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On certain types of manifolds with f-structure

By Richard S. MILLMAN*

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1. Let M^n be a smooth (C^∞) n-manifold, let $T_m M$ be the tangent space at $m \in M$ and let $\chi(M)$ be the set of all smooth vector fields on M. An f-structure on M is a tensor, f_M, of type (1, 1) on M such that (1) $f_M^3(X)+f_M(X)=0$ for all $X \in \chi(M)$; and (2) f_M has constant nullity on M. An f-manifold is a manifold M together with an f-structure. If $\gamma: M_1 \to M_2$ then $(\gamma_m)_*$ or $\dot{\gamma}_m$ will denote the differential of γ at $m \in M$. We may occasionally omit the point m if there is no danger of confusion. If M_1 (resp. M_2) is an f-manifold with f-structure f_1 (resp. f_2) then γ is an f-map if $f_2\gamma_*(X)=\gamma_*(f_1X)$ for all $X \in \chi(M)$. The idea of combining two f-manifolds to obtain an f-structure on their product is very useful. Morimoto [3] (see also Sasaki [4]) used this idea to define a product on two almost contact manifolds which is an almost complex structure, generalizing the Calabi-Eckmann manifolds. Another way of defining a product shows that if M is a complex manifold then $M \times R$ is an almost contact manifold in which M imbeds. We will define a certain kind of f-structure (which we call a Cousin structure) on $M \times G$ (where M is an f-manifold and G is a Lie group with an f-structure, f_G) which will generalize the almost contact structure on $M \times R$ given above. Another motivation (and the reason for the name) is that if both f_M and f_G are complex structures then these structures are Weil's generalization of the classical Cousin problem of several complex variables ([1]). This second special case has been studied in [2]. We will exhibit all the Cousin structures explicitly (Theorem A), calculate the Nijenhuis tensor of the Cousin structure (Proposition 1) and then work an example in Section 3.

2. The product of M_1 and M_2 is the f-manifold $M_1 \underline{\times} M_2$ where the f-structure on $M_1 \times M_2$ is defined by $f((X_1, X_2))=(f_1X_1, f_2X_2)$ for $X_1 \in \chi(M_1)$, $X_2 \in \chi(M_2)$. We shall assume for the remainder of the paper that M has an f-structure f_M and the Lie group G has an f-structure f_G. If f is an f-structure on $M \times G$ such that $\pi: M \times G \to M$ (which is projection) and $\alpha: (M \times G) \underline{\times} G$

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$\rightarrow M\times G$ (which is right action of G on $M\times G$) are f-maps then f is called a *Cousin structure* on $M\times G$.

Let R_λ be right multiplication by $\lambda\in G$. Let $\Lambda^p(M, \hat{G})$ be the space of $\hat{G}$-valued p-forms on M (where $\hat{G}$ is the Lie algebra of G) and for $\omega\in\Lambda^1(M, \hat{G})$ define $l: M\times G\rightarrow R$

$$l(m, \lambda)=\dim \{(A, B)\in T_{m,\lambda}(M\times G) \mid f_M A=0 \text{ and } f_G B=-\dot{R}_\lambda\omega(A)\} . \tag{2.1}$$

We say that $\omega\in\Lambda^1(M, \hat{G})$ is an *admissible* 1-*form* if l is a constant function and if for all $m\in M$ and $A\in T_m M$

$$f_G^2\omega(A)+f_G\omega(f_M A)+\omega(f_M^2 A)+\omega(A)=0 . \tag{2.2}$$

The set of admissible forms will be denoted by $\mathfrak{a}$.

f_G is *bi-invariant* if both R_λ (right multiplication by $\lambda\in G$) and L_λ (left multiplication by $\lambda\in G$) are f-maps.

THEOREM A. *$M\times G$ admits a Cousin structure if and only if f_G is bi-invariant. Furthermore, if there is a Cousin structure on $M\times G$, then there is a one-to-one correspondence between Cousin structures on $M\times G$ and admissible 1-forms given as follows: If $\omega\in\mathfrak{a}$ then define f^ω to be the f-structure on $M\times G$ givin by:*

For $m\in M$, $\lambda\in G$, $A\in T_m M$, $B\in T_\lambda G$:

$$f^\omega_{m,\lambda}(A, B)=(f_M A, f_G B+(R_\lambda)_*\omega(A)) . \tag{2.3}$$

PROOF. (a) It is a routine calculation to show that if f_G is bi-invariant and ω is identically zero then f^ω is a Cousin structure. We will show that any Cousin structure must take the form (2.3) with $\omega\in\mathfrak{a}$ and f_G bi-invariant.

(b) Let f be a Cousin structure. We may write $f_{m,\lambda}(A, B)=(\bar{f}_{m,\lambda}(A, B), \tilde{f}_{m,\lambda}(A, B))$ where $\bar{f}_{m,\lambda}(A, B)\in T_m M$ and $\tilde{f}_{m,\lambda}(A, B)\in T_\lambda G$ and $m\in M$, $\lambda\in G$. Now $\dot{\pi}f(A, B)=\bar{f}(A, B)$ but $f_M\dot{\pi}(A, B)=f_M A$ hence by definition of an f-structure $\bar{f}(A, B)=f_M A$ or:

$$f_{m,\lambda}(A, B)=(f_M A, \tilde{f}_{m,\lambda}(A, B)) . \tag{2.4}$$

(c) Let $\alpha: (M\times G)\underline{\times} G\rightarrow M\times G$ be given by $\alpha(m, \lambda, g)=(m, \lambda g)$ then by using the Leibniz formula it is easy to see that (if L_λ is left translation by $\lambda\in G$)

$$\dot{\alpha}_{m,\lambda,g}(A, B, C)=(A, \dot{R}_g B+\dot{L}_\lambda C) \tag{2.5}$$

for $A\in T_m M$, $B\in T_\lambda G$ and $C\in T_g G$. If $P=M\times G$ and $f_{P\underline{\times} G}$ is the product f-structure on $P\underline{\times} G$ then

$$\dot{\alpha}_{m,\lambda,g} f_{P\underline{\times} G}(A, B, C)=\dot{\alpha}_{m,\lambda,g}(f_M A, \tilde{f}_{m,\lambda}(A, B), f_G C)$$

hence by (2.5)

(2.6) $$\dot{\alpha}_{m,\lambda,g} f_{P\times G}(A, B, C) = (f_M A,\ \dot{R}_g \tilde{f}_{m,\lambda}(A, B) + \dot{L}_\lambda f_G C)\,.$$

On the other hand, from (2.5):

$$f_{m,\lambda g} \dot{\alpha}_{m,\lambda,g}(A, B, C) = f_{m,\lambda g}(A,\ \dot{R}_g B + \dot{L}_\lambda C)$$

so:

(2.7) $$f_{m,\lambda g} \dot{\alpha}_{m,\lambda,g}(A, B, C) = (f_M A,\ \tilde{f}_{m,\lambda g}(A,\ \dot{R}_g B + \dot{L}_\lambda C))\,.$$

Comparing (2.6) and (2.7) we see $\dot{\alpha} f_{P\times G} = f_P \dot{\alpha}$ if and only if

(2.8) $$\dot{R}_g \tilde{f}_{m,\lambda}(A, B) + \dot{L}_\lambda f_G C = \tilde{f}_{m,\lambda g}(A,\ \dot{R}_g B + \dot{L}_\lambda C)\,.$$

Setting $A = B = 0$ and $\lambda = e$ in (2.8), we obtain:

(2.9) $$f_G C = \tilde{f}_{m,g}(0, C)\,.$$

Setting $B = C = 0$ and $\lambda = e$ in (2.8) we obtain

$$\dot{R}_g \tilde{f}_{m,e}(A, 0) = \tilde{f}_{m,g}(A, 0)\,.$$

Let $\omega_m(A) = \tilde{f}_{m,e}(A, 0) \in \hat{G}$ then $\omega \in \Lambda^1(M, \hat{G})$ and

(2.10) $$\dot{R}_g \omega_m(A) = \tilde{f}_{m,g}(A, 0)\,.$$

From (2.9) and (2.10) we conclude

(2.11) $$\begin{aligned} \tilde{f}_{m,\lambda}(A, B) &= \tilde{f}_{m,\lambda}(A, 0) + \tilde{f}_{m,\lambda}(0, B) \\ \tilde{f}_{m,\lambda}(A, B) &= \dot{R}_\lambda \omega_m(A) + f_G B\,. \end{aligned}$$

Equation (2.3) is now immediate from (2.4) and (2.11). We shall now see that f_G is bi-invariant. Putting (2.11) into (2.8) yields

$$\dot{R}_g \dot{R}_\lambda \omega_m(A) + \dot{R}_g f_G B + \dot{L}_\lambda f_G C = \dot{R}_{\lambda g} \omega_m(A) + f_G(\dot{R}_g B + \dot{L}_\lambda C)$$

hence

$$\dot{R}_g f_G B + \dot{L}_\lambda f_G C = f_G \dot{R}_g B + f_G \dot{L}_\lambda C$$

for all $B \in T_\lambda G$, $C \in T_g G$. Setting $B = 0$ shows that L_λ is an f_G-map and setting $C = 0$ shows that R_g is an f_G-map so f_G is bi-invariant.

(d) We show that if f has the form (2.3) then $f^3 + f = 0$ if and only if (2.2) holds.

$$f^2(A, B) = (f_M^2 A,\ f_G^2 B + f_G \dot{R}_\lambda \omega(A) + \dot{R}_\lambda \omega(f_M A))$$

thus

$$\begin{aligned} (f^3 + f)(A, B) = (f_M^3 A + f_M A,\ & f_G^3 B + f_G^2 \dot{R}_\lambda \omega(A) + f_G \dot{R}_\lambda \omega(f_M A) \\ & + \dot{R}_\lambda \omega(f_M^2 A) + \dot{R}_\lambda \omega(A) + f_G B) \end{aligned}$$

so $f^3 + f = 0$ if and only if

$$f_G^2 \dot{R}_\lambda \omega(A) + f_G \dot{R}_\lambda \omega(f_M A) + \dot{R}_\lambda \omega(f_M^2 A) + \dot{R}_\lambda \omega(A) = 0\,.$$

Exploiting the right invariance of f_G yields equation (2.2).

(e) We need only show that f has constant nullity if and only if l of equation (2.1) is constant; however, $f_{m,\lambda}(A, B)=0$ if and only if $f_M A=0$ and $f_G B+\dot{R}_\lambda\omega(A)=0$ so this is immediate. Q. E. D.

If $G=R$ and M is a complex manifold then the Cousin structure with $\omega=0$ is the standard almost contact structure on $M\times R$ mentioned in the introduction.

Recall the Nijenhuis torsion tensor N of f is given by [5] (for $X, Y\in \mathfrak{X}(M)$)

$$N(X, Y)=[fX, fY]-f[fX, Y]-f[X, fY]+f^2[X, Y]. \tag{2.12}$$

Before calculating the Nijenhuis torsion of a Cousin structure we need the following lemmas:

LEMMA 1. *For any $\lambda\in G$, $A\in \mathfrak{X}(M)$ and $B\in \mathfrak{X}(G)$ and right invariant f-structure f_G on G:*

$$f_G[\dot{R}_\lambda\omega(A), B]=[\dot{R}_\lambda\omega(A), f_G B].$$

PROOF. If $\varphi_t=R_{\exp t\omega(A)\lambda}$ then φ_t is the one parameter subgroup of G generated by $\dot{R}_\lambda\omega(A)$ hence

$$\begin{aligned} f_G[\dot{R}_\lambda\omega(A), B] &= f_G \lim_{t\to 0}\frac{1}{t}(B-(\varphi_t)_* B)\\ &= \lim_{t\to 0}\frac{1}{t}(f_G B-(\varphi_t)_* f_G B)\\ &= [\dot{R}_\lambda\omega(A), f_G B] \end{aligned}$$

where the second equality follows from the right invariance of f_G. Q. E. D.

We shall use the next result (whose proof is immediate) many times in what follows and so shall not mention it explicitly:

LEMMA. *If $A_1, A_2\in \mathfrak{X}(M)$, $B_1, B_2\in \mathfrak{X}(G_2)$ then*

$$[(A_1, B_1), (A_2, B_2)]=([A_1, A_2]+(B_1A_2-B_2A_1), A_1B_2-A_2B_1+[B_1, B_2]).$$

Let $\{e_1, \cdots, e_r\}$ be any basis for $\hat{G}$. If $\omega=\sum\omega^k e_k$ where $\omega^k\in\Lambda^1(M, R)$ for $k=1, \cdots, r$. If $A_1, A_2\in \mathfrak{X}(M)$, then by $A_1\omega(A^2)$ we mean the $\hat{G}$-valued function on M, $A_1\omega(A_2)=\sum_{k=1}^{r}(A_1\omega^k(A_2))e_k$. By $R_\lambda\omega(A)$ we mean the vector field on $M\times G$ whose value at (m, λ) is $(0, \dot{R}_\lambda\omega_m(A))\in T_{m,\lambda}(M\times G)$.

LEMMA 2. $[(A_1, 0), R_\lambda\omega(A_2)]_{m,\lambda}=(0, \dot{R}_\lambda(A_1\omega(A_2))_m)$ *for all* $A_1, A_2\in \mathfrak{X}(M)$.

PROOF. $[(A_1, 0), R_\lambda\omega(A_2)]=[(A_1, 0), (0, \dot{R}_\lambda\omega(A_2))]$ which by the above discussion is:

$$(0-\dot{R}_\lambda\omega(A_2)A_1, A_1\dot{R}_\lambda\omega(A_2))=(0, \dot{R}_\lambda A_1\omega(A_2)). \qquad \text{Q. E. D.}$$

LEMMA 3. *If $\varphi:\hat{G}\to\hat{G}$ is a linear transformation and $\omega\in\Lambda^1(M, \hat{G})$ then $A_1\varphi(\omega A_2))=\varphi(A_1\omega(A_2))$.*

PROOF. We shall write $\varphi=(\varphi_k^j)$ with respect to the basis $\{e_1, \cdots, e_r\}$ of $\hat{G}$

and $A_1\omega(A_2)=\sum_{k=1}^{r} A_1\omega^k(A_2)e_k$. Thus $\varphi(A_1\omega(A_2))=\sum A_1\omega^k(A_2)\varphi(e_k)$ since for each $m\in M$ $(A_1\omega^k(A_2))_m \in R$ and φ is R-linear. We see therefore that $\varphi(A_1\omega(A_2)) =\sum_{k,j}(A_1\omega^k(A_2))\varphi_k^j e_j$. On the other hand, $\varphi(\omega(A_2))=\sum\omega^k(A_2)\varphi(e_k)=\sum\omega^k(A_2)\varphi_k^j e_j$ where φ_k^j are independent of $m\in M$ so $A_1\varphi(\omega(A_2))=\sum_{k,j}(A_1\omega^k(A_2))\varphi_k^j e_j$.

Q. E. D.

If $\omega\in\Lambda^1(M,\hat{G})$ we shall define $\Omega^\omega\in\Lambda^2(M,\hat{G})$ to be the *complex Nijenhuis defect* and $\Psi^\omega\in\Lambda^2(M,\hat{G})$ to be the *f-Nijenhuis defect* where for $A_1, A_2\in\chi(M)$

(2.13a) $\Omega^\omega(A_1, A_2)=[\omega(A_1), \omega(A_2)]+2d\omega(f_M A_1, A_2)+2d\omega(A_1, f_M A_2)$

(2.13b) $\Psi^\omega(A_1, A_2)=A_2\omega(f_M A_1)-A_1\omega(f_M A_2)+\omega(f_M[A_1, A_2])-2f_G d\omega(A_1, A_2)$.

PROPOSITION 1. *If N_M (resp. N_G) represent the Nijenhuis torsion tensor of M (resp. of G) and N^ω is the Nijenhuis torsion tensor of the Cousin structure f^ω then for $A_1, A_2\in\chi(M)$ and $B_1, B_2\in\chi(G)$*

$$N^\omega_{m,\lambda}((A_1, B_1), (A_2, B_2))=(N_M(A_1, A_2), N_G(B_1, B_2)) +\dot{R}_\lambda\{\Omega^\omega(A_1, A_2)+\Psi^\omega(A_1, A_2)\} .$$

PROOF. Let $X=(A_1, B_1)$ and $Y=(A_2, B_2)$ in (2.12) then

$$\begin{aligned}
&N^\omega((A_1, B_1), (A_2, B_2))\\
&=([f_M A_1, f_M A_2], [f_G B_1, f_G B_2]+[\dot{R}_\lambda\omega(A_1), f_G B_2]\\
&\qquad +[f_G B_1, \dot{R}_\lambda\omega(A_2)]+[\dot{R}_\lambda\omega(A_1), \dot{R}_\lambda\omega(A_2)])\\
&\quad +[(f_M A_1, 0), R_\lambda\omega(A_2)]-[(f_M A_2, 0), R_\lambda\omega(A_1)]\\
&\quad -(f_M[f_M A_1, A_2], f_G[f_G B_1, B_2]+f_G[\dot{R}_\lambda\omega(A_1), B_2]+\dot{R}_\lambda\omega[f_M A_1, A_2])\\
&\quad +f_G[(A_2, 0), R_\lambda\omega(A_1)]\\
&\quad -(f_M[A_1, f_M A_2], f_G[B_1, f_G B_2]+f_G[B_1, \dot{R}_\lambda\omega(A_2)]+\dot{R}_\lambda\omega([A_1, f_M A_2]))\\
&\quad -f_G[(A_1, 0), R_\lambda\omega(A_2)]\\
&\quad +(f_M^2[A_1, A_2], f_G^2[B_1, B_2]+f_G\dot{R}_\lambda\omega([A_1, A_2])+\dot{R}_\lambda\omega(f_M[A_1, A_2])\\
&=(N_M(A_1, A_2), N_G(B_1, B_2)+[\dot{R}_\lambda\omega(A_1), \dot{R}_\lambda\omega(A_2)]-\dot{R}_\lambda\omega([f_M A_1, A_2])\\
&\qquad -\dot{R}_\lambda\omega([A_1, f_M A_2])+f_G\dot{R}_\lambda\omega([A_1, A_2])+\dot{R}_\lambda\omega(f_M[A_1, A_2]))\\
&\quad +[(f_M A_1, 0), R_\lambda\omega(A_2)]-[(f_M A_2, 0), R_\lambda\omega(A_1)]\\
&\quad +f_G[(A_2, 0), R_\lambda\omega(A_1)]-f_G[(A_1, 0), R_\lambda\omega(A_2)]\\
&\quad +(0, [\dot{R}_\lambda\omega(A_1), f_G B_2]+[f_G B_1, \dot{R}_\lambda\omega(A_2)]\\
&\qquad -f_G[\dot{R}_\lambda\omega(A_1), B_2]-f_G[B_1, R_\lambda\omega(A_2)]) .
\end{aligned}$$

The sum of the last four terms is zero by Lemma 1. Applying Lemma 2 we obtain:

$$(2.14)\qquad \begin{aligned} &N^{\omega}((A_1, B_1), (A_2, B_2)) \\ &\quad =(N_M(A_1, A_2), N_G(B_1, B_2)+\dot{R}_\lambda\{[\omega(A_1), \omega(A_2)] \\ &\qquad -\omega([f_M A_1, A_2])-\omega([A_1, f_M A_2])+f_G\omega([A_1, A_2])+\omega(f_M[A_1, A_2]) \\ &\qquad +(f_M A_1)\omega(A_2)-(f_M A_2)\omega(A_1)+f_G A_2\omega(A_1)-f_G A_1\omega(A_2)\}).\end{aligned}$$

On the other hand it is well-known that

$$2d\omega(A_1, A_2)=A_1\omega(A_2)-A_2\omega(A_1)-\omega[A_1, A_2]$$

so that

$$\begin{aligned} &2\{-f_G d\omega(A_1, A_2)+d\omega(f_M A_1, A_2)+d\omega(A_1, f_M A_2)\} \\ &\quad =-f_G A_1\omega(A_2)+f_G A_2\omega(A_1)+f_G\omega([A_1, A_2])+f_M A_1\omega(A_2)-A_2\omega(f_M A_1) \\ &\qquad -\omega[f_M A_1, A_2]+A_1\omega(f_M A_2)-(f_M A_2)\omega(A_1)-\omega[A_1, f_M A_2].\end{aligned}$$

Plugging this into 2.14 yields:

$$\begin{aligned} &N^{\omega}((A_1, B_1), (A_2, B_2)) \\ &\quad =(N_M(A_1, A_2), N_G(B_1, B_2)+\dot{R}_\lambda\{[\omega(A_1), \omega(A_2)] \\ &\qquad +\omega(f_M([A_1, A_2]))-\omega f_M([A_1, A_2])+A_2\omega(f_M A_1)-A_1\omega(f_M A_2) \\ &\qquad +2(d\omega(f_M A_1, A_2)+d\omega(A_1, f_M A_2)-f_G d\omega(A_1, A_2))\}) \\ &\quad =(N_M(A_1, A_2), N_G(B_1, B_2)+\dot{R}_\lambda\{\Omega^{\omega}(A_1, A_2)+\Psi^{\omega}(A_1, A_2)\}).\qquad \text{Q.E.D.}\end{aligned}$$

COROLLARY. *If both f_M and f_G are complex structures then f^{ω} is a complex structure if and only if its complex Nijenhuis defect is zero.*

PROOF. If both f_M and f_G are complex structures then equation (2.2) yields $\omega(f_M A)=-f_G\omega(A)$ from which it is clear that $\Psi^{\omega}=0$ and $N^{\omega}=\dot{R}_\lambda\Omega^{\omega}$.
Q. E. D.

We remark that in the above case the condition $\Omega^{\omega}=0$ (when extended to complex tangent vectors) is precisely the condition $\bar{\partial}\omega=\frac{i}{4}[\omega, \omega]$ which is the result (obtained in a different way) of [2, Theorem 2.3.5].

3. We shall assume in this section that M has an almost complex structure J and G has a bi-invariant almost contact structure $\Sigma=(\varphi, \xi, \eta)$; that is, φ is a tensor field of type (1, 1), $\xi\in\mathfrak{X}(G)$ and η is a (real-valued) 1-form on G such that $\eta(\xi)=1$, $\varphi^2=-I+\xi\otimes\eta$, $\varphi\circ\eta=0$, $\dot{R}_\lambda\varphi=\varphi\dot{R}_\lambda$ and $\dot{L}_\lambda\varphi=\varphi\dot{L}_\lambda$ for all $\lambda\in G$. Morimoto [3] has shown that every Lie group G of odd dimension admits a right invariant such structure and if G is reductive of odd dimension (in particular, if G is compact and odd dimensional) then G admits an integrable such structure.

The further requirement that G be compact and f_G be bi-invariant greatly restricts G because:

PROPOSITION 2. *If G is a compact Lie group with integrable bi-invariant almost contact structure then G is isomorphic to a torus.*

PROOF. Define

$$J_{p,q}(X_p, Y_q) = (\varphi(X_p) - \eta(Y_q)\xi_p, \varphi(Y_q) + \eta(X_p)\xi_q)$$

where $X_p \in T_pG$, $Y_q \in T_qG$ and $p, q \in G$, then J is an integrable almost complex structure [3, Theorem 2]. This means that $G \times G$ is a compact complex Lie group, hence abelian. We see therefore that G is abelian and so G is a torus. Q. E. D.

Because of the above proposition we shall not make the restrictive assumption that G be compact.

If $\omega \in \Lambda^1(M, \hat{G})$ then ω is admissible if and only if equation (2.2) holds (because l is always constant). Thus ω is admissible if and only if $\varphi^2(\omega(A)) + \varphi(\omega(JA)) = 0$. This last equation is equivalent to:

$$-\omega(A) + \eta(\omega(A))\xi + \varphi(\omega(JA)) = 0 \tag{3.1a}$$

or

$$\omega(JA) = \eta(\omega(JA))\xi - \varphi(\omega(A)). \tag{3.1b}$$

PROPOSITION 3. *If $\Sigma^\omega = (\varphi^\omega, \xi^\omega, \eta^\omega)$ is the almost contact structure on $M \times G$ given by the Cousin structure f^ω (where ω satisfies (3.1)) then for $A, A_1, A_2 \in \chi(M)$, $B \in T_\lambda G$*

(a) $\varphi^\omega(A, B) = (JA, \varphi(B) + (R_\lambda)_* \omega(A))$

(b) $\xi^\omega = (0, \xi)$ *and* $\eta^\omega(A, B) = \eta(B) + \eta(\omega(JA))$

(c) $\Omega^\omega(A_1, A_2) = [\omega(A_1), \omega(A_2)] + 2d\omega(JA_1, A_2) + 2d\omega(A_1, JA_2)$

(d) $\Psi^\omega(A_1, A_2) = \{A_2\eta(\omega(JA_1)) - A_1\eta\omega(JA_2) + \eta\omega(J[A_1, A_2])\}\xi_e$.

PROOF. (a) Clearly $\eta^\omega(\xi^\omega) = 1$ so we need only check $(\varphi^\omega)^2(X) = -X + \eta^\omega(X)\xi^\omega$ if $X \in T_{m,\lambda}(M \times G)$. If $X = (A, B)$ then

$$(\varphi^\omega)^2(A, B) = (-A, \varphi^2(B) + \varphi(\dot{R}_\lambda\omega(A)) + \dot{R}_\lambda\omega(JA))$$

or

$$= (-A, -B + \eta(B)\xi + \dot{R}_\lambda(\varphi(\omega(A)) + \omega(JA))$$

applying equation (3.1b)

$$(\varphi^\omega)^2(A, B) = -(A, B) + (\eta(B) + \eta(\omega(JA)))(0, \xi)$$

hence

$$(\varphi^\omega)^2(A, B) = -(A, B) + \eta^\omega(A, B)\xi^\omega$$

and so (a) and (b) are proven. (c) is obvious.

(d) Applying (2.13b) we have

$$\Psi^{\omega}(A_1, A_2) = A_2\omega(JA_1) - A_1\omega(JA_2) + \omega(J[A_1, A_2]) - 2\varphi d\omega(A_1, A_2)\,.$$

We now use (3.1b):

$$= A_2\{\eta(\omega(JA_1))\xi - \varphi(\omega(A_1))\} - A_1\{\eta(\omega(JA_2))\xi - \varphi(\omega(A_2))\}$$
$$+ \eta(\omega(J[A_1, A_2]))\xi - \varphi\omega([A_1, A_2]) - 2\varphi d\omega(A_1, A_2)\,.$$

Applying Lemma 3,

$$= \{A_2\eta\omega(JA_1) - A_1\eta\omega(JA_2) + \eta\omega(J[A_1, A_2])\}\xi$$
$$+ \varphi\{A_1\omega(A_2) - A_2\omega(A_1) - \omega([A_1, A_2])\} - 2\varphi d\omega(A_1, A_2)$$
$$= \{A_2\eta\omega(JA_1) - A_1\eta\omega(JA_2) + \eta\omega(J[A_1, A_2])\}\xi\,. \qquad \text{Q. E. D.}$$

We call the Cousin structure f^{ω} *trivial* if $\omega = 0$. We shall now show that non-trivial Cousin structures exist on $M \times G$ if G is abelian. (Since one is usually interested in compact almost contact manifolds, Proposition 2 dictates the assumption that G be abelian.) Let $\{e_1, \cdots, e_r\}$ be a basis for $\hat{G}$ such that $\eta(e_i) = 0$, $i = 1, \cdots, r-1$ and $e_r = \xi(e)$ (so $\eta(e_r) = 1$). This can be done since the kernel of η_g has dimension $r-1$ for each $g \in G$ [**4**; Vol. I, p. 1-4].

PROPOSITION 4. *If* $\omega_1 \in \Lambda^1(M, \hat{G})$ *and* $\omega(A) = \dfrac{\omega_1(JA) - \varphi(\omega_1(A))}{2}$ *then* ω *is admissible.*

PROOF. We show that equation (3.1a) is valid.

$$2(-\omega(A) + \eta(\omega(A)\xi + \varphi(\omega(JA))$$
$$= -\omega_1(JA) + \varphi(\omega_1(A)) + \eta(\omega_1(JA) - \varphi(\omega_1(A)))\xi + \varphi(\omega_1(-A) - \varphi\omega_1(JA))$$
$$= -\omega_1(JA) + \eta(\omega_1(JA)) - (-\omega_1(JA) + \eta(\omega_1(JA))\xi)$$

which is zero since $\eta \circ \varphi = 0$. Q. E. D.

PROPOSITION 5. *If* $\tilde{\omega} \in \Lambda^1(M, R)$ *such that* $d\tilde{\omega} = 0$ *and define* $\omega, \omega_1 \in \Lambda^1(M, \hat{G})$ *as* $\omega_1(A) = \tilde{\omega}(A)\sum_{k=1}^{r-1} e_k$ *and* $\omega(A) = \dfrac{\omega_1(JA) - \varphi(\omega_1(A))}{2}$ *then*

(a) *ω is admissible*

(b) $\Omega^{\omega}(A_1, A_2) = [\omega(A_1), \omega(A_2)] + \tilde{\omega}(N_J(A_1, A_2))\sum_{k=1}^{r-1} e_k$

(c) $\Psi^{\omega}(A_1, A_2) = 0$.

PROOF. (a) is a special case of Proposition 4. To prove (b) we use Proposition 3c to compute Ω^{ω}. Note that

$$2\omega(A) = \tilde{\omega}(JA)\sum_{k=1}^{r-1} e_k - \tilde{\omega}(A)\sum_{k=1}^{r-1}\sum_{h=1}^{r-1} \varphi_k^h e_h\,.$$

Let $\alpha \in \Lambda^1(M, R)$ be defined by $\alpha(A) = \tilde{\omega}(JA)$. Then $d\omega = d\alpha \sum_k e_k$ since $d\tilde{\omega} = 0$. Thus we must compute

$$
\begin{aligned}
&2d\alpha(JA_1, A_2)+2d\alpha(A_1, JA_2)\\
&\quad =JA_1\tilde{\omega}(JA_2)-A_2\tilde{\omega}(J^2A_1)-\tilde{\omega}(J[JA_1, A_2])\\
&\qquad +A_1\tilde{\omega}(J^2A_2)-JA_2\tilde{\omega}(JA_1)-\tilde{\omega}(J[A_1, JA_2])\\
&\quad =2d\tilde{\omega}(A_2, A_1)+2d\tilde{\omega}(JA_1, JA_2)-\tilde{\omega}([A_1, A_2])+\tilde{\omega}([JA_1, JA_2])\\
&\qquad -\tilde{\omega}(J[JA_1, A_2])-\tilde{\omega}(J[A_1, JA_2]).
\end{aligned}
$$

The first two terms are zero since $d\tilde{\omega}=0$ and the last four terms are exactly $\tilde{\omega}(N_J(A_1, A_2))$.

(c) Because of the computation in part (a) we know that $\eta(\omega(B))=0$ for all $B\in\chi(M)$ hence Proposition 3d implies $\Psi^{\omega}\equiv 0$. Q. E. D.

The following corollary is immediate from Propositions 1, 3, 4 and 5.

COROLLARY. *If M is a complex manifold and G an abelian Lie group with integrable bi-invariant almost contact structure then there exists an integrable non-trivial Cousin structure on $M\times G$.*

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Richard S. MILLMAN
Department of Mathematics
Southern Illinois University
Carbondale, Illinois 62901
U.S.A.