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A correction to "Theta series and automorphic forms on GL_2"

By Hideo SHIMIZU

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The proof of Lemma 2 in [2] is not correct if the base field F is of characteristic 2. We give here a proof which is valid for any characteristic. Rather than proving the lemma in its original form, it is more convenient to prove it for every irreducible admissible representation (not necessarily pre-unitary).

1. We keep the notation in [2]. Let π be an irreducible admissible representation of $\mathcal{H}(\mathcal{K}_A^\times)$ in a vector space $\mathcal{V}$. By [**1**, Proposition 9.1] π is the tensor product of irreducible admissible representations π_v of $\mathcal{H}(\mathcal{K}_v^\times)$ in the space $\mathcal{V}_v$. Let $\mathfrak{d}=\otimes\mathfrak{d}_v$ be an irreducible representation of K^1, $\mathfrak{d}_v$ being an irreducible representation of K_v^1. Almost all $\mathfrak{d}_v$ are the identity representations. We define $\mathcal{V}(\mathfrak{d})$ or $\mathcal{V}_v(\mathfrak{d}_v)$ in the same way as in [**2**, §3, No. 3].

LEMMA. *$\mathcal{V}_v(\mathfrak{d}_v)$ is finite dimensional. Moreover, if the restriction of π_v to K_v contains the identity representation and if $\mathfrak{d}_v=1$, then $\mathcal{V}_v(\mathfrak{d}_v)$ is one-dimensional.*

PROOF. If $\mathcal{K}_v$ is a division algebra, the first assertion is trivial since π_v itself is finite dimensional, and the second assertion follows from the fact that if π_v contains the identity representation of K_v, then π_v is one-dimensional.

Let us assume that $\mathcal{K}_v^\times=GL_2(F_v)$.

i) π_v is a representation of principal series (special representations are included). In this case π_v is realized in a subspace of some $\mathcal{B}(\mu_1, \mu_2)$. Since $GL_2(F_v)=TK_v^1$, a function in $\mathcal{B}(\mu_1, \mu_2)$ is determined by its restriction to K_v^1. A function φ on K_v^1 which transforms according to $\mathfrak{d}_v$ by $\varphi(g)\rightarrow\varphi(gk)$ ($k\in K_v^1$) is a linear combination of the coefficients of $\mathfrak{d}_v$. Hence they form a finite dimensional space. This proves the first assertion. The second assertion is obvious.

ii) π_v is absolutely cuspidal (F_v is a non-archimedean local field). Let the notation be the same as in [2, §5, No. 6], and assume that Ψ is a character of F_v of conductor $\mathfrak{o}_v$. Put

$$H_m=\{k\in K_v^1 \mid k\equiv 1 \pmod{\mathfrak{p}^m}\}\,.$$

It is enough to prove that the space $\mathcal{V}_m$ of all H_m-invariant functions in

$\mathcal{S}(F_v^\times)$ is finite dimensional. Put $\varphi'=\pi_v(w)\varphi$ for $\varphi\in\mathcal{V}_m$. Since H_m is normal in K_v^1, φ' is again in $\mathcal{V}_m$. As in the proof of [2, Lemma 14], we see immediately that the supports of φ and φ' are contained in $\mathfrak{p}^{-m}$. Furthermore, if we write

$$a_n(\nu)=\int_{\mathfrak{o}_v^\times}\varphi'(\varpi^n\varepsilon)\nu(\varepsilon)d\varepsilon$$

$$b_n(\nu)=\eta_v(\varpi)^n\int_{\mathfrak{o}_v^\times}\varphi(\varpi^{-n}\varepsilon)\nu^{-1}\nu_0^{-1}(\varepsilon)d\varepsilon\,,$$

we obtain

$$\sum_{n=-\infty}^{\infty}t^n a_n(\nu)=\Big(\sum_{n=-\infty}^{\infty}t^n b_n(\nu)\Big)\Big(\sum_{n=-\infty}^{-2}t^n C_n(\nu)\Big)$$

for all $t\in\boldsymbol{C}$ with $|t|=1$ and for all characters ν of $\mathfrak{o}_v^\times$; $a_n(\nu)=0$ if $n<-m$ and $b_n(\nu)=0$ if $n>m$. Hence $a_n(\nu)=0$ if $n>m-2$. It implies that the support of φ' is contained in $\mathfrak{p}^{-m}-\mathfrak{p}^{m-1}$. Interchanging the role of φ and φ', we see that the same is true for φ.

Let ν be a character of $\mathfrak{o}_v^\times$ and $M=M(\nu)$ the smallest integer such that $\nu(1+\mathfrak{p}^M)=1$. By [1, Proposition 2.16.6], if M is large enough, then $C_n(\nu)\neq 0$ if and only if $n=-2M$. For such a ν, we have

$$a_{n-2M}(\nu)=b_n(\nu)C_{-2M}(\nu)$$

and hence $b_n(\nu)=0$ if $n-2M<-m$ i.e. if $n<2M-m$. Hence, if $2M-m>m$, then $b_n(\nu)=0$ for all n. We see that there is an integer M_0 such that $b_n(\nu)=0$ for all n and for all ν with $M(\nu)>M_0$. It follows that $\varphi(\varpi^{-n}\varepsilon)$ is, as a function of ε, a linear combination of $\nu\nu_0(\varepsilon)$ with $M(\nu)\leqq M_0$. Evidently such functions with support in $\mathfrak{p}^{-m}-\mathfrak{p}^{m-1}$ form a finite dimensional space. This proves the lemma. (Note that no absolutely cuspidal representation contains the identity representation of K_v.)

2. Since $\mathcal{V}(\mathfrak{d})=\otimes\mathcal{V}_v(\mathfrak{d}_v)$, the lemma shows that $\mathcal{V}(\mathfrak{d})$ is finite dimensional. If $\mathcal{V}$ is the space of K-finite functions in an irreducible closed subspace $\mathcal{L}$ of $L_0^2(\eta,\mathcal{K}_A^\times)$, we have $\mathcal{L}(\mathfrak{d})=\mathcal{V}(\mathfrak{d})$. In fact, for every $\varphi\in\mathcal{L}$ and for every $\varepsilon>0$, there exists a $\varphi'\in\mathcal{V}$ such that $\|\varphi-\varphi'\|<\varepsilon$. Then

$$\|E(\mathfrak{d})\varphi-E(\mathfrak{d})\varphi'\|<\varepsilon\int_{K^1}|\chi_{\mathfrak{d}}|\,dk$$

and hence $\mathcal{L}(\mathfrak{d})$ is the closure of $\mathcal{V}(\mathfrak{d})$. Since $\mathcal{V}(\mathfrak{d})$ is finite dimensional and closed, the both spaces are the same. This completes the proof of [2, Lemma 2].

References

[1] H. Jacquet and R. P. Langlands, Automorphic forms on $GL(2)$, Lecture notes in Mathematics, No. 114, Springer, 1970.
[2] H. Shimizu, Theta series and automorphic forms on GL_2, J. Math. Soc. Japan, **24** (1972), 638-683.

Hideo SHIMIZU
Department of Mathematics
College of General Education
University of Tokyo
Komaba, Meguro-ku
Tokyo, Japan