>0$, there exists an n_1 such that

$$I_2<\varepsilon, \qquad \text{if} \quad n>n_1.$$

If we take $N=\max(n_0,\ n_1)$, then

$$E\Big|n\int_{-\infty}^{\infty} X(x+u,\ \omega)K(nu)du - X(x,\ \omega)\int_{-\infty}^{\infty} K(u)du\Big|^2$$

$$\leqq I_1+I_2<2\varepsilon, \qquad \text{if} \quad n>N,$$

which proves the theorem.

COROLLARY 3. *If*

$$\text{(4.1)} \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{M(\rho(s,\ t))}{(1+|s|^{1+\alpha})(1+|t|^{1+\alpha})}\,dsdt<\infty, \qquad \alpha>0,$$

(3.3)″ *Let there exist a monotone decreasing function* $K_0(u)$ *which satisfies the conditions:*

$$|K(u)| \leqq K_0(u), \qquad \int_{-\infty}^{\infty} K_0(u)du<\infty,$$

$$K_0(u)=O(|u|^{-(1+\alpha)}), \qquad as \quad |u|\to\infty,$$

then for every x the relation (4.3) *holds.*

The proof will be done in a quite similar way.

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