J. Math. Soc. Japan
Vol. 26, No. 2, 1974

Ramanujan's formulas for L-functions

(To the memory of Professor Sigekatu Kuroda)

By Koji KATAYAMA

(Received Aug. 24, 1972)
(Revised May 9, 1973)

Let $\zeta(s)$ be the Riemann's zeta-function. It is famous that $\zeta(2\nu)$, $0<\nu\in \boldsymbol{Z}$, is represented in terms of Bernoulli number and $\pi^{2\nu}$ and so is rational up to $\pi^{2\nu}$. But the numerical nature of $\zeta(2\nu+1)$, $\nu\geqq 1$, has long been unknown. As far as the author knows, only Ramanujan's formula*) is one involving $\zeta(2\nu+1)$.

Let χ be a non-principal primitive character mod k and $L(s, \chi)$ a Dirichlet L-function associated with χ. Then it is known that $L(2\nu, \chi)$, $\nu\geqq 1$, for even χ and $L(2\nu+1, \chi)$, $\nu\geqq 1$, for odd χ are represented by the generalized Bernoulli numbers in the sense of Leopoldt up to $\pi^{2\nu}$ and $\pi^{2\nu+1}$, respectively**). Analogously to the case of $\zeta(s)$, the numerical properties of $L(2\nu+1, \chi)$ for even χ and of $L(2\nu, \chi)$ for odd χ are unknown. Thus we are naturally led to ask "Ramanujan's formulas" for these values.

Now the purpose of the present paper is to formulate and prove "Ramanujan's formulas" for L-functions. Put

$$T_\chi=\sum_{h=0}^{k-1}\chi(h)e^{2\pi ih/k}.$$

Then for any $n>0$, we have

$$\chi(n)T_{\bar{\chi}}=\sum_{h=0}^{k-1}\bar{\chi}(h)e^{2\pi inh/k}. \tag{0}$$

We define, for $0<a\in \boldsymbol{Z}$ and for $x>0$,

$$F_1(a, x, \chi)=\sum_{m=1}^{\infty}\frac{\chi(m)}{m^a}\frac{1}{e^{2\pi mx}-1}$$

and

$$F_2(a, x, \chi)=\sum_{h=0}^{k-1}\bar{\chi}(h)\sum_{n=1}^{\infty}\frac{1}{n^a}\frac{e^{2\pi nxh/k}}{e^{2\pi nx}-1}.$$

Then our formulas are formulated as follows:

---

*) See for example [2].

**) The value $L(1, \chi)$ with odd or even χ is given in finite type at p. 336 of Borevich and Shafarevich's book "Number Theory, Academic Press, (1966)".

(1) χ (non-principal): even, $\nu \geqq 0$;

$$\frac{1}{2} L(2\nu+1, \chi)+F_1(2\nu+1, x, \chi)-(-x^2)^{\nu} T_{\bar{\chi}}^{-1} F_2(2\nu+1, x^{-1}, \chi)$$
$$=\pi^{-1} \sum_{j=-1}^{\nu-1}(-1)^j x^{2j+1} \zeta(2j+2) L(2\nu-2j, \chi),$$

(2) χ: odd, $\nu \geqq 1$;

$$\frac{1}{2} L(2\nu, \chi)+F_1(2\nu, x, \chi)-(ix)^{2\nu-1} T_{\bar{\chi}}^{-1} F_2(2\nu, x^{-1}, \chi)$$
$$=\pi^{-1} \sum_{j=-1}^{\nu-1}(-1)^j x^{2j+1} \zeta(2j+2) L(2\nu-1-2j, \chi).$$

Note that the formula (1), taken χ to be principal, does not give Ramanujan's formula for $\zeta(2\nu+1)$ because of the orthogonality of χ.

In his letter dated October 4, 1972, Professor C. L. Siegel kindly showed the author a simple proof of Ramanujan's formula for $\zeta(2\nu+1)$, $\nu \geqq 1$, based on the following two identities:

$$\frac{(n^2)^{\nu}-(-m^2x^2)^{\nu}}{n^2-(-m^2x^2)}=(n^2)^{\nu-1}+(n^2)^{\nu-2}(-m^2x^2)+\cdots+(-m^2x^2)^{\nu-1}, \tag{3}$$

$$\sum_{n=1}^{\infty} \frac{1}{x^2+n^2}=\frac{\pi}{x} \frac{1}{e^{2\pi x}-1}+\frac{\pi}{2x}-\frac{1}{2x^2}. \tag{4}$$

Also he suggested the author that his method will work well in proving (1). It was an easy matter to derive (1) and (2) following him. The author expresses here his hearty thanks to Professor Siegel for his kind suggestion.

We shall prove (1) and (2) following Siegel's idea***). Then besides (4), we need the following:

$$\frac{e^{2\pi ux}}{e^{2\pi x}-1}=\frac{1}{2\pi x}-\frac{1}{2\pi i} \sum_{n=1}^{\infty}\left\{\frac{e^{2\pi inu}}{xi+n}+\frac{e^{-2\pi inu}}{xi-n}\right\}, \tag{5}$$
$$0<u<1.$$

A proof of this (and also of (4)) can be seen in C. L. Siegel [**3**], pp. 35-36.

LEMMA 1.

$$F_2(a, x, \chi)=\begin{cases} \dfrac{T_{\bar{\chi}} x}{\pi} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{a-1}} \sum_{m=1}^{\infty} \frac{\chi(m)}{n^2x^2+m^2} & \textit{for even } \chi, \\ -\dfrac{T_{\bar{\chi}}}{\pi i} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^a} \sum_{m=1}^{\infty} \frac{m\chi(m)}{n^2x^2+m^2} & \textit{for odd } \chi. \end{cases}$$

Proof is by the straight-forward calculation; namely, we have, by (0), (5) and the orthogonality of χ,

***) After reading the first version of the present paper, my student Miss Amenomori got a proof of (1) and (2), by the function theoretic method as was employed by Grosswald in [**1**].

$$F_2(a, x, \chi) = -\frac{T_{\bar{\chi}}}{2\pi i}\sum_{n=1}^{\infty}\frac{1}{n^a}\sum_{m=1}^{\infty}\left\{\frac{\chi(m)}{nxi+m}+\frac{\chi(-m)}{nxi-m}\right\}$$

If χ is even, i. e., $\chi(m)=\chi(-m)$, the last sum is equal to

$$\sum_{m=1}^{\infty}\frac{2nxi\chi(m)}{-n^2x^2-m^2}$$

and if χ is odd, i. e., $\chi(-m)=-\chi(m)$,

$$\sum_{m=1}^{\infty}\frac{-2m\chi(m)}{-n^2x^2-m^2}.$$

Thus we get the Lemma.

We have by the definition and by (4),

$$(6)\qquad \frac{-1}{2xi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\chi(m)}{m^{a+1}}\left\{\frac{1}{mxi+n}+\frac{1}{mxi-n}\right\}$$
$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\chi(m)}{m^a}\frac{1}{m^2x^2+n^2}$$
$$=\frac{\pi}{x}F_1(a+1, x, \chi)+\frac{\pi}{2x}L(a+1, \chi)-\frac{1}{2x^2}L(a+2, \chi).$$

It is necessary to prove the order of m, n of the first double sum can be changed. For $a\geqq 2$, the sum is absolutely convergent and we can change the order of m, n of the sum.

LEMMA 2. *Let u, v be real numbers such that $0<u<1$, $0\leqq v<1$. Then the order of m, n of the sum*

$$(7)\qquad \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\frac{\chi(m)e^{2\pi i(mv+nu)}}{m(mxi+n)}$$

can be changed.

PROOF. Our proof goes on an analogous way to Siegel's estimation for a certain double series in [3], pp. 31-32.

We consider the following partial sum

$$(8)\qquad \sum_{m=m_1}^{m_2}\frac{\chi(m)e^{2\pi imv}}{m}\sum_{n=n_1}^{n_2}\frac{e^{2\pi inu}}{mxi+n}.$$

First take $n_1>0$. Putting

$$c_k=\frac{e^{2\pi iku}}{e^{2\pi iu}-1}\qquad \text{for any } k\in \boldsymbol{Z},$$

we have

$$e^{2\pi inu}=c_{n+1}-c_n$$

and

$$|c_n|=|1-e^{2\pi iu}|^{-1}=\alpha,$$

which is independent of n. Then

$$(8)=\sum_{m=m_1}^{m_2}\frac{\chi(m)e^{2\pi imv}}{m}\sum_{n=n_1}^{n_2}\frac{c_{n+1}-c_n}{mxi+n}$$

$$=\sum_{m=m_1}^{m_2}\frac{\chi(m)e^{2\pi imv}}{m}\left\{\sum_{n=n_1+1}^{n_2}c_n\left(\frac{1}{mxi+n-1}-\frac{1}{mxi+n}\right)\right.$$

$$\left.+\frac{c_{n_2+1}}{n_2+mxi}-\frac{c_{n_1}}{n_1+mxi}\right\}.$$

Here

$$\left|\frac{c_{n_2+1}}{n_2+mxi}\right|\leqq\frac{\alpha}{n_2}$$

and

$$(8')\qquad \left|\frac{c_{n_1}}{n_1+mxi}\right|\leqq\frac{\frac{\alpha}{\sqrt{2}}}{(mn_1x)^{1/2}}.$$

Further, we have

$$\left|\frac{1}{mxi+n-1}-\frac{1}{mxi+n}\right|=\left|\int_{n-1}^{n}\frac{d\mu}{(\mu+mxi)^2}\right|$$

$$\leqq\int_{n-1}^{n}\frac{d\mu}{\mu^2+m^2x^2}.$$

Thus

$$\sum_{n=n_1}^{n_2}\frac{c_{n+1}-c_n}{mxi+n}=\sum_{n=n_1+1}^{n_2}c_n\left(\frac{1}{mxi+n-1}-\frac{1}{mxi+n}\right)$$

$$+\frac{c_{n_2+1}}{n_2+mxi}-\frac{c_{n_1}}{n_1+mxi}$$

is majorized by

$$\frac{\frac{\alpha}{\sqrt{2}}}{(mxn_1)^{1/2}}+\frac{\alpha}{n_2}+\alpha\sum_{n=n_1+1}^{n_2}\int_{n-1}^{n}\frac{d\mu}{\mu^2+m^2x^2}.$$

Letting $n_2\to\infty$, we majorize this by

$$\frac{\frac{\alpha}{\sqrt{2}}}{(mxn_1)^{1/2}}+\alpha\int_{n_1}^{\infty}\frac{d\mu}{\mu^2+m^2x^2}=\frac{\frac{\alpha}{\sqrt{2}}}{(mxn_1)^{1/2}}+\frac{\alpha}{mx}\left[\arctan\frac{mx}{\mu}\right]_{n_1}^{\infty}.$$

Since arc tan is a monotone increasing function, $-\frac{1}{2}\pi<\arctan x<\frac{1}{2}\pi$ and $\arctan x<x$ for $x>0$, we see that

$$\left[\arctan\frac{mx}{\mu}\right]_{n_1}^{\infty}<\mathrm{Min}\left(\frac{1}{2}\pi,\frac{mx}{n_1}\right)$$

$$<\sqrt{\frac{1}{2}\pi\frac{mx}{n_1}}.$$

Thus we get

$$\frac{\alpha}{x}\frac{1}{m}\Big[\arc\tan\frac{mx}{\mu}\Big]_{n_1}^{\infty}<\frac{\alpha}{x}\frac{1}{m}\sqrt{\frac{\pi}{2}\frac{mx}{n_1}}=\frac{\sqrt{\pi}\,\alpha}{\sqrt{2x}}\frac{1}{m^{1/2}}\frac{1}{n_1^{1/2}}.$$

For $-n_2<-n_i<0$, we have the same estimation for the partial sum (8). Then letting $m_1=1$, $m_2\to\infty$, we see that

$$(7)-\sum_{m=1}^{\infty}\sum_{-n_1}^{n_1}\frac{\chi(m)e^{2\pi i(mv+nu)}}{m(mxi+n)}$$
$$<\beta(\sum_{m=1}^{\infty}m^{-3/2})|n_1|^{-1/2}.$$

with some constant β. The left hand side of this is equal to

$$(7)-\sum_{n=-n_1}^{n_1}\sum_{m=1}^{\infty}(***).$$

Tending n_1 to ∞, we get our Lemma.

In the above proof, we replace m by $|m|$ in the formula (8) and in what follows and also replace $\sum_{m=1}$ by $\sum_{m=m_1}$. Letting $m_1\to-\infty$, $m_2\to+\infty$, we have the following

COROLLARY 1. *The order of* m, n *in the sum*

$$\sum_{m=-\infty}^{\infty}{}'\sum_{n=-\infty}^{\infty}\frac{\chi(m)e^{2\pi i(mv+nu)}}{m(mxi+n)}$$

with $0<u<1$, $0\leqq v<1$, *can be changed.*

Also in the same way as above, we have

COROLLARY 2. *The order of* m, n *in the sum*

$$\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\frac{\chi(m)e^{2\pi i(mv+nu)}}{m^2(mxi+n)}$$

with $0<u<1$, $0\leqq v<1$, *can be changed.*

We prove our formulas (1), (2) only on the basis of Lemma 2 and Corollary 2. Then as a result (start with the sum over m, n in this order in the proofs of (1) and (2) below), we have the following

COROLLARY 3. *Let* χ *be a non-principal character. The order of* m, n *of the sums in Lemma* 2 *and Corollary* 2 *can be changed for* $u=0$.

(i) Proof of (1). Here χ is assumed to be even. Multiply both hands of (3) by $\chi(m)(mn)^{-2\nu}$ and sum over all positive integers n, m in this order. Then we have

$$(9)\qquad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\chi(m)}{m^{2\nu}(m^2x^2+n^2)}-(ix)^{2\nu}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\chi(m)}{n^{2\nu}(n^2+m^2x^2)}$$
$$=L(2\nu,\chi)\zeta(2)+(-x^2)L(2\nu-2,\chi)\zeta(4)+\cdots+(-x^2)^{\nu-1}L(2,\chi)\zeta(2\nu).$$

The second double sum of the left hand side is equal to

$$\frac{1}{x^2 T_{\bar{\chi}}} \sum_{h=0}^{k-1} \bar{\chi}(h) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{e^{2\pi imh/k}}{n^{2\nu}(n^2/x^2+m^2)}.$$

By Lemma 2 (m, n, u in the Lemma are now $n, m, h/k$) and the remark mentioned just before Lemma 2, we can change the order of m, n of the above sum. Then by Lemma 1, the second sum of (9) is equal to

$$T_{\bar{\chi}}^{-1} \pi x F_2(2\nu+1, x^{-1}, \chi).$$

By (6), the first sum of (9) is equal to

$$\frac{\pi}{x} F_1(2\nu+1, x, \chi) + \frac{\pi}{2x} L(2\nu+1, \chi) - \frac{1}{2x^2} L(2\nu+2, \chi).$$

Putting these in (9), we have

$$\frac{\pi}{x} F_1(2\nu+1, x, \chi) + \frac{\pi}{2x} L(2\nu+1, \chi) - \frac{1}{2x^2} L(2\nu+2, \chi)$$
$$+ \frac{(-x^2)^{\nu-1}\pi x}{T_{\bar{\chi}}} F_2(2\nu+1, x^{-1}, \chi)$$
$$= \sum_{j=0}^{\nu-1} (-x^2)^j \zeta(2j+2) L(2\nu-2j, \chi).$$

For the case $\nu=0$, the last sum is not necessary. Using $\zeta(0) = -\frac{1}{2}$, we have the formula (1).

(ii) Proof of (2). In this case, χ is assumed to be odd. Multiplying both sides of (3) by $\chi(m)m(mn)^{-2\nu}$ and sum over all positive integers n, m in this order. Then we have

$$(10) \qquad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\chi(m)}{m^{2\nu-1}} \frac{1}{n^2+m^2x^2} - \frac{(xi)^{2\nu}}{x^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\chi(m)}{n^{2\nu}} \frac{m}{m^2+(n^2/x^2)}$$
$$= \sum_{j=0}^{\nu-1} (xi)^{2j} \zeta(2j+2) L(2\nu-1-2j, \chi).$$

The second sum for $\nu=1$ is equal to

$$T_{\bar{\chi}}^{-1} \sum_{h=0}^{k-1} \bar{\chi}(h) \sum_{m=1}^{\infty} \frac{1}{m^2} \sum_{n=-\infty}^{\infty}{}' \frac{e^{2\pi inh/k}}{m+(ni/x)}.$$

Note that the term corresponding to $h=0$ vanishes. By Corollary 2 to Lemma 2 (m, n, u in the Lemma are now $n, m, h/k$), we can change the order of m, n in this sum. Then returning to general ν, by Lemma 1, we see that the second sum of (10) is equal to

$$-\frac{\pi i}{T_{\bar{\chi}}} F_2(2\nu, x^{-1}, \chi).$$

Further by (6), the first sum of (10) is equal to

$$\frac{\pi}{x}F_1(2\nu, x, \chi)+\frac{\pi}{2x}L(2\nu, \chi)-\frac{1}{2x^2}L(2\nu+1, \chi).$$

Putting these in (10), we get our formula (2).

The case $\nu=0$ in (1) is interesting in connection with Ramanujan's arithmetical function $\tau(n)$. In fact, we can define arithmetical functions, analogous to $\tau(n)$, which are to be called "Ramanujan's τ-functions with χ". We shall deal with these functions in a forthcoming paper.

References

[1] E. Grosswald, Die Werte der Riemannschen Zeta-Funktion an ungeraden Argumentstellen, Nachr. der Akad. Wiss., Göttingen, 1970, 9-13.

[2] K. Katayama, On Ramanujan's formulas for values of Riemann zeta-function at positive odd integers, Acta Arithmetica, XXII, 1973, 149-155.

[3] C. L. Siegel, Lectures on advanced analytic number Theory, Tata Inst. of Fund. Res., Bombay, 1961.

Koji KATAYAMA
Department of Mathematics
Tsuda College
Kodaira, Tokyo
Japan