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A characterization of the groups $PSL(3, 2^n)$ and $PSp(4, 2^n)$

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1. It is the purpose of this paper to demonstrate a characterization theorem for the simple groups $PSL(3, 2^n)$[1] and $PSp(4, 2^n)$, $n \geqq 2$. Let G denote any one of these groups, then it is not difficult to verify that G satisfies the following hypothesis:

HYPOTHESIS A. G is a finite group with $O(G)=1=O_2(G)$ in which each maximal 2-local subgroup M which is not 2-closed satisfies the following conditions:

(1) $C_M(O_2(M)) \leqq O_2(M)$;

(2) $M/O_2(M)$ is a TI-group;

(3) every involution of $O_2(M)$ is contained in the center of some S_2-subgroup of M.

Recall that a finite group is called a TI-group if any two distinct S_2-subgroups have only the identity element in common. It is obvious that the simple TI-groups $PSL(2, 2^n)$, $Sz(2^n)$ and $PSU(3, 2^n)$, $n \geqq 2$, satisfy Hypothesis A. Indeed, every 2-local subgroup of a TI-group is 2-closed. In addition, there are simple groups which by accident satisfy Hypothesis A; that is, $PSL(2, q)$ with $q=2^n \pm 1>5$, $PSL(3, 3)$ and the Mathieu group M_{11}.

Our objective is to prove the converse.

THEOREM A. *Let G be a non-identity group satisfying Hypothesis* A. *Then G contains a normal subgroup K isomorphic to one of the simple groups on the following list:*

(1) $PSL(2, q)$ *with* $q=2^n \pm 1>5$, $PSL(3, 3)$, M_{11};

(2) $PSL(2, 2^n)$, $Sz(2^n)$, $PSU(3, 2^n)$, $n \geqq 2$;

(3) $PSL(3, 2^n)$, $PSp(4, 2^n)$, $n \geqq 2$.

Furthermore, we have $C_G(K)=1$ and therefore G is isomorphic to a subgroup of the automorphism group of K.

Though different in appearence, our theorem is quite similar in nature to Suzuki's classification of C-groups (finite groups in which the centralizer of every involution is 2-closed) [**14**], and this paper may be considered to be

1) In general, we follow the notation and terminology of [**6**].

a successful attempt to apply his method to characterizations of presently known finite simple groups (in particular, the groups of Lie type defined over the fields of characteristic 2) by the structure of their maximal 2-local subgroups. Also the classification is necessary for the proof of our theorem. In addition, we require the following classification theorems:

(1) Suzuki's classification of TI-groups [**13**];

(2) Bender's classification of finite groups having a strongly embedded subgroup [**2**];

(3) Goldschmidt's work on finite groups whose S_2-subgroup contains a strongly closed Abelian subgroup [**5**];

(4) Gorenstein and Walter's classification of finite groups with dihedral S_2-subgroups [**8**];

(5) Alperin, Brauer and Gorenstein's classification of finite groups with semi-dihedral S_2-subgroups [**1**].

Of course, we use implicitly the fundamental theorem of Feit and Thompson [**4**]. By virtue of these classification theorems, Theorem A will be proved, once we establish the following result:

THEOREM B. *Let G be a group satisfying Hypothesis* A. *Then one of the following conditions holds:*

(1) $O^2(G) \neq G$;

(2) *G has dihedral or semi-dihedral S_2-subgroups;*

(3) *G is a C-gronp;*

(4) $O^{2'}(G)$ *is isomorphic to* $PSp(4, 2^n)$, $n \geqq 2$.

It is easily seen that Hypothesis A is inherited by subgroups of index 2, so the case that $O^2(G) \neq G$ causes no difficulty. Thus the substantials of the proof of Theorem A exist in constructing a normal subgroup of odd index isomorphic to $PSp(4, 2^n)$.

The organization of the paper is as follows. In Section 2, we describe some useful properties of the family of 2-subgroups defined by Suzuki [**12**]. Section 3 is devoted to the study of the groups satisfying a hypothesis weaker than Hypothesis A. We collect there various technical lemmas. In Section 4, we study the groups G which satisfy Hypothesis A but not satisfy the conclusions (1)—(3) of Theorem B. In particular, we restrict the structure of S_2-subgroups and maximal 2-local subgroups of G considerably, and determine the fusion of involutions and the way two distinct S_2-subgroups intersect non-trivially. That section contains the heart of the proof of our theorem. In Section 5, we follow Suzuki's C-group paper closely, and construct an appropriate (B, N)-pair in G on the basis of Section 4. Then by using D. G. Higman's geometric characterization of $PSp(4, q)$ [**10**, Theorem 2], we identify G with an odd extension of $PSp(4, 2^n)$, thereby proving Theorem B. The final section contains some concluding remarks.

We use the following terminology and notation:

$H \leqq G$	H is contained in G;
$H < G$	$H \leqq G$ but $H \neq G$;
$N(H)$	the normalizer of H in G, in case there is no danger of confusion;
$C(H)$	the centralizer of H in G, in case there is no danger of confusion;
$O'(G) = O^{2'}(G)$	the subgroup of G generated by the 2-elements of G;
G^2	the subgroup of G generated by the squares of the elements of G;
$\mathcal{S}(G)$	the set of the S_2-subgroups of G;
S_2-intersection	an intersection of two distinct S_2-subgroups;
$m(G)$	the 2-rank of G;
$C^*(x) = C^*_G(x)$	$\{y;\ y \in G,\ x^y = x^{\pm 1}\}$;
$\boldsymbol{Z}_2$	a group of order 2;
$\boldsymbol{D}_8$	a dihedral group of order 8.

All groups are assumed to be finite from now on.

2. Let G be a finite group. As in [**14**], we define $\mathcal{H}$ to be the family of 2-subgroups H of G satisfying the following conditions:

(1) $H \neq 1$;

(2) $N(H)$ is not 2-closed;

(3) $H = O_2(N(H))$.

The following properties of the family $\mathcal{H}$ are known and will be frequently used henceforth. The proof of these statements will be found in [**12**, pp. 437-438, a remark p. 442].

(1) Every maximal S_2-intersection $\neq 1$ of G is contained in $\mathcal{H}$.

(2) Let N be a maximal 2-local subgroup which is not 2-closed, then $O_2(N) \in \mathcal{H}$.

(3) If $H \in \mathcal{H}$ and $N(H) \leqq K \leqq G$, then H is the intersection of the S_2-subgroups of K containing H.

(4) If D is a 2-subgroup $\neq 1$ and $N(D)$ is not 2-closed, then there is H in $\mathcal{H}$ with $D \leqq H$ and $N(D) \leqq N(H)$.

(5) G is a TI-group if and only if $\mathcal{H}$ is empty.

The following result is of basic importance to this paper.

(2.1) *Let G be a group with $O(G) = 1 = O_2(G)$. If $H \in \mathcal{H}$ and $H \leqq P \in \mathcal{S}(G)$, then $N_P(H)$ contains an element $\neq H$ of $\mathcal{H}$.*

To prove this, we recall from [**7**, Theorem 3] that if G is a group with $O_2(G) = 1$ and with only one conjugacy class of maximal 2-local subgroups, then G has a strongly embedded subgroup. For a later use, we prove a generalization of this fact.

(2.2) *Let G be a group of even order and let $\{N_1, \cdots, N_k\}$ be a complete set of representatives of the conjugacy classes of maximal 2-local subgroups of G. Set $M=\langle N_1, \cdots, N_k\rangle$. If $M\neq G$, then M is a strongly embedded subgroup of G.*

PROOF. It is obvious that M has even order. Suppose, by way of contradiction, that $|M\cap M^x|$ is even for some x in $G-M$, and take x so that $|M\cap M^x|_2$ is maximal. Let $P\in \mathcal{S}(M\cap M^x)$, then $N(P)\leqq M^y$ for some y in G. Let $P\leqq Q\in \mathcal{S}(M)$ and suppose that $P\neq Q$, then $P<N_Q(P)\leqq M\cap M^y$, and so $y\in M$ by the maximality of $|M\cap M^x|_2$. But then if $P\leqq R\in \mathcal{S}(M^x)$, we have $P<N_R(P)\leqq M\cap M^x$, contrary to the fact that $P\in \mathcal{S}(M\cap M^x)$. Therefore P is an S_2-subgroup of M. Since M contains the normalizer of an S_2-subgroup of G, we conclude that $N(P)\leqq M$. Since $P^{x^{-1}}\leqq M$, we have $P^m=P^{x^{-1}}$ for some m in M, whence $x\in M\cdot N(P)=M$. This contradiction completes the proof.

We now prove (2.1). If $N_P(H)\neq P$, then for every x in $N_P(N_P(H))-N_P(H)$ we have $H\neq H^x\in \mathcal{H}$ and $H^x\leqq N_P(H)$. Hence we can assume that H is normal in P. Suppose then that H is the only element of $\mathcal{H}$ that is contained in $P=N_P(H)$. Then $N(P)<N(H)$, and so every maximal 2-local subgroup of G is conjugate to $N(H)$. Since $O_2(G)=1$, $N(H)$ is strongly embedded in G by (2.2). In addition, we have $m(P)\geqq 2$ by the transfer theorem of Burnside or the theorem of Brauer and Suzuki [3]. Since $O(G)=1$, it follows from the theorem of Bender [2] that G is a TI-group, whence $\mathcal{H}$ is empty. This contradiction completes the proof.

The following result is of use in studying intersections of S_2-subgroups of G when we have enough information on the family $\mathcal{H}$.

(2.3) *Let P, Q be distinct S_2-subgroups of G such that $P\cap Q\neq 1$, then there is a sequence $P_0=P, P_1, \cdots, P_n=Q$ of S_2-subgroups of G satisfying the following conditions:*

(1) *$H_i=P_{i-1}\cap P_i$ is a tame intersection of P_{i-1} and P_i, $1\leqq i\leqq n$;*

(2) *$H_i\in \mathcal{H}$, $1\leqq i\leqq n$;*

(3) *$N(H_i)/H_i$ has a strongly embedded subgroup, $1\leqq i\leqq n$;*

(4) *$P\cap Q=H_1\cap\cdots\cap H_n$.*

PROOF. Set $H=P\cap Q$. If H is a maximal S_2-intersection, the sequence P, Q satisfies the requirements. Hence we can proceed by induction on $|P:H|$. Take S_2-subgroups R, S of G so that $N_P(H)\leqq N_R(H)\in \mathcal{S}(N(H))$ and $N_Q(H)\leqq N_S(H)\in \mathcal{S}(N(H))$. Note that $|H|<|P\cap R|$ and $|H|<|S\cap Q|$. If $R=S$, then $P\neq R\neq Q$ and we can apply the induction hypothesis. If $R\neq S$, then by an obvious induction argument we can assume that $R\cap S=H$, in which case H is a tame intersection of R and S, and $H\in \mathcal{H}$. If $N(H)/H$ has no strongly embedded subgroups, there is a sequence $R_0=R, R_1, \cdots, R_k=S$ of

S_2-subgroups of G such that $R_{i-1} \neq R_i$ and $H < R_{i-1} \cap R_i$, $1 \leqq i \leqq k$. Hence we can again apply the induction hypothesis. The proof is complete.

(2.4) *Let P, Q be as in* (2.3), *then there are a sequence $P_0 = P, P_1, \cdots, P_n = Q$ of S_2-subgroups of G and a sequence $H_1, \cdots, H_n$ of elements of $\mathcal{H}$ satisfying the following conditions:*

(1) $P_{i-1} \neq P_i$, $1 \leqq i \leqq n$;

(2) $H_i \nleqq H_{i+1}$ *and* $H_{i+1} \nleqq H_i$, $1 \leqq i \leqq n-1$;

(3) $H_i \leqq P_{i-1} \cap P_i$, $1 \leqq i \leqq n$;

(4) $N(H_i)/H_i$ *has a strongly embedded subgroup* $1 \leqq i \leqq n$;

(5) $P \cap Q = H_1 \cap \cdots \cap H_n$.

PROOF. By the preceding lemma, there are a sequence $P_0 = P, P_1, \cdots, P_n = Q$ of S_2-subgroups of G and a sequence $H_1, \cdots, H_n$ of elements of $\mathcal{H}$ satisfying (1), (3), (4) and (5). Suppose for instance that $H_i \geqq H_{i+1}$. If $P_{i-1} \neq P_{i+1}$, delete P_i and H_i. If $P_{i-1} = P_{i+1}$, delete P_i, P_{i+1}, H_i and H_{i+1}. In either case, remaining sequences satisfy the requirements.

We remark here that (2.4) can be used to prove the following refinement of (2.1).

(2.5) *Let G be a group with $O(G) = 1 = O_2(G)$, and let $\mathcal{H}_0$ be the subset of $\mathcal{H}$ consisting of all H in $\mathcal{H}$ such that $N(H)/H$ has a strongly embedded subgroup. Then either $\mathcal{H}_0$ is empty, in which case G is a TI-group, or else each S_2-subgroup of G contains at least two elements of $\mathcal{H}_0$.*

3. In this section, we continue the study of the family $\mathcal{H}$ defined in the preceding section in the case that G is a group satisfying the following hypothesis.

HYPOTHESIS A′. G is a group with $O(G) = 1 = O_2(G)$ in which each maximal 2-local subgroup M which is not 2-closed satisfies the following conditions:

(1) $C_M(O_2(M)) \leqq O_2(M)$;

(2) $M/O_2(M)$ is a TI-group;

(3) $\Omega_1(O_2(M)) \leqq Z(O_2(M))$.

All propositions in this section will be proved under this hypothesis, but some of them are valid under the weaker one in which the assumption that $\Omega_1(O_2(M)) \leqq Z(O_2(M))$ is not made.

(3.1) *If $H \in \mathcal{H}$, the following conditions hold:*

(1) *$N(H)$ is a maximal 2-local subgroup;*

(2) *H is a maximal S_2-intersection;*

(3) *if $H \leqq P \in \mathcal{S}(G)$, then $N_P(H) \in \mathcal{S}(N(H))$;*

(4) *if x is a central involution of G contained in H, there is $P \in \mathcal{S}(G)$ such that $x \in Z(P) \leqq H \leqq P$.*

PROOF. Let M be a maximal 2-local subgroup containing $N(H)$. Then $O_2(M)\leqq H$ by a property of $\mathcal{H}$. Since $M/O_2(M)$ is a TI-group and $N(H)$ is not 2-closed, we have $O_2(M)=H$, and hence (1) holds. Also $C(H)\leqq H$ and $\Omega_1(H)\leqq Z(H)$ by our hypothesis. Let D be a maximal S_2-intersection containing H. Since D is also contained in $\mathcal{H}$, we have $\Omega_1(D)\leqq Z(D)\leqq H$, and so $\Omega_1(D)=\Omega_1(H)$. Since both $N(D)$ and $N(H)$ are maximal 2-local subgroups by (1), it follows that $N(D)=N(H)$. Thus $H=D$ is a maximal S_2-intersection. (3) is a well-known property of a maximal S_2-intersection. Let x be a central involution of G contained in H. Since $\Omega_1(H)\leqq Z(H)$, we can take $P\in\mathcal{S}(G)$ such that $H\leqq P\leqq C(x)$. As remarked above, $C(H)\leqq H$ and therefore $x\in Z(P)\leqq H$, q.e.d.

(3.2) *If $P\in\mathcal{S}(G)$ and a is a central involution of G contained in $P-Z(P)$, then $C_P(a)\in\mathcal{H}$.*

PROOF. Let $C_P(a)\leqq Q\in\mathcal{S}(C(a))$. Then $P\neq Q\in\mathcal{S}(G)$ and $C_P(a)=P\cap Q$. Let D be maximal among S_2-intersections of the form $P\cap R$, $P\neq R\in\mathcal{S}(G)$, containing $P\cap Q$. Then it is known that D is a maximal S_2-intersection. By (3.1), $\Omega_1(D)\leqq Z(D)$, so $C_P(a)=D\in\mathcal{H}$, q.e.d.

(3.3) *Let a be an involution of G such that $C(a)$ is not 2-closed. Then $O_2(C(a))\in\mathcal{H}$.*

PROOF. By a property of $\mathcal{H}$, there is H in $\mathcal{H}$ such that $O_2(C(a))\leqq H$ and $C(a)\leqq N(H)$. By (3.1), $\Omega_1(H)\leqq Z(H)$, so $H\leqq C(a)$ and therefore $O_2(C(a))=H$, q.e.d.

(3.4) *Let P, Q be distinct S_2-subgroups of G such that $Z(P)\cap Z(Q)\neq 1$. Then $\langle Z(P), Z(Q)\rangle\leqq P\cap Q\in\mathcal{H}$ and $\langle P, Q\rangle\leqq N(P\cap Q)$.*

PROOF. Let a be an involution of $Z(P)\cap Z(Q)$. Then $C(a)$ is not 2-closed, and so $H=O_2(C(a))$ is contained in $\mathcal{H}$ by (3.3). Clearly, $\langle P, Q\rangle\leqq N(H)$ and $\langle Z(P), Z(Q)\rangle\leqq H$. Since H is a maximal S_2-intersection by (3.1), we have $H=P\cap Q$ and therefore the lemma holds.

It will be convenient to introduce the notation $\mathcal{H}(X)$ to denote the set of elements of $\mathcal{H}$ contained in the subgroup X of G.

(3.5) *Let $P\in\mathcal{S}(G)$ and let H_i, $1\leqq i\leqq 2$, be distinct elements of $\mathcal{H}(P)$. If a is an involution of $H_1\cap H_2$, then $C_P(a)\in\mathcal{S}(C(a))$. In particular, $\Omega_1(Z(P))$ is strongly closed in $H_1\cap H_2$ with respect to G.*

PROOF. Since $\Omega_1(H_i)\leqq Z(H_i)$, we have $\langle H_1, H_2\rangle\leqq C_P(a)$. Take $Q\in\mathcal{S}(G)$ so that $C_P(a)\leqq C_Q(a)\in\mathcal{S}(C(a))$. Since H_i is a maximal S_2-intersection and $H_1\neq H_2$, we have $P=Q$ and so $C_P(a)\in\mathcal{S}(C(a))$, q.e.d.

(3.6) *Let $P\in\mathcal{S}(G)$ and let $H\in\mathcal{H}(P)$. Set $Q=N_P(H)$. If $Q_1\in\mathcal{S}(N(H))$ and x is an involution of $Z(Q_1)-Z(Q)$, then $C_P(x)=H$.*

PROOF. Let $Q_1\leqq P_1\in\mathcal{S}(G)$ and $C_{P_1}(x)\in\mathcal{S}(C(x))$. Take R in $\mathcal{S}(G)$ so that $C_P(x)\leqq C_R(x)\in\mathcal{S}(C(x))$. If $P\neq R$, then as H is a maximal S_2-intersection, we have $H=C_P(x)=P\cap R$ and the assertion holds in this case. Suppose then that $P=R$, in which case $C_P(x)\in\mathcal{S}(C(x))$. As a consequence $C(x)$ is not 2-closed, so $O_2(C(x))\in\mathcal{H}$ by (3.3). Furthermore

$$O_2(C(x))\leqq C_P(x)\cap C_{P_1}(x)\leqq P\cap P_1=H,$$

whence $O_2(C(x))=H$. Thus $C(x)\leqq N(H)$ and therefore $C_P(x)=C_Q(x)$ and $C_{P_1}(x)=C_{Q_1}(x)$, which, however, is not possible as $x\in Z(Q_1)-Z(Q)$.

As an immediate consequence, we have

(3.7) *Let P, H and Q be as in the preceding lemma. If $H\neq H_1\in\mathcal{H}(P)$ and $Q_1\in\mathcal{S}(N(H))$, then $\Omega_1(Z(Q_1))\cap H_1\leqq\Omega_1(Z(Q))$.*

We next prove

(3.8) *Let P_i, $1\leqq i\leqq 3$, be S_2-subgroups of G such that $P_1\neq P_2\neq P_3$, and let H_i, H_{i+1} be distinct elements of $\mathcal{H}(P_i)$, $1\leqq i\leqq 3$. Then we have $\bigcap_{i=1}^{4}H_i=1$.*

PROOF. Suppose false, and take an involution d of $D=\bigcap_{i=1}^{4}H_i$. Then by (3.5), $\langle H_i, H_{i+1}\rangle\leqq C_{P_i}(d)\in\mathcal{S}(C(d))$. Furthermore $C_{P_1}(d)\neq C_{P_2}(d)$ because $H_1\neq H_2$ and $P_1\neq P_2$, and so $O_2(C(d))\in\mathcal{H}$ by (3.3). However

$$O_2(C(d))\leqq C_{P_1}(d)\cap C_{P_2}(d)\leqq P_1\cap P_2=H_2,$$

whence $O_2(C(d))=H_2$. Likewise we have $O_2(C(d))=H_3$, which contradicts our assumption that $H_2\neq H_3$. Hence (3.8) holds.

(3.9) *Let $P\in\mathcal{S}(G)$ and let H_i, $1\leqq i\leqq 2$, be distinct elements of $\mathcal{H}(P)$. If $N_P(H_i)\neq Q_i\in\mathcal{S}(N(H_i))$, $1\leqq i\leqq 2$, then $Z(Q_1)\cap Z(Q_2)=1$.*

PROOF. By (2.1), $|\mathcal{H}(Q_i)|>1$, so let $H_i\neq K_i\in\mathcal{H}(Q_i)$, $1\leqq i\leqq 2$. If $Q_i\leqq P_i\in\mathcal{S}(G)$, then clearly $P_1\neq P\neq P_2$. Therefore $Z(Q_1)\cap Z(Q_2)\leqq K_1\cap H_1\cap H_2\cap K_2=1$ by the preceding lemma, q. e. d.

We now proceed to the study of the S_2-intersections of G. In view of (2.4) and (3.1), we are naturally led to the

DEFINITION. A *path* $(P_i; H_i; m)$ is, by definition, a pair of a sequence $\{P_i\}_{0\leqq i\leqq m}$ of S_2-subgroups of G and a sequence $\{H_i\}_{1\leqq i\leqq m}$ of elements of $\mathcal{H}$ satisfying the following conditions:

(1) $P_{i-1}\neq P_i$, $1\leqq i\leqq m$;

(2) $H_i\neq H_{i+1}$, $1\leqq i\leqq m-1$;

(3) $H_i=P_{i-1}\cap P_i$, $1\leqq i\leqq m$.

We shall call m the *length* of the path. The path is *proper* if $\bigcap_{i=1}^{m}H_i\neq 1$. The path *joins* P and Q if $P_0=P$ and $P_m=Q$.

Our aim here is to establish the uniqueness of the proper path joining two fixed S_2-subgroups of G. We first prove the following result.

(3.10) *Let $(P_i; H_i; m)$ be a path joining S_2-subgroups P and Q of G. Then the following conditions hold:*

(1) *if* $m \leqq 3$, *then* $P \neq Q$;

(2) *if* $(Q_i; K_i; n)$ *is another path joining* P *and* Q *such that* $\left(\bigcap_{i=1}^{m} H_i\right) \cap \left(\bigcap_{i=1}^{n} K_i\right) \neq 1$, *then* $m = n$ *and* $P_i = Q_i$, $1 \leqq i \leqq m$ (*and therefore* $H_i = K_i$ *for each* i).

PROOF. In either case, $m \leqq 3$ by our assumption or (3.8). We note also that we need only consider the case $n \leqq m$ in proving (2), and consequently the assertion is obvious if $m = 1$.

Assume then that $m = 2$. Since $H_1 \neq H_2$, we have $P \neq Q$. Suppose that $P \cap Q = K \in \mathscr{H}$, then $1 \neq H_1 \cap H_2 = H_1 \cap H_2 \cap K$ and $H_1 \neq K \neq H_2$ because $Q \neq P_1 \neq P$. This, however, contradicts (3.8). Therefore $P \cap Q$ is not contained in $\mathscr{H}$, and consequently $n = 2$. But then $H_i = K_i$, $1 \leqq i \leqq 2$, by (3.8) and so $P_1 = Q_1$. Thus the assertion holds if $m = 2$.

Assume next that $m = 3$. Since Q and P_1 are joined by a path of length 2 while $P \cap P_1 \in \mathscr{H}$, $P \neq Q$ as was proved above. Since $P_1 \neq P_2 \neq Q$, $H_1 \neq H_2 \neq H_3$ and $H_1 \cap H_2 \cap H_3 \cap K_n \neq 1$, (3.8) yields that $H_3 = K_n$. Consequently, we have that $n > 1$ and that $H_1 \neq H_2 \neq H_3 \neq K_{n-1}$. Since $H_1 \cap H_2 \cap H_3 \cap K_{n-1} \neq 1$, it follows from (3.8) that $P_2 = Q_{n-1}$. We can now apply the argument of the preceding paragraph to conclude that $n = 3$ and $P_1 = Q_1$. The proof is complete.

We finally prove

(3.11) *Let* P, Q *be* S_2-*subgroups of* G. *Then the following conditions hold:*

(1) *there is at most one proper path joining* P *and* Q;

(2) P *and* Q *are joined by a proper path if and only if* $P \neq Q$ *and* $P \cap Q \neq 1$;

(3) *if* $(P_i; H_i; m)$ *is a proper path joining* P *and* Q, *then* $m \leqq 3$ *and* $P \cap Q = \bigcap_{i=1}^{m} H_i$.

PROOF. Let $(P_i; H_i; m)$ be a proper path joining P and Q, so that $1 \neq \bigcap_{i=1}^{m} H_i \leqq P \cap Q$. Then (3.8) yields that $m \leqq 3$, and therefore $P \neq Q$ by the preceding lemma. Conversely, (2.4) shows if $P \neq Q$ and $P \cap Q \neq 1$ that P and Q are joined by a proper path $(Q_i; K_i; n)$ such that $P \cap Q = \bigcap_{i=1}^{n} K_i$. Thus (2) holds. Furthermore the above argument and (3.11, 2) show that (1) and (3) also hold.

4. Henceforth, we assume that G is a group satisfying the following conditions:

(1) G satisfies Hypothesis A;

(2) $O^2(G) = G$;

(3) an S_2-subgroup of G is neither dihedral nor semi-dihedral;

(4) G is not a C-group.

The purpose of this section is to establish various properties of G that will be necessary in the succeeding one. The following is the list of them.

Let $P \in \mathcal{S}(G)$, then the following conditions hold:

(4.1) $|P| = q^4$, *q is a power of 2 and $q > 2$;*

(4.2) *$Z(P)$ and $P/Z(P)$ are elementary Abelian groups of order q^2;*

(4.3) $|\mathcal{H}(P)| = 2$.

Let $\mathcal{H}(P) = \{H_1, H_2\}$, then the following conditions hold:

(4.4) *H_i, $1 \leqq i \leqq 2$, are elementary Abelian normal subgroups of P of order q^3;*

(4.5) *every elementary Abelian subgroup of P is contained in either H_1 or H_2;*

(4.6) *$P = H_1 H_2$ and $H_1 \cap H_2 = Z(P)$;*

(4.7) *$O'(N(H_i))/H_i$ is a central extension of $PSL(2, q)$ by a group of odd order, $1 \leqq i \leqq 2$;*

(4.8) *$N(H_1) \cap N(H_2) = N(P)$;*

(4.9) *for each i, $1 \leqq i \leqq 2$, H_i has a subgroup $\hat{H}_i$ of order q such that $Z(Q) \cap Z(R) = \hat{H}_i$ for any distinct S_2-subgroups Q, R of G containing H_i;*

(4.10) *$Z(P) = \hat{H}_1 \times \hat{H}_2$;*

(4.11) *there exist precisely three conjugacy classes of involutions of G, and we can take representatives of these classes respectively from $Z(P) - \hat{H}_1 - \hat{H}_2$, $\hat{H}_1 - \{1\}$ and $\hat{H}_2 - \{1\}$;*

(4.12) *$C(x) \leqq N(P)$ for each x in $Z(P) - \hat{H}_1 - \hat{H}_2$;*

(4.13) *$C(x) \leqq N(H_i)$ for each $x \neq 1$ in $\hat{H}_i$ and each i;*

(4.14) *if $h_i \in H_i - Z(P)$, $1 \leqq i \leqq 2$, then $[h_1, h_2] \in Z(P) - \hat{H}_1 - \hat{H}_2$.*

Let $Q \in \mathcal{S}(G)$, then the following conditions hold:

(4.15) *there exists at most one path of length $\leqq 3$ joining P and Q;*

(4.16) *P and Q are joined by a path of length $\leqq 3$ if and only if $P \neq Q$ and $P \cap Q \neq 1$.*

Let $(P_i; K_i; m)$ be a path of length m joining P and Q, then the following conditions hold:

(4.17) *if $m = 2$, $P \cap Q = K_1 \cap K_2 = Z(P_1)$;*

(4.18) *if $m = 3$, $P \cap Q = K_1 \cap K_2 \cap K_3 = Z(P_1) \cap Z(P_2) = \hat{K}_2$;*

(4.19) *if $m = 4$, $P \cap Q$ contains no involutions that are conjugate to the elements of $Z(P) - \hat{H}_1 - \hat{H}_2$.*

Finally, we have

(4.20) *precisely $1 + 2(q + q^2 + q^3)$ S_2-subgroups of G intersect P in the non-identity elements.*

The outline of the proof is as follows. In addition to the family $\mathcal{H}$, two other families of 2-subgroups of G will come into our consideration. Namely, for each subgroup X of G, we define

$$\mathscr{Z}(X)=\{\Omega_1(Z(P))\,;\ P\in\mathcal{S}(G)\text{ and }\Omega_1(Z(P))\leqq X\}$$

and

$$\mathscr{H}'(X)=\{H;\ H\in\mathscr{H}(X)\text{ and }|\mathscr{Z}(H)|>1\}\,.$$

The families $\mathscr{Z}(G)$ and $\mathscr{H}'(G)$ will be denoted simply by $\mathscr{Z}$ and $\mathscr{H}'$. Still more important than these is the family $\mathscr{K}(P)$, where $P\in\mathcal{S}(G)$, consisting of pairs (H_1, H_2) of distinct elements of $\mathscr{H}'(P)$ such that $N_P(H_1)=N_P(H_2)$. We first show that $\mathscr{K}(P)$ is in fact not empty. In proving this, Goldschmidt's 2-fusion theorem [**5**] will be used significantly. Taking (H_1, H_2) from $\mathscr{K}(P)$, we then investigate the structure of $Q=N_P(H_i)$ and $N(H_i)$, $1\leqq i\leqq 2$. We prove among other things that there is a power q of 2 such that $q^3\leqq|Q|\leqq q^4$, that $|P:Q|\leqq 2$, and that $\mathscr{H}'(P)=\{H_1, H_2\}$. If $q=2$, then $|P|\leqq 32$ and it will not be difficult to derive a contradiction, so we assume $q\neq 2$. We then show that $\mathscr{H}=\mathscr{H}'$. In proving this, Corollary 4 or 5 of [**5**] will be useful. At this stage, we can count the number of S_2-subgroups of G which have nontrivial intersections with P, and in particular show that there is $R\in\mathcal{S}(G)$ such that $P\cap R=1$. On the other hand, we can show that $Z(Q)$ has subgroups $\hat{H}_i$, $1\leqq i\leqq 2$, of index q such that every involution of $Z(Q)-\hat{H}_1-\hat{H}_2$ has a 2-closed centralizer. It then follows if $P\neq Q$ that the subgroups $Z(P)$, $\hat{H}_1$ and $\hat{H}_2$ partition $Z(Q)$, which, however, is not possible as $q\neq 2$. Thus we conclude that $P=Q$. Once this is established, it is not difficult to verify (4.1)—(4.20).

Before entering into details, a few remarks are in order. First, it is clear that Hypothesis A implies Hypothesis A$'$ of Section 3, so that the results there are applicable. Secondly, since G is not a TI-group, $\mathscr{H}$ is not empty in G. Lastly, if $H\in\mathscr{H}$ then $N(H)/H$ is a TI-group by (3.1), and hence the structure theorem of TI-groups will be essential. We quote the following result from [**13**, Theorems 2, 3 and 6].

THEOREM. *Let N be a TI-group which is not 2-closed, then N is solvable if and only if $m(N)=1$. If N is nonsolvable, then $O'(N)/Z(O'(N))$ is isomorphic to one of the groups $PSL(2, q)$, $Sz(q)$ and $PSU(3, q)$, q a power of 2 and $q>2$.*

We begin by proving an analogue of [**14**, Lemma 8].

(4.21) *Assume that the elements of $\mathscr{Z}$ are TI-sets. If $P\in\mathcal{S}(G)$, $X\in\mathscr{Z}$ and $P\cap X\neq 1$, then $X\leqq P$.*

PROOF. If $Z(P)\cap X\neq 1$, the assertion follows from (3.4). Otherwise, let $1\neq x\in P\cap X$. Then $H=C_P(x)\in\mathscr{H}$ by (3.2). Hence, there is $Y\in\mathscr{Z}(H)$ such that $x\in Y$, by (3.1). Since $x\in X\cap Y$, $X=Y\leqq P$ by our assumption, q.e.d.

(4.22) *If $H\in\mathscr{H}'$, then $|H|\neq 4$.*

PROOF. Let $H\leqq P\in\mathcal{S}(G)$. Since $|\mathscr{Z}(H)|>1$, $H-Z(P)$ contains a central involution x of G. By (3.2), $C_P(x)=H$, so $|H|\neq 4$ by the lemma of Suzuki [**11**, Lemma 4].

(4.23) *If $H \in \mathcal{H}$, $N(H)$ is nonsolvable and P, Q are distinct S_2-subgroups of $N(H)$, then the following conditions hold:*

(1) *the subgroup $\Omega_1(Z(P)) \cap \Omega_1(Z(Q))$ does not depend upon the choice of P and Q;*

(2) *if $\Omega_1(Z(P)) \neq \Omega_1(Z(Q))$, then*

$$|P: H| = |\Omega_1(Z(P)): \Omega_1(Z(P)) \cap \Omega_1(Z(Q))|.$$

PROOF. Since $N(H)$ is nonsolvable, it follows from the structure theorem of TI-groups that P acts by conjugation transitively on $\mathcal{S}(N(H))-\{P\}$. Hence, $\Omega_1(Z(P)) \cap \Omega_1(Z(Q)) = \Omega_1(Z(P)) \cap \Omega_1(Z(R))$ for any $R \in \mathcal{S}(N(H))-\{P\}$, which implies the validity of (1).

To prove (2), we set $Z = \Omega_1(Z(P)) \cap \Omega_1(Z(Q))$, $q = |P: H|$, $r = |\Omega_1(Z(P))|$, $s = |Z|$ and $t = |\Omega_1(H)|$. Hypothesis A and (1) show that $\Omega_1(H)-Z$ is the direct union of $(q+1)$ subsets $\Omega_1(Z(R))-Z$, R ranging over all S_2-subgroups of $N(H)$. Consequently, $t-s = (q+1)(r-s)$ whence $t = qr+r-qs$. In particular, $t-r$ $(\neq 0)$ is divisible by qs and hence $r \geqq qs$. Therefore, setting $t' = t/s$ and $r' = r/s$, we have that $t' = qr'+r'-q$ and that $r' \geqq q$. As $t' > r' \geqq q$ and $qr' > q$, it follows that $r' = q$, q. e. d.

(4.24) *Assume that the elements of $\mathcal{Z}$ are not TI-sets. Then, for any $H \in \mathcal{H}$, $N(H)$ contains an S_2-subgroup of G.*

PROOF. Let $H \leqq P \in \mathcal{S}(G)$ and set $X = \Omega_1(Z(P))$. By our assumption, there is Y in $\mathcal{Z}$ such that $X \neq Y$ and $X \cap Y \neq 1$. Take $Q \in \mathcal{S}(G)$ so that $Y = \Omega_1(Z(G))$. Then $H_1 = P \cap Q \in \mathcal{H}$ and $P \leqq N(H_1)$ by (3.4). In particular, we can assume that $H \neq H_1$. Furthermore, we can assume that $|\mathcal{Z}(H)| > 1$, because otherwise $P \leqq N(X) = N(H)$. Let $X \neq Z \in \mathcal{Z}(H)$. Then $H_1 \cap Z = X \cap Z$ by (3.5). Suppose $X \cap Z = 1$, then, as Z is not cyclic by our assumption, (4.23) yields that

$$|P: H_1| = |X: X \cap Y| < |X| = |Z| \leqq |P: H_1|,$$

a contradiction. Thus $X \cap Z \neq 1$. We can take $R \in \mathcal{S}(G)$ so that $H \leqq R$ and $Z = \Omega_1(Z(R))$ (see the proof of (3.1, 4)). Clearly, $P \cap R = H$, so the assertion follows from (3.4).

(4.25) *We have $|\mathcal{H}'(P)| \geqq 2$ for each $P \in \mathcal{S}(G)$.*

PROOF. If $\mathcal{H}(P) = \mathcal{H}'(P)$, the assertion follows from (2.1), so we assume that there is $H \in \mathcal{H}(P)$ such that $|\mathcal{Z}(H)| = 1$. Since such H is unique, we are reduced to showing that $|\mathcal{H}(P)| \geqq 3$. Suppose, by way of contradiction, that $\mathcal{H}(P) = \{H, H_1\}$ with $H \neq H_1$. Let x be a central involution of G contained in P. If x is not contained in $Z(P)$, then $C_P(x) \in \mathcal{H}(P)$ by (3.2) and so $x \in H$ or $x \in H_1$. Since $\Omega_1(H) = \Omega_1(Z(P))$ by Hypothesis A, every central involution of G contained in P is necessarily contained in H_1. On the other hand, H_1 is normal in P, so Hypothesis A implies that every involution of H_1 is a central involution of G. We conclude that $A=\Omega_1(H_1)$ is a strongly closed Abelian subgroup of P with respect to G. By the 2-fusion theorem of Goldschmidt [**5**], the normal closure K of A is the direct product of the simple groups isomorphic to $PSL(2, q)$ with $q\equiv 0$, 3 or 5 (mod 8), $Sz(q)$, $PSU(3, q)$, q a power of 2, or a group of type JR, and if $A\leqq Q\in\mathcal{S}(K)$ then $A=\Omega_1(Q)$. Since every 2-local subgroup of G is 2-constrained (and core-free) by a result of Gorenstein [**7**, Theorem 4], K is in fact a simple TI-group. Let $y\in N(H)-N(P)$ and set $H_2=H_1^y$. Then $\Omega_1(Z(P))=\Omega_1(Z(P))^y\leqq A\cap A^y$ and so $\langle A, A^y\rangle$ is a 2-subgroup of K. Thus $A=A^y$ by the above remark, which, however, is a contradiction because $H_1\neq H_2$.

(4.26) *$\mathcal{K}(P)$ is not empty for each $P\in\mathcal{S}(G)$.*

PROOF. If $H\in\mathcal{H}'(P)$ is not normal in P, take x in $N_P(N_P(H))-N_P(H)$. Then clearly $(H, H^x)\in\mathcal{K}(P)$. If every element of $\mathcal{H}'(P)$ is normal in P, the assertion follows from the preceding lemma.

(4.27) *We have $\mathcal{Z}(H)\neq\mathcal{Z}(N(H))$ for each $H\in\mathcal{H}$.*

PROOF. Suppose, by way of contradiction, that $\mathcal{Z}(H)=\mathcal{Z}(N(H))$, and let $P\in\mathcal{S}(N(H))$, then as $\mathcal{Z}(P)=\mathcal{Z}(H)$, we have $N(P)\leqq N(\langle\mathcal{Z}(H)\rangle)=N(H)$. Consequently, $P\in\mathcal{S}(G)$. Let $H\neq H_1\in\mathcal{H}(P)$, then (3.5) yields $\mathcal{Z}(H_1)=\mathcal{Z}(H\cap H_1)=\{\Omega_1(Z(P))\}$, whence $H_1=O_2(N(\Omega_1(Z(P))))$. Since H_1 was arbitrary, we conclude that $\mathcal{H}(P)=\{H, H_1\}$, contrary to (4.25). Hence the lemma holds.

(4.28) *Suppose that $P\in\mathcal{S}(G)$ has a cyclic center. Then no element of $\mathcal{H}'(P)$ is normal in P.*

PROOF. Suppose that $H\in\mathcal{H}'(P)$ is normal in P. Then Hypothesis A implies that every involution of H is central in G. Since $|\mathcal{Z}(H)|>1$, $\Omega_1(H)$ is not cyclic and so H contains a four-group A normalized by P. Let $a\in A-Z(P)$, then $H=C_P(a)$ by (3.2). On the other hand, $C_P(a)=C_P(A)$ whence $|P:H|=2$. Since $O^2(G)=G$, it follows from the transfer lemma of Thompson [**15**, Lemma 5.38] that every involution of G is central in G. Consequently, if $H\neq H_1\in\mathcal{H}'(P)$, then $\Omega_1(H_1)\cap H=\Omega_1(H_1\cap H)=\Omega_1(Z(P))$ by (3.5). Hence $\Omega_1(H_1)$ is a four-group and $H_1\cap H$ is either cyclic or generalized quaternion. Furthermore, we have $H_1=\Omega_1(H_1)(H_1\cap H)$ and so $H_1^2=(H_1\cap H)^2\neq 1$ as H_1 is not a four-group by (4.22). Hence $\Omega_1(H_1^2)=\Omega_1((H_1\cap H)^2)=\Omega_1(Z(P))$ whence $P\leqq N(\Omega_1(H_1^2))=N(H_1)$. We now apply the above argument to H_1 and conclude that $H^2=(H_1\cap H)^2=H_1^2$, which, however, is not possible as $H\neq H_1$.

(4.29) *If P, Q are distinct S_2-subgroups of G and $\Omega_1(Z(P))=\Omega_1(Z(Q))$, then the following conditions hold:*

(1) *elements of $\mathcal{Z}$ are TI-sets;*

(2) *$P\cap Q\in\mathcal{H}-\mathcal{H}'$.*

PROOF. Firstly, $H=P\cap Q\in\mathcal{H}$ and $P\leqq N(H)$ by (3.4). Moreover, if $X\in\mathcal{Z}$ and $X\cap H\neq 1$, then $X\leqq H$ by (3.4). Hence (1) is a consequence of (2).

There is an element X of $\mathcal{Z}(P)$ such that $X \not\leqq H$, by (4.27). By the above remark, $X \cap H = 1$. Consequently, if X is not cyclic, then $N(H)$ is nonsolvable and hence (2) holds by (4.23). If X is cyclic, (2) holds by the preceding lemma.

(4.30) *Let $H \in \mathcal{H}'$ and let Q_i, $1 \leqq i \leqq 2$, be distinct S_2-subgroups of $N(H)$. Then $\Omega_1(Z(Q_1)) \neq \Omega_1(Z(Q_2))$.*

PROOF. Let $Q_i \leqq P_i \in \mathcal{S}(G)$, $1 \leqq i \leqq 2$. Then $\Omega_1(Z(P_1)) \neq \Omega_1(Z(P_2))$ by (4.29, 2). On the other hand, $\mathcal{Z}(Z(Q_i)) = \{\Omega_1(Z(P_i))\}$, $1 \leqq i \leqq 2$, by (2.1) and (3.5). Hence $\Omega_1(Z(Q_1)) \neq \Omega_1(Z(Q_2))$ as asserted.

For the remainder of this section, we fix the following notation:

$$P \in \mathcal{S}(G);$$

$$(H_1, H_2) \in \mathcal{K}(P);$$

$$N_i = N(H_i), \quad 1 \leqq i \leqq 2:$$

$$Q = N_P(H_i), \quad 1 \leqq i \leqq 2;$$

$$X = \Omega_1(Z(Q)).$$

Furthermore, in a few lemmas below, let $Q \neq Q_i \in \mathcal{S}(N_i)$, and set $X_i = \Omega_1(Z(Q_i))$, $1 \leqq i \leqq 2$. Note that $X_i \neq X$ by (4.30), and that $X_1 \cap H_2 = X_1 \cap X$ and $X_2 \cap H_1 = X_2 \cap X$ by (3.7).

(4.31) *The following conditions hold:*

(1) $|Q : H_1| = |Q : H_2|$;

(2) $|Q : H_i| = |X : X_i \cap X| \geqq |X_i \cap X|$, $1 \leqq i \leqq 2$;

(3) $Q = X_1 H_2 = X_2 H_1$.

PROOF. Note first that $X_1 \cap X_2 = 1$ by (3.9). Hence, if $i \neq j$, $|X : X_i \cap X| \geqq |X_j \cap X|$. Assume first that N_i is nonsolvable for each i, $1 \leqq i \leqq 2$. Then, by (4.23),

$$\begin{aligned} |Q : H_1| &= |X_1 : X_1 \cap X| \\ &\leqq |Q : H_2| \\ &= |X_2 : X_2 \cap X| \\ &\leqq |Q : H_1| . \end{aligned}$$

Therefore the assertion holds in this case.

Assume next that either N_1 or N_2, say N_2, is solvable. Then $m(Q/H_2) = 1$ and so $|X_1 : X_1 \cap X| = 2 = |\Omega_1(H_1) : X|$ as $\Omega_1(H_1) \cap H_2 = X$ by Hypothesis A and (3.7). This implies first that N_1 is solvable by (4.23), and next that $|\Omega_1(H_1) : X_1 \cap X| = 4$. Therefore, setting $C = C_{N_1}(X_1 \cap X)$, we have that C has precisely three S_2-subgroups (see (4.30)). On the other hand, C/H_1 is a TI-group and so $|\mathcal{S}(C)| \equiv 1 \pmod{|Q : H_1|}$. Thus $|Q : H_1| = 2$. Since N_1 is solvable, we also have $|Q : H_2| = 2$. Hence the lemma holds in this case as well.

NOTATION. We define $q = |Q : H_i|$, $1 \leqq i \leqq 2$.

(4.32) *The following conditions hold:*

(1) H_i *is elementary Abelian,* $1 \leqq i \leqq 2$;

(2) $Q = H_1 H_2$ *and* $H_1 \cap H_2 = X$;

(3) $|X| \geqq 4$.

PROOF. By the preceding lemma, $H_i = X_i(H_1 \cap H_2)$, whence $H_1^2 = (H_1 \cap H_2)^2 = H_2^2$. Since $H_1 \neq H_2$, we must have $H_1^2 = 1 = H_2^2$, proving (1). As $Q = H_1 H_2$ by (4.31), it follows from (1) that $H_1 \cap H_2 = X$. Suppose $|X| = 2$, then, as $|X| = q|X_1 \cap X|$ by (4.31), we have $q = 2$ and $X_1 \cap X = 1$. But then by (2), $|H_1| = q|X| = 4$, contrary to (4.22). Hence $|X| \geqq 4$.

(4.33) *The following conditions hold:*

(1) *every elementary Abelian subgroup of* Q *is contained in either* H_1 *or* H_2;

(2) $\mathcal{H}'(Q) = \{H_1, H_2\}$;

(3) *either* $P = Q$ *or* $|N_P(Q) : Q| = 2$ *and* H_1 *is conjugate to* H_2 *in* $N_P(Q)$.

PROOF. Let $h_1 \in H_1 - X$. As $H_1 = X_1 X$ by (4.31, 3) and (4.32, 2), h_1 can be written as $h_1 = x_1 x$ with $x_1 \in X_1 - X$ and $x \in X$, whence $C_Q(h_1) = C_Q(x_1) = H_1$ by (3.6). Likewise we have $C_Q(h_2) = H_2$ for each $h_2 \in H_2 - X$. It then follows from (4.32) that every involution of Q is contained in either H_1 or H_2. Hence (1) holds. Consequently, if $H \in \mathcal{H}'(Q)$, we have either $|\mathcal{Z}(H \cap H_1)| > 1$ or $|\mathcal{Z}(H \cap H_2)| > 1$, whence $H = H_1$ or H_2 by (3.5). (3) is an immediate consequence of (2).

(4.34) *For each* i, $1 \leqq i \leqq 2$, H_i *has a subgroup* $\hat{H}_i$ *such that* $Z(R) \cap Z(S) = \hat{H}_i$ *for any distinct* S_2*-subgroups* R, S *of* N_i. *Furthermore the following conditions hold:*

(1) $\hat{H}_1 \cap \hat{H}_2 = 1$;

(2) $|\hat{H}_i| \leqq q$, $1 \leqq i \leqq 2$;

(3) *if* $1 \neq x \in \hat{H}_i$, *then* $C(x) \leqq N(H_i)$.

PROOF. In proving the first part, we need only consider the case $i = 1$, and must show that $Z(R) \cap X$ does not depend upon the choice of $R \in \mathcal{S}(N_1)$, $R \neq Q$. If $q > 2$, N_1 is nonsolvable by (4.31) and the assertion holds by (4.23). Assume then that $q = 2$. Then $|X| = 4$ and $Z(R) \cap X$ is a subgroup of X of order 2 for each $R \in \mathcal{S}(N_i) - \{Q\}$ and each i, by (4.31) and (4.32). Suppose that $R, S \in \mathcal{S}(N_1) - \{Q\}$ and $Z(R) \cap X \neq Z(S) \cap X$. Then $Z(T) \cap X$ does not depend upon the choice of $T \in \mathcal{S}(N_2) - \{Q\}$, by (3.9). Hence H_1 is not conjugate to H_2 in $N_P(Q)$ whence $P = Q$ by the preceding lemma. But then $P \cong \boldsymbol{Z}_2 \times \boldsymbol{D}_8$ and either $O(G) \neq O_{2',2}(G)$ or $O^2(G) \neq G$ by the theorem of Harada [**9**], contrary to our hypothesis. Hence the assertion holds in this case as well.

(1) and (2) are consequences of (3.9) and (4.31). Let $1 \neq x \in \hat{H}_1$, then $C_P(x) \in \mathcal{S}(C(x))$ by (3.5). Likewise, $C_{P_1}(x) \in \mathcal{S}(C(x))$ if $Q_1 \leqq P_1 \in \mathcal{S}(G)$. As $C(x)$ is not 2-closed, $O_2(C(x)) \in \mathcal{H}$. But $O_2(C(x)) \leqq C_P(x) \cap C_{P_1}(x) \leqq P \cap P_1 = H_1$,

whence $O_2(C(x))=H_1$. Hence $C(x)\leqq N(H_1)$. The proof of (4.34) is complete.

(4.35) *If $x\in X-\hat{H}_1-\hat{H}_2$ and $C(x)\nleqq N(P)$, then $\mathcal{H}\neq\mathcal{H}'$ and $x\in Z(P)$.*

PROOF. Let $y\in C(x)-N(P)$. Suppose first that the elements of $\mathcal{Z}$ are not TI-sets. Then $P=Q$ and $\mathcal{H}=\mathcal{H}'$ by (4.24) and (4.29). Therefore $\mathcal{H}(P)=\{H_1, H_2\}$ by (4.33, 2). On the other hand, $x=x^y\in X\cap X^y$, so $P\cap P^y\in\mathcal{H}$ by (3.4). Consequently, $x\in X\cap X^y=\hat{H}_1$ or $\hat{H}_2$ by the preceding lemma, contrary to our assumption. Hence the elements of $\mathcal{Z}$ are TI-sets.

By (3.5), $C_P(x)$ and $C_P(x)^y$ are distinct S_2-subgroups of $C(x)$, so $H=O_2(C(x))\in\mathcal{H}$ by (3.3). Suppose that x is not contained in $Z(P)$. Then $H\in\mathcal{H}'$ because otherwise $\Omega_1(H)=Z(P)$, and we have $N_P(H)=C_P(x)\neq P$ because x is central in some S_2-subgroup of $N(H)$. Thus if $z\in N_P(N_P(H))-N_P(H)$, then $(H, H^z)\in\mathcal{K}(P)$. As a consequence, $\mathcal{H}'(N_P(H))=\{H, H^z\}$ by (4.33, 2). But $H_i\leqq C_P(x)=N_P(H)$, so $H=H_1$ or H_2 and hence $N_P(H)=Q$. Again we have $x\in\hat{H}_1$ or $\hat{H}_2$, contrary to our assumption. Hence $x\in Z(P)$. Since $x\in Z(P)\cap Z(P)^y$ and elements of $\mathcal{Z}$ are TI-sets, $Z(P)=Z(P)^y$ and hence $P\cap P^y\in\mathcal{H}-\mathcal{H}'$ by (4.29). The proof is complete.

(4.36) *We have $\mathcal{H}'(N_P(Q))=\{H_1, H_2\}$.*

PROOF. In view of (4.33, 2), we may assume that $P\neq Q$, so that elements of $\mathcal{Z}$ are TI-sets by (4.24). The proof differs according as $Z(P)$ is cyclic or not.

Assume first that $Z(P)$ is not cyclic. Let $Z\in\mathcal{Z}(N_P(Q))$. Since $|N_P(Q):Q|=2$ by (4.33, 3), $Q\cap Z\neq 1$ and hence $H_i\cap Z\neq 1$, $i=1$ or 2, as every involution of Q is contained in H_1 or H_2. Thus $Z\leqq H_1$ or H_2 by (4.21). Consequently, if $H\in\mathcal{H}'(N_P(Q))$, then $|\mathcal{Z}(H\cap H_i)|>1$ for $i=1$ or 2 and so $H=H_1$ or H_2 by (3.5). Hence the assertion holds in this case.

Assume next that $Z(P)$ is cyclic. Let $H\in\mathcal{H}'(N_P(Q))$. By (4.28), H is not normal in P and so $(H, H')\in\mathcal{K}(P)$ for some $H'\in\mathcal{H}'(P)$. As a consequence, $|Z(N_P(H))|\geqq 4$ by (4.32, 3). Let $N_P(H)\neq R\in\mathcal{S}(N(H))$. If $H_1\neq H\neq H_2$, (3.9) yields that $Z(R)\cap Z(Q_i)=1$ for each $Q_i\in\mathcal{S}(N_i)-\{Q\}$ and each i, $1\leqq i\leqq 2$. Hence $(Z(R)-\{1\})\cap Q=(Z(R)-\{1\})\cap(X-\hat{H}_1-\hat{H}_2)$. Since $|Z(R)|\geqq 4$ and $|N_P(Q):Q|=2$, $Z(R)\cap Q$ contains an element $x\neq 1$. Since $R\leqq C(x)$ but $R\nleqq N(P)$, (4.35) forces $\langle x\rangle=Z(P)$. Therefore if $R\leqq S\in\mathcal{S}(G)$, then $\langle x\rangle=Z(S)$ by (2.1) and (3.5). But then $H=P\cap S\in\mathcal{H}-\mathcal{H}'$ by (4.29), contrary to $H\in\mathcal{H}'(N_P(Q))$. Hence we must have $H=H_1$ or H_2 in this case as well.

(4.37) *We have $|P:Q|\leqq 2$ and $\mathcal{H}'(P)=\{H_1, H_2\}$.*

PROOF. By the preceding lemma, $N_P(N_P(Q))$ permutes H_1 and H_2, and so normalizes $Q=H_1H_2$. Hence $N_P(N_P(Q))=N_P(Q)$, which implies that $N_P(Q)=P$. Therefore the assertions follow from (4.33) and (4.36).

(4.38) *We have $q>2$.*

PROOF. Suppose that $q=2$. Then $|X|=4$ and $|\hat{H}_i|=2$, so $Q\cong\boldsymbol{Z}_2\times\boldsymbol{D}_8$.

We will show that either $O(G)\neq O_{2',2}(G)$ or $O^2(G)\neq G$, which obviously contradicts our hypothesis. Harada's theorem disposes of the case $P=Q$, so we assume that $P\neq Q$, in which case elements of $\mathcal{Z}$ are TI-sets by (4.24) and hence $X\neq Z(P)$. Set $\langle b_i\rangle=\hat{H}_i$, $1\leqq i\leqq 2$, and $c=b_1b_2$. Since b_1 is conjugate to b_2 in P by (4.33, 3), $Z(P)=\langle c\rangle$. By the same reason, $[Q, Q]=\langle c\rangle$. On the other hand, $P/X\cong \boldsymbol{D}_8$, and so there is an element d of $P-Q$ such that $d^2\in X$ and consequently $d^2\in Z(P)$. Replacing d by b_1d, if necessary, we may assume that $d^2=1$. Take $a_1\in H_1-X$ and set $a_2=a_1^d$. Then $a_2\in H_2-X$, so $[a_1, a_2]=c$ by (4.33, 1). We can now calculate $C_P(d)$. It is easy to see that $|C_P(d)|=8$. Suppose $O^2(G)=G$, then d is conjugate to an element of X by Thompson's lemma. In particular, $|C(d)|$ is divisible by 16. Thus $C_P(d)$ is not an S_2-subgroup of $C(d)$, which implies that $C_P(d)\in\mathcal{H}$ (see the proof of (3.2)). Moreover, d is not contained in $Z(P)$, so $C_P(d)\in\mathcal{H}'$, contradicting the preceding lemma. Therefore we have $O^2(G)\neq G$ in this case.

Since Q/H_i is elementary Abelian of order q, (4.38) has the following immediate consequence.

(4.39) *$O'(N(H_i))/H_i$ is a central extension of $PSL(2, q)$ by a group of odd order, $1\leqq i\leqq 2$.*

(4.40) *We have $\mathcal{H}=\mathcal{H}'$.*

PROOF. Suppose $\mathcal{H}\neq\mathcal{H}'$, then $\mathcal{H}(P)\neq\mathcal{H}'(P)$ and elements of $\mathcal{Z}$ are TI-sets by (4.29). Let $H\in\mathcal{H}(P)-\mathcal{H}'(P)$ and set $N=N(H)$. Note that $N(Q)=N(P)\leqq N(Z(P))=N$ by (4.37).

Suppose that $H\leqq Q$. Then $X\leqq H$ and hence $X=Z(P)$ because $\Omega_1(H)=Z(P)$. Since elements of $\mathcal{Z}$ are TI-sets, $\hat{H}_i=1$, $1\leqq i\leqq 2$, and $Z(R)\cap H=1$ if $Q\neq R\in\mathcal{S}(N_1)$, by (4.21). As $2<q=|Z(R)|$, N is nonsolvable, so $P=Q$ because $H<Q$, $|P:Q|\leqq 2$ and Q is normal in $N(P)$. However this contradicts the following general lemma.

(4.41) *Let N be a group with $C_N(O_2(N))\leqq O_2(N)$ and with class 2 S_2-subgroups. Then for any distinct S_2-subgroups S, T of N, we have $Z(S)\neq Z(T)$.*

PROOF. Suppose that $Z(S)=Z(T)$. Set $M=\langle S, T\rangle$ and $Z=Z(S)=Z(T)$. Then $C_M(O_2(M))\leqq O_2(M)$ and $Z\leqq Z(M)$. Consequently, setting $\bar{M}=M/Z$, we obtain that $C_{\bar{M}}(O_2(\bar{M}))\leqq O_2(\bar{M})$. Since $\bar{M}$ has Abelian S_2-subgroups, this yields that $\bar{M}$ is 2-closed, contrary to the fact that $S\neq T$, q.e.d.

Hence $H\nleqq Q$. Therefore $|P:Q|=2$ and so $P=QH$. Since $N_H(H_1)\nleqq H_1$, $H_1<H_1N_H(H_1)\leqq Q$. Moreover $H_1N_H(H_1)$ is normal in $N_{N_1}(Q)$, whence $H_1N_H(H_1)=Q$. Consequently, $P=H_1H$ and $H_1\cap H=Z(P)$ as $\Omega_1(H)=Z(P)$. In particular, P/H is an elementary Abelian group of order $\geqq q$, and so $O'(N)/H$ is a central extension of $PSL(2, r)$, $r\geqq q$, by a group of odd order. Since $H\leqq HX<P$ and HX is normal in $N(P)$, we conclude that $HX=H$ whence $X=Z(P)$. Set $\bar{N}=N/X$ and $M=C(X)$. Note that $M\leqq N$. Since $O_2(M)=H$

and $C_{\bar M}(O_2(\bar M)) \leqq O_2(\bar M)$, $Z(\bar P) \leqq \bar H$, which implies that the second center Z_2 of P is contained in H. On the other hand, $[Z_2, H_1] \leqq X \leqq H_1$ and therefore $Z_2 \leqq Q \cap H$. Since $H_1 < H_1Z_2$ and H_1Z_2 is normal in $N_{N_1}(Q)$, it follows that $H_1Z_2 = Q$ whence $Z_2 = Q \cap H$. Consequently, $\bar H$ is Abelian and $\bar P$ has class 2. Furthermore Z_2 is not normal in N by (4.41) applied to $\bar M$, which implies that $\bar H$ is elementary. Suppose that $\bar N$ has a subgroup $\bar K = K/X$ of index 2. Then, as $O^2(N/H) = N/H$, $|H: H \cap K| = 2$. Since Z_2 is not normal in N, we conclude that $|Z_2: Z_2 \cap K| = 2$. But then $H_1 < H_1(Z_2 \cap K) < Q$ and $H_1(Z_2 \cap K)$ is normal in $N_{N_1}(Q)$, a contradiction. Hence $O^2(\bar N) = \bar N$ and consequently every involution of $\bar P$ is conjugate to an element of $\bar Q$ in $\bar N$, by Thompson's lemma. Since $\bar H$ is elementary, $\bar Q$ is not strongly closed in $\bar P$ with respect to $\bar N$. However $\bar Q$ is weakly closed in $\bar P$ with respect to $\bar N$, because $\mathcal{H}'(P) = \{H_1, H_2\}$ and $Q = H_1H_2$. We can now apply Corollary 4 or 5 of [5], and conclude that $m([\bar Q, \bar t]) \leqq 1$ for each involution $\bar t$ of $\bar P - \bar Q$, whereas $|[\bar Q, \bar t]| = q$ as $\bar t$ interchanges $\bar H_1$ and $\bar H_2$. This contradiction completes the proof.

The lemma (4.40) combined with (4.35) and (4.37) has the following immediate consequences.

(4.42) *$C(x) \leqq N(P)$ for each $x \in X - \hat H_1 - \hat H_2$.*

(4.43) *$\mathcal{H}(P) = \{H_1, H_2\}$.*

We next eliminate the C-group case.

(4.44) *We have $\hat H_i \neq 1$ for each i, $1 \leqq i \leqq 2$.*

PROOF. Suppose false, then each element $\neq 1$ of X has a 2-closed centralizer by (4.42). Since $|P: Q| \leqq 2$, every involution of G is conjugate to an element of X by Thompson's lemma. Consequently, G is a C-group, contrary to our hypothesis. Hence $\hat H_i \neq 1$, $1 \leqq i \leqq 2$.

(4.45) *A path is proper if and only if its length is not larger than* 3.

PROOF. Let $(P_i; K_i; m)$ be a path. We have already shown in Section 3 that it is proper only if $m \leqq 3$. Therefore we must show that it is proper whenever $m \leqq 3$. It clearly holds if $m \leqq 2$, so assume $m = 3$. Since $\mathcal{H}(P_i) = \mathcal{H}'(P_i) = \{K_i, K_{i+1}\}$, $(K_i, K_{i+1}) \in \mathcal{K}(P_i)$, $1 \leqq i \leqq 2$. Set $Q_i = N_{P_i}(K_i) = N_{P_i}(K_{i+1})$, $1 \leqq i \leqq 2$. Then, by (4.32, 2), (4.34) and the preceding lemma, $\bigcap_{i=1}^{3} K_i = (K_1 \cap K_2) \cap (K_2 \cap K_3) = Z(Q_1) \cap Z(Q_2) = \hat K_2 \neq 1$, as desired.

(4.46) *Precisely $1 + 2(q + q^2 + q^3)$ S_2-subgroups of G have non-identity intersections with P. There is $R \in \mathcal{S}(G)$ such that $P \cap R = 1$.*

PROOF. By (3.11) and the preceding lemma, there is a one to one correspondence between the set of the S_2-subgroups $\neq P$ of G which intersect P in the non-identity elements and the set of paths of length $\leqq 3$ having P as one end. Since $|\mathcal{H}(P)| = 2$ and $N(H_i)$ has precisely $1+q$ S_2-subgroups, the number of such paths is clearly equal to $2(q + q^2 + q^3)$. Thus the first part of the lemma holds.

Since $N_{N_1}(Q)=N(P)\cap N_1$, $|N(P):N_{N_1}(Q)|=|P:Q|$. On the other hand, $|N_1:N_{N_1}(Q)|=1+q$. Consequently, $|P:Q|\cdot|G:N(P)|$ is divisible by $1+q$, and therefore $|G:N(P)|$ is divisible by $1+q$. Since $1+2(q+q^2+q^3)$ is not divisible by $1+q$, the second part of the lemma holds as well.

(4.47) *Involutions of* $X-\hat{H}_1-\hat{H}_2$ *are conjugate to each other in* G.

PROOF. In view of (4.42) and (4.46), it will suffice to prove the following (perhaps well-known) result.

(4.48) *Let* G *be a group containing* S_2*-subgroups* P *and* R *such that* $P\cap R=1$. *Then the involutions of* G *with* 2*-closed centralizers form a single conjugacy class.*

PROOF. Let $\mathcal{C}$ denote the set of the involutions of G with 2-closed centralizers. Take a from $\mathcal{C}$ so that $C_P(a)\in\mathcal{S}(C(a))$. We will show that every element b of $\mathcal{C}$ is conjugate to a in G. Replacing b by its conjugate element, if necessary, we can assume that $C_R(b)\in\mathcal{S}(C(b))$. If b is not conjugate to a, there is an involution c commuting with both a and b. Since $C(a)$ and $C(b)$ are 2-closed, we get $c\in P\cap R$, whereas $P\cap R=1$ by our assumption. Hence b is conjugate to a, q. e. d.

(4.49) *We have* $P=Q$.

PROOF. Suppose false. Then the elements of $\mathcal{Z}$ are TI-sets, by (4.24). Consequently, by (4.29) and (4.40), the centralizer of every central involution of G is 2-closed. Therefore $Z(P)\cap\hat{H}_i=1$, $1\leqq i\leqq 2$, whence every involution of $X-\hat{H}_1-\hat{H}_2$ is central in G by (4.47). But then (3.5) and (4.34, 1) yield that X is partitioned by $\hat{H}_1$, $\hat{H}_2$ and $Z(P)$. Thus, setting $|\hat{H}_i|=r$, $1\leqq i\leqq 2$, we have $|Z(P)|=qr-2r+2$. This is obviously a contradiction because $r\neq 1$ and $q>2$.

(4.50) *The following conditions hold;*

(1) $N(P)=N_1\cap N_2$;

(2) $X=\hat{H}_1\times\hat{H}_2$ *and* $|\hat{H}_i|=q$, $1\leqq i\leqq 2$;

(3) *the sets* $\hat{H}_1-\{1\}$, $\hat{H}_2-\{1\}$ *and* $X-\hat{H}_1-\hat{H}_2$ *are the conjugacy classes of* $N(P)$ *contained in* $X-\{1\}$.

PROOF. Since $|\mathcal{H}(P)|=2$, $|N(P):N(P)\cap N_i|\leqq 2$. Hence, $N(P)\leqq N_i$ by (4.49), which proves (1). As a consequence, $N(P)$ normalizes each $\hat{H}_i$. By (4.47), (4.49) and the Burnside lemma, elements of $X-\hat{H}_1-\hat{H}_2$ are conjugate to each other in $N(P)$. As $\hat{H}_1\hat{H}_2\nleqq\hat{H}_1\cup\hat{H}_2$, it follows that $X=\hat{H}_1\hat{H}_2=\hat{H}_1\times\hat{H}_2$ and therefore $|\hat{H}_i|=q$, $1\leqq i\leqq 2$. Let h_i, $h_i'\in\hat{H}_i-\{1\}$, $1\leqq i\leqq 2$, and set $x=h_1h_2$, $x'=h_1'h_2'$. Since $N(P)$ is transitive on $X-\hat{H}_1-\hat{H}_2$, we have $x'=x^n$ for some $n\in N(P)$. Since $N(P)$ also normalizes $\hat{H}_i-\{1\}$, it follows that $h_i'=h_i^n$ for each i. Hence (3) holds.

Before proceeding further, we remark here that we have already established almost all of the statements (4.1)—(4.20). Note that every involution of G is conjugate to an element of $X=Z(P)$. Hence, (4.11) is immediate from

(4.50, 3) and the Burnside lemma. Also, (4.15)—(4.18) are the consequences of (3.11), (4.45), (4.6) and (4.9). Thus it remains only to prove (4.14) and (4.19).

(4.51) *If* $h_i \in H_i - X$, $1 \leqq i \leqq 2$, *then* $[h_1, h_2] \in X - \hat{H}_1 - \hat{H}_2$.

PROOF. Since $H_i = X_i X$, there is $x_i \in X_i - X$ such that $h_i \in x_i X$. Suppose, for instance, that $[h_1, h_2] \in \hat{H}_1$, then $[x_1, x_2] \in \hat{H}_1$, whence x_2 normalizes $\langle \hat{H}_1, x_1 \rangle$. But then $\hat{H}_1 < X_1 \cap X_1^{x_2}$, contradicting (4.9). Hence the lemma holds.

(4.52) *Let* $(P_i; K_i; 4)$ *be a path of length* 4 *joining* $P_0 = P$ *and* P_4. *Then* $P \cap P_4$ *contains no elements that are conjugate to the elements of* $X - \hat{H}_1 - \hat{H}_2$.

PROOF. If $P \cap P_4 = 1$, there is nothing to prove, so assume that $P \cap P_4 \neq 1$. By (4.15), $P \neq P_4$ and therefore P and P_4 are joined by a path $(Q_i; L_i; n)$ of length $n \leqq 3$. Suppose that $n \leqq 2$. Then the uniqueness of a path of length $\leqq 3$ joining P and P_3 yields first that $K_4 \neq L_n$ and next that $K_3 = K_4$, a contradiction. Therefore $n = 3$. But then $P \cap P_4 = \hat{L}_2$ contains no elements that are conjugate to the elements of $X - \hat{H}_1 - \hat{H}_2$. The proof of (4.52) is complete.

5. We continue assuming that G is a group satisfying the conditions stated at the beginning of the preceding section, so that G satisfies (4.1)—(4.20). Furthermore, we fix the following notation throughout the section.

$P \in \mathcal{S}(G)$;

$Z = Z(P)$;

$\{H_1, H_2\} = \mathcal{H}(P)$;

$N_i = N(H_i)$, $1 \leqq i \leqq 2$;

$q = |P : H_i|$, $1 \leqq i \leqq 2$;

$\mathcal{C}$: the class of the involutions of G conjugate to the elements of $Z - \hat{H}_1 - \hat{H}_2$;

$B = N(P)$;

K: a complement of P in B;

$N = N(K)$;

$W = N/K$.

(5.1) *For each* $w \in N$, *we have* $BwB \cap N = Kw$.

PROOF. We first consider the case that $w = 1$. As $B \cap N = C_P(K)K$, we must show that $C_P(K) = 1$. Set $C = C_P(K)$. Since K acts by conjugation transitively on each of the sets $\hat{H}_1 - \{1\}$, $\hat{H}_2 - \{1\}$ and $Z - \hat{H}_1 - \hat{H}_2$, $C \cap Z = 1$. Since the square of every element of P is contained in Z, it follows that C is elementary Abelian. Hence C is contained in either H_1 or H_2. Let h be an arbitrary element of C and set $D = [h, P]$. Then D is a K-invariant subgroup of Z of order $\leqq q$. In addition, $D \cap \hat{H}_i = 1$, $1 \leqq i \leqq 2$, by (4.14). Since K is transitive on $Z - \hat{H}_1 - \hat{H}_2$ which consists of $(q-1)^2$ elements, we must have $D = 1$. Consequently $C \leqq Z$ and so $C = 1$, as required.

We next consider the general case. Let x be an element of $BwB \cap N$.

Since $BwB=BwP$, x is written as $x=bwy$ with $b\in B$ and $y\in P$. As $K, K^y \leqq B\cap B^x$, the Schur-Zassenhaus theorem applied to $B\cap B^x$ shows that $P\cap P^x$ contains an element z such that $K^y=K^z$. By the preceding paragraph, $N\cap P=1$, whence $y=z\in P\cap P^x=(P\cap P^w)^y$. We conclude that $y\in P\cap P^w$. Hence $x=by^{w^{-1}}\cdot w\in Bw$, and so $x\in Kw$ by the preceding paragraph. The proof is complete.

(5.2) *For each i, $1\leqq i\leqq 2$, N contains an involution w_i such that $N_i=B\cup Bw_iB$.*

PROOF. By symmetry, we need only consider the case that $i=1$. Set $M=O'(N_1)$. Since M/H_1 is a central extension of $PSL(2, q)$ by a group of odd order, $M\cap K$ normalizes exactly one element, say P_1, of $\mathcal{S}(N_1)-\{P\}$. It also follows from the structure of M/H_1 that N_1 has an element x such that $P^x=P_1$ and $x^2\in H_1$. Since $M\cap K$ is normal in K, K normalizes P_1 and so $K, K^x\leqq B\cap B^x$. Therefore by the Schur-Zassenhaus theorem applied to $B\cap B^x$, H_1 has an element h such that $K^{xh}=K$. Setting $w=xh$, we have that $w\in N\cap N_1$ and that $P^w=P_1$. Since N_1 acts by conjugation doubly transitively on the set $\mathcal{S}(N_1)$, the above implies that $N_1=B\cup BwB$. Furthermore, since H_1 is elementary Abelian,

$$w^2=x^2[x, h]\in H_1\cap N=H_1\cap(B\cap N)=1\,,$$

by the preceding lemma. Therefore w satisfies all the requirements.

NOTATION.

$$P_i=P^{w_i},\ 1\leqq i\leqq 2;$$

$$Z_i=Z(P_i),\ 1\leqq i\leqq 2.$$

(5.3) *We have $G=\langle B, N\rangle$.*

PROOF. Since $\mathcal{H}(P)=\{H_1, H_2\}$ and $B<N_1$, every maximal 2-local subgroup of G is conjugate to either N_1 or N_2. Furthermore, G has no strongly embedded subgroups, because G has three conjugancy classes of involutions. Hence, by (2.2) and (5.2), $G=\langle N_1, N_2\rangle=\langle B, w_1, w_2\rangle=\langle B, N\rangle$, q. e. d.

(5.4) *If $1\leqq i\leqq 2$, $1\leqq j\leqq 2$ and $i\neq j$, the following conditions hold:*

(1) $w_iBw_i\leqq B\cup Bw_iB$;

(2) $w_iBw_j=KZw_iw_jZ$;

(3) $w_iBw_iw_j\leqq Bw_jB\cup Bw_iw_jB$;

(4) $w_iBw_jw_i=KZw_iw_jw_iZ_i$;

(5) $w_iBw_iw_jw_i\leqq Bw_jw_iB\cup Bw_iw_jw_iB$;

(6) $w_iBw_jw_iw_j\leqq B(w_iw_j)^2B\cup Bw_jw_iw_jB$;

(7) $w_iB(w_iw_j)^2\leqq Bw_jw_iw_jB\cup B(w_iw_j)^2B$.

PROOF. Without loss, $i=1$ and $j=2$. (1) is an immediate consequence of (5.2). To prove (2), we note that $P=Z_1Z_2$. We have

$$\begin{aligned} w_1Bw_2 &= w_1KZ_1Z_2w_2 \\ &= Kw_1Z_1Z_2w_2 \\ &= KZw_1Z_2w_2 \\ &= KZw_1w_2Z\,, \end{aligned}$$

proving (2). The equation (4) can be verified in a similar way. (3) is an immediate consequence of (1) and (2). Indeed,

$$\begin{aligned} w_1Bw_1w_2 &\leqq (B\cup Bw_1B)w_2 \\ &\leqq Bw_2B\cup B(w_1Bw_2) \\ &\leqq Bw_2B\cup Bw_1w_2B\,. \end{aligned}$$

We can verify (5) in a similar way by using (2), (3) and (4). It remains to prove (6) and (7). Since $w_1Bw_2w_1w_2=w_1Bw_1(w_1w_2)^2$, it will suffice to prove the following:

$$N_1(w_1w_2)^2\leqq Bw_2w_1w_2B\cup B(w_1w_2)^2B\,.$$

Let $x=n(w_1w_2)^2$ be an arbitrary element of $N_1(w_1w_2)^2$, so that $n\in N_1$. Setting $H=H_1^{w_2w_1w_2}=H_1^{(w_1w_2)^2}$, we see that $P^{w_2w_1w_2}$ and $P^{(w_1w_2)^2}$ are distinct S_2-subgroups of $N(H)$. Since $P^n\leqq N(H_1)$, we also have $P^x\leqq N(H)$. If $P^x=P^{w_2w_1w_2}$, then $x\in Bw_2w_1w_2$, so we assume $P^x\neq P^{w_2w_1w_2}$. It then follows from the structure of $N(H)/H$ that $P^{w_2w_1w_2}$ contains an element y such that $P^x=P^{(w_1w_2)^2y}$. Since

$$P^{w_2w_1w_2}=(H_2H_1)^{w_2w_1w_2}=H_2^{w_1w_2}H\,,$$

we can assume that $y\in H_2^{w_1w_2}$. We can then assume that $y\in \hat{H}_1^{w_2}$, because

$$H_2^{w_1w_2}=(\hat{H}_1Z_2)^{w_1w_2}=\hat{H}_1^{w_2}Z^{w_2w_1w_2}\leqq \hat{H}_1^{w_2}H\,.$$

But $\hat{H}_1^{w_2}\leqq Z^{w_2}=Z_2\leqq B$ whence $y\in B$. Hence $x\in B(w_1w_2)^2B$, as required.

(5.5) *Let x be an element of G. Then $|P\cap P^x|=q^i$, $0\leqq i\leqq 4$. Moreover the following conditions hold:*

(1) *if $|P\cap P^x|=q^3$, then $x\in Bw_1B\cup Bw_2B$;*

(2) *if $|P\cap P^x|=q^2$, then $x\in Bw_2w_1B\cup Bw_1w_2B$;*

(3) *if $|P\cap P^x|=q$, then $x\in Bw_1w_2w_1B\cup Bw_2w_1w_2B$.*

PROOF. Assume that $P\neq P^x$ and $P\cap P^x\neq 1$. Then, by (4.16)—(4.18), P and P^x are joined by a path $(Q_i\,;K_i\,;k)$ of length k $(\leqq 3)$ and $|P\cap P^x|=q^{4-k}$. We can assume without loss that $K_1=H_1$.

Assume first that $k=1$. Then $P\neq P^x\leqq N_1$ and so B contains an element b such that $P^x=P_1^b=P^{w_1b}$. Hence $x\in Bw_1B$ in this case.

Assume next that $k=2$. There is an element b of B such that $Q_1^b=P_1$ and so, replacing x by xb, we can assume that $Q_1=P_1$. Then $K_2=H_2^{w_1}$ and hence $N(K_2)=(B\cup Bw_2B)^{w_1}$. Since $B^{w_1}=N(P_1)$, there is an element b of B

such that $P^x = P_1^{w_1(w_2b)w_1} = P^{w_2bw_1}$. Consequently, $x \in Bw_2Bw_1 \leqq Bw_2w_1B$ by (5.4, 2).

Assume finally that $k=3$. As was shown above, B contains an element b such that $Q_2^b = P^{w_2w_1}$. Therefore, replacing x by xb, we can assume that $Q_2 = P^{w_2w_1}$. Then the uniqueness of the path of length 2 between P and Q_2, (4.15), yields that $Q_1 = P_1$ and that $K_2 = H_2^{w_1} = H_2^{w_2w_1}$. Consequently, $K_3 = H_1^{w_2w_1}$ and hence $N(K_3) = (B \cup Bw_1B)^{w_2w_1}$. Since $B^{w_2w_1} = N(Q_2)$, B has an element b such that $P^x = Q_2^{w_1w_2w_1bw_2w_1} = P^{w_1bw_2w_1}$. Hence $x \in Bw_1Bw_2w_1 \leqq Bw_1w_2w_1B$ by (5.4, 4). The proof is complete.

(5.6) *If $1 \leqq i \leqq 2$, $1 \leqq j \leqq 2$ and $i \neq j$, the following conditions hold:*

(1) $P \cap P^{w_i} = H_i$;

(2) $P \cap P^{w_iw_j} = Z_j$;

(3) $P \cap P^{w_iw_jw_i} = Z_i \cap Z_j^{w_i} = \hat{H}_j^{w_i}$;

(4) $P \cap P^{(w_iw_j)^2} = 1$.

PROOF. Without loss $i=1$ and $j=2$. (1) is immediate from the definition of w_i. Since $(P, P_2, P^{w_1w_2}; H_2, H_1^{w_2}; 2)$ is a path joining P and $P^{w_1w_2}$, (2) is a consequence of (4.17). (3) can be proved in a similar way. It follows from (2) and (3) that $Kw_1w_2 \neq Kw_2w_1$ and that $Kw_1w_2w_1 \neq Kw_2w_1w_2$. Therefore the coset $K(w_1w_2)^2$ differs from any of the following seven cosets:

$$K,\ Kw_1,\ Kw_2,\ Kw_1w_2,\ Kw_2w_1,\ Kw_1w_2w_1,\ Kw_2w_1w_2\,.$$

Hence (4) is an immediate consequence of (5.1) and (5.5).

(5.7) *Let u be an involution of G such that $P \cap P^u \neq 1$. Then either $u \in P$ or u is contained in an S_2-subgroup of G which is joined to P by a path of length $\leqq 2$.*

PROOF. If $P = P^u$, then $u \in P$. Otherwise, there is a path $(Q_i; K_i; k)$ of length k $(\leqq 3)$ joining P and P^u. Since u is an involution, $(Q_i^u; K_i^u; k)$ is also a path of the same length joining P and P^u. Now the uniqueness of the path yields that $u \in N(K_1)$, $u \in N(Q_1)$ or $u \in N(K_2)$ according as $k=1$, $k=2$ or $k=3$. Hence (5.7) holds.

NOTATION. $\mathcal{C}(P)$ will denote the set of the elements of $\mathcal{C}$ satisfying the conclusion of (5.7).

(5.8) *Let x, y be distinct elements of $\mathcal{C}$ and let $z \in Z - \hat{H}_1 - \hat{H}_2$. If $xy \in C(z)$, then $x, y \in \mathcal{C}(P)$.*

PROOF. Set $u = xy$. When u has even order, let v be an involution of $\langle u \rangle$. Otherwise, let v be an involution of $\langle zx \rangle$. Such v exists, because $z \in C(u)$ but $x \in C^*(u) - C(u)$ when u has odd order. In either case, $x \in C(v)$ and $v \in P$ by (4.12). If $v \in Z$, then $x \in C(v) \leqq N_1$ or N_2 by (4.8), (4.12) and (4.13), and the assertion holds in this case. If v is not contained in Z, then there is an S_2-subgroup Q $(\neq P)$ of N_i, $i=1$ or 2, such that $v \in Z(Q)$. Hence $x \in C(v) \leqq N_i$ or $N(H)$ where $H_i \neq H \in \mathcal{H}(Q)$, and the assertion holds in this case as well.

(5.9) *We have* $|\mathcal{C}(P)|=(q-1)^2(1+2(q+q^2+q^3))$.

PROOF. We devide $\mathcal{C}(P)$ into three subsets; that is

$\mathcal{C}_0(P)=\mathcal{C}\cap P$;

$\mathcal{C}_1(P)$: the set of the elements of $\mathcal{C}(P)-\mathcal{C}_0(P)$ contained in an S_2-subgroup of G which is joined to P by a path of length 1;

$\mathcal{C}_2(P)=\mathcal{C}(P)-\mathcal{C}_0(P)-\mathcal{C}_1(P)$.

Let H be an element of $\mathscr{H}$, then $\mathcal{C}\cap H$ is the direct union of the sets $\mathcal{C}\cap Z(Q)$, Q ranging over all S_2-subgroups of $N(H)$, whence $|\mathcal{C}\cap H|=(q+1)(q-1)^2$. Hence

$$|\mathcal{C}_0(P)|=2(q+1)(q-1)^2-(q-1)^2=(2q+1)(q-1)^2 .$$

If Q, R are distinct S_2-subgroups of G joined to P by a path of length 1, then $Q\cap R\leqq P$ by (4.17). Hence $\mathcal{C}_1(P)$ is the direct union of the sets $\mathcal{C}\cap(Q-P)$, Q ranging over all S_2-subgroups joined to P by a path of length 1. Thus $|\mathcal{C}_1(P)|=2q^2(q-1)^2$. Let Q be an S_2-subgroup of G joined to P by a path $(Q_i; K_i; 2)$ of length 2. Then $\mathcal{C}_2(P)\cap Q=\mathcal{C}\cap(Q-K_2)$ by (4.17) and (4.18), whence $|\mathcal{C}_2(P)\cap Q|=q(q-1)^2$. Furthermore if R is another S_2-subgroup of G joined to P by a path of length 2, then either $Q\cap R\leqq K_2$ or $Q\cap R\cap\mathcal{C}$ is empty by (4.17)—(4.19). Therefore $\mathcal{C}_2(P)$ is the direct union of the sets $\mathcal{C}_2(P)\cap Q$ where Q ranges over all such S_2-subgroups. Thus $|\mathcal{C}_2(P)|=2q^3(q-1)^2$. Hence (5.9) holds.

(5.10) *Let Q be an S_2-subgroup of G such that $P\cap Q=1$. Then there is an element u of $\mathcal{C}-\mathcal{C}(P)$ such that $Q=P^u$.*

PROOF. Let u be an element of $\mathcal{C}-\mathcal{C}(P)$. Then, by (5.7) and the definition of $\mathcal{C}(P)$, $P\cap P^u=1$. Let z be an element of $Z-\hat{H}_1-\hat{H}_2$, then $|B:C(z)|=(q-1)^2$ and so each coset of B is a union of $(q-1)^2$ distinct cosets of $C(z)$. By (5.8), each coset of $C(z)$ contains at most one element of $\mathcal{C}-\mathcal{C}(P)$, whence Bu contains at most $(q-1)^2$ elements of $\mathcal{C}-\mathcal{C}(P)$. Consequently, the set $\{P^u; u\in\mathcal{C}-\mathcal{C}(P)\}$ contains at least $|\mathcal{C}-\mathcal{C}(P)|/(q-1)^2$ distinct elements. By the preceding lemma,

$$|\mathcal{C}-\mathcal{C}(P)|/(q-1)^2=|G:B|-(1+2(q+q^2+q^3))$$

which is equal to the number of S_2-subgroups Q such that $P\cap Q=1$ by (4.20). Hence the assertion holds.

(5.11) *The following conditions hold:*

(1) *W is a dihedral group of order 8 and consists of eight cosets K, Kw_1, Kw_2, Kw_1w_2, Kw_2w_1, $Kw_1w_2w_1$, $Kw_2w_1w_2$ and $K(w_1w_2)^2$;*

(2) *(B, N) is a (B, N)-pair of G.*

PROOF. Let x be an element of N. If $P\cap P^x\neq 1$, then x is contained in one of the above cosets except $K(w_1w_2)^2$ by (5.1) and (5.5). If $P\cap P^x=1$, then by (5.10), there is an element u of $\mathcal{C}-\mathcal{C}(P)$ such that $P^x=P^u$. Since $P\cap P^x=1$, $B\cap B^u=K$ and therefore $u\in N(K)=N$, whence $u\in Kx$. Consequently, if $x\in N$ and $P\cap P^x=1$ then Kx is an involution of W. Hence W contains precisely two elements of order larger that two. Since W is a non-Abelian group of order at least 8 by (5.6), it is easily seen that W is a dihedral group of order 8 and then that W consists of the eight cosets listed above. In particular $K(w_1w_2)^2=K(w_2w_1)^2$, and hence (B, N) satisfies all the axioms of a (B, N)-pair.

We have arrived at the goal.

(5.12) *$O'(G)$ is isomorphic to $PSp(4, q)$.*

PROOF. We have $G=BNB$ by (5.11). Furthermore

$$N_1w_2N_1=Bw_2B\cup Bw_1w_2B\cup Bw_2w_1B\cup Bw_1w_2w_1B$$

and

$$N_1w_2w_1w_2N_1=Bw_2w_1w_2B\cup B(w_1w_2)^2B\,.$$

Therefore

$$G=N_1\cup N_1w_2N_1\cup N_1w_2w_1w_2N_1\,.$$

Hence, when represented as a permutation group on the set Ω of the cosets of N_1, G is a transitive rank 3 group. Since $|N_1:B|=1+q$, each coset of N_1 contains precisely $1+q$ cosets of B. Hence, by using (5.6), we can compute the subdegrees of (G, Ω). They are 1, $(1+q)q$ and q^3. Since $H_2K\leqq N_1\cap N_1^{w_2}$ and $|N_1:H_2K|=(1+q)q$, $N_1\cap N_1^{w_2}\leqq H_2K\leqq B$. Hence $\bigcap_{n\in N}N^n\leqq\bigcap_{n\in N}B^n=K$ by (5.6, 4). Consequently,

$$\bigcap_{x\in G}N^x\leqq\bigcap_{x\in G}K^x\leqq O(G)=1\,,$$

which implies that the representation of G on Ω is faithful. We have

$$N_1w_2N_1=N_1w_2B\cup N_1w_2w_1B=N_1w_2Z_1\cup N_1w_2w_1Z_2\,.$$

Hence $\hat{H}_1=Z\cap Z_1$ fixes every coset of N_1 contained in $N_1w_2N_1$. Therefore by the theorem of D. G. Higman [**10**], G contains a normal subgroup G_0 isomorphic to $PSp(4, q)$. Since $|P|=q^4$, $|G:G_0|$ is odd and therefore $G_0=O'(G)$.

This completes the proof of Theorem B.

6. For certain reasons, we may consider the structure of maximal 2-local subgroups to be a local structure of the simple groups of Lie type defined over the fields of characteristic 2 (for brevity call them groups of characteristic 2) which best reflects their global structure. Therefore we may propose the problem of characterizing groups of characteristic 2 by the structure of their maximal 2-local subgroups, and view Theorem A from the standpoint of this problem. As was remarked by Gorenstein [7], the groups of characteristic 2 are expected to be almost characterized by the property that each their maximal 2-local subgroup M satisfies the condition (1) of Hypothesis A. Therefore a natural formulation of the problem will be as follows: assume the condition (1) for each maximal 2-local subgroup M of a simple group G and assume that $M/O_2(M)$ is isomorphic to $M^*/O_2(M^*)$ (or its odd extension) for some maximal 2-local subgroup M^* of a group G^* of characteristic 2, and then prove that G is actually isomorphic to G^*. For instance, we have the following problem in connection with groups of characteristic 2 and Lie rank 2:

SUBPROBLEM. Classify the groups in which each maximal 2-local subgroup M which is not 2-closed satisfies the conditions (1) and (2) of Hypothesis A.

An answer to this problem would of course contain Theorem A as a special case.

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