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On the Euler integral representations of hypergeometric functions in several variables

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§1. Statement of the problem.

It is well known that the hypergeometric function $F(\alpha, \beta, \gamma, x)$ defined by the series

(1.1) $$F(\alpha, \beta, \gamma, x)=\sum_{m=0}^{\infty} \frac{(\alpha, m)(\beta, m)}{(\gamma, m)(1, m)} x^{m}$$

has the Euler integral representation

(1.2) $$F(\alpha, \beta, \gamma, x)=\frac{\Gamma(\gamma)}{\Gamma(\beta) \Gamma(\gamma-\beta)} \int_{0}^{1} z^{\beta-1}(1-z)^{\gamma-\beta-1}(1-x z)^{-\alpha} d z,$$

where (a, k) denotes the factorial function

$$a(a+1) \cdots(a+k-1)=\frac{\Gamma(a+k)}{\Gamma(a)}$$

Series (1.1) is meaningful provided that $\gamma \neq 0,-1,-2, \cdots$, and then the radius of convergence is one except in the case when either α or β is a non positive integer. On the other hand, integral (1.2) is convergent if

(1.3) $$0<\operatorname{Re} \beta<\operatorname{Re} \gamma,$$

and then integral (1.2) is holomorphic with respect to x in $\boldsymbol{C}-[1, \infty)$, $\boldsymbol{C}$ being the set of complex numbers.

It is natural to attempt weakening restriction (1.3) in the integral representation. Indeed, two methods for that are known: method of double contour integrals and that of the finite part for divergent integrals. The former is usually carried out as follows. Suppose that $x \in \boldsymbol{C}-[1, \infty)$. Let a be a point in $\boldsymbol{C}-\{0,1,1 / x\}$, say lying on the real axis between 0 and 1, and let l_{0} and l_{1} be loops in the positive direction at a in $\boldsymbol{C}-\{0,1,1 / x\}$, l_{0} encircling only $z=0$ and l_{1} encircling only $z=1$. Form the contour C consisting of l_{0}, l_{1}, l_{0}^{-1} and l_{1}^{-1} in this order:

$$C=l_{0} l_{1} l_{0}^{-1} l_{1}^{-1},$$

l_{0}^{-1} and l_{1}^{-1} being the inverse loops of l_{0} and l_{1} respectively. Then take a branch of the integrand on C in such a way that $\arg z$ and $\arg(1-z)$ are continuous in z and are reduced to zero at the starting point of C and $\arg(1-xz)$ is continuous in (x, z) and is reduced to zero at $x=0$ and the starting point of C. Finally integrate $z^{\beta-1}(1-z)^{\gamma-\beta-1}(1-xz)^{-\alpha}$ along C.

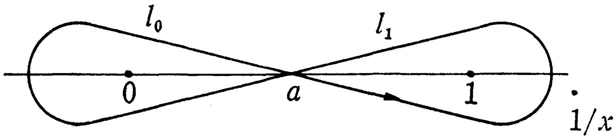

Fig. 1.

We can take as l_0 and l_1 respectively a loop consisting of the segment from a to ρ, the circle $|z|=\rho$ and the segment from ρ to a, and a loop consisting of the segment from a to $1-\rho$, the circle $|z-1|=\rho$ and the segment from $1-\rho$ to a, where ρ is a sufficiently small positive number. We obtain, by letting $\rho\to 0$,

$$\begin{aligned}(1.4)\qquad &\int_C z^{\beta-1}(1-z)^{\gamma-\beta-1}(1-xz)^{-\alpha}dz\\ &=-(1-e^{2\pi i\beta})(1-e^{2\pi i(\gamma-\beta)})\int_0^1 z^{\beta-1}(1-z)^{\gamma-\beta-1}(1-xz)^{-\alpha}dz\end{aligned}$$

under restriction (1.3).

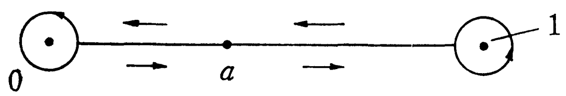

Fig. 2.

The integral (1.4) can be given an interpretation in the following way. For a given $x\in \boldsymbol{C}-[1, \infty)$, let D be an open rectangle in $\boldsymbol{C}$ with summits $(-\rho, i\rho)$, $(-\rho, -i\rho)$, $(1+\rho, -i\rho)$ and $(1+\rho, i\rho)$, ρ being a sufficiently small positive number. We suppose that the point $1/x$ is not contained in D but the loop C is contained in D. For simplicity we suppose that 1, β and $\gamma-\beta$ are linearly independent over the rationals. Then the Riemann surface of the integrand $z^{\beta-1}(1-z)^{\gamma-\beta-1}(1-xz)^{-\alpha}$ restricted on $X=D-\{0,1\}$ is the covering surface $\tilde{X}$ of X corresponding to the commutator subgroup of the fundamental group of X, $\pi_1(X)$. The 1-form $z^{\beta-1}(1-z)^{\gamma-\beta-1}(1-xz)^{-\alpha}dz$ can be naturally regarded as a holomorphic 1-form $\tilde{\omega}$ on $\tilde{X}$, which we denote symbolically by $\tilde{z}^{\beta-1}(1-\tilde{z})^{\gamma-\beta-1}(1-x\tilde{z})^{-\alpha}d\tilde{z}$. Let $\tilde{a}$ be the point of $\tilde{X}$ over a corresponding to the value of the integrand at the starting point of C and let $\tilde{C}$ be the lift of C on $\tilde{X}$ starting from $\tilde{a}$. Then the integral (1.4) can be written as

$$(1.5)\qquad \int_{\tilde{C}}\tilde{z}^{\beta-1}(1-\tilde{z})^{\gamma-\beta-1}(1-x\tilde{z})^{-\alpha}d\tilde{z}\,.$$

Since the class of $C=l_0l_1l_0^{-1}l_1^{-1}$ is a commutator of $\pi_1(X)$, the path $\tilde{C}$ is a closed path and hence a cycle in $\tilde{X}$. It should be noted that the value of integral (1.5) depends only on the homology class of the cycle $\tilde{C}$ and that X is homeomorphic to $\boldsymbol{C}-\{0, 1\}$.

The homology of $\tilde{X}$ is well-known. Indeed, the space X has the same homotopy type as the bouquet of two circles which may be supposed to be l_0 and l_1. The fundamental group of the bouquet is a free group generated by l_0 and l_1. Thus the homology group $H_1(X, \boldsymbol{Z})$, the quotient group of $\pi_1(X)$ by its commutator subgroup, is a free abelian group generated by l_0 and l_1, and is canonically identified with the group of covering transformations of the covering $\tilde{X}\to X$. It follows that the covering surface $\tilde{X}$ has the homotopy type of the space

$$\tilde{X}_0=\{(t_1, t_2)\in \boldsymbol{R}^2 \mid t_1 \text{ or } t_2\in \boldsymbol{Z}\} .$$

We thus see that the homology group $H_1(\tilde{X}, \boldsymbol{Z})$ is a free $\boldsymbol{Z}(H_1(X, \boldsymbol{Z}))$-module generated by $\tilde{C}$, where $\boldsymbol{Z}(H_1(X, \boldsymbol{Z}))$ is the group ring of $H_1(X, \boldsymbol{Z})$ over $\boldsymbol{Z}$. It is clear that the homology class $\tilde{C}$ corresponds, through a homotopy equivalence between $\tilde{X}$ and $\tilde{X}_0$, to the class of a loop $\tilde{C}_0$ as indicated in Fig. 3.

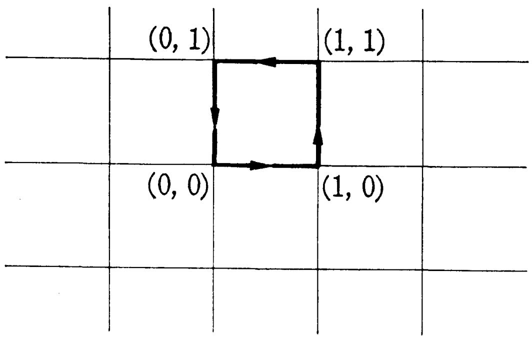

Fig. 3.

We can give another interpretation of integral (1.5) using cohomology with coefficients in a local system. First we observe the following relation

$$\text{(1.6)}\qquad \int_{\tilde{C}l_0^m l_1^n} \tilde{z}^{\beta-1}(1-\tilde{z})^{\gamma-\beta-1}(1-x\tilde{z})^{-\alpha}d\tilde{z}$$
$$=e^{2\pi i(m\beta+n(\gamma-\beta))}\int_{\tilde{C}} \tilde{z}^{\beta-1}(1-\tilde{z})^{\gamma-\beta-1}(1-x\tilde{z})^{-\alpha}d\tilde{z} ,$$

where l_0 and l_1 act on $\tilde{X}$ by covering transformations on the right as usual and $\tilde{C}l_0^m l_1^n$ is the transform of $\tilde{C}$ under the transformation $l_0^m l_1^n$. Let $\mathcal{S}$ denote the local system on X with stalk $\boldsymbol{C}$ associated to the representation

$$\theta:\ \pi_1(X) \longrightarrow H_1(X, \boldsymbol{Z}) \longrightarrow \boldsymbol{C}^*$$

given by

$$\theta(l_0)=e^{2\pi i\beta}\,,\qquad \theta(l_1)=e^{2\pi i(\gamma-\beta)}\,.$$

The chain group of X with coefficients in $\mathcal{S}$ is, by definition, $C_*(\tilde{X})\underset{\theta}{\otimes}\boldsymbol{C}$ and the cycle $\tilde{C}l_0^m l_1^n\underset{\theta}{\otimes}1$ equals $\tilde{C}\underset{\theta}{\otimes}e^{2\pi i(m\beta+n(\gamma-\beta))}$ in that group. Using a homotopy equivalence between $\tilde{X}$ and $\tilde{X}_0$, it is easily seen that the homology group $H_1(X, \mathcal{S})$ is isomorphic to $\boldsymbol{C}$ and is generated by the class $[\tilde{C}]$ of $\tilde{C}\underset{\theta}{\otimes}1$. The above relation (1.6) shows that the integration of $\tilde{z}^{\beta-1}(1-\tilde{z})^{\gamma-\beta-1}(1-x\tilde{z})^{-\alpha}d\tilde{z}$ over the cycles of $C_1(\tilde{X})\underset{\theta}{\otimes}\boldsymbol{C}$ and hence over the homology classes of $H_1(X, \mathcal{S})$ is well-defined in such a way that

$$\int_{[\tilde{C}]}\tilde{z}^{\beta-1}(1-\tilde{z})^{\gamma-\beta-1}(1-x\tilde{z})^{-\alpha}d\tilde{z}$$
$$=\int_{\tilde{C}}\tilde{z}^{\beta-1}(1-\tilde{z})^{\gamma-\beta-1}(1-x\tilde{z})^{-\alpha}d\tilde{z}\,.$$

If we denote by $\mathcal{S}^{-1}$ the local system defined by the representation θ^{-1}, then $H_1(X, \mathcal{S})$ and $H^1(X, \mathcal{S}^{-1})$ are dual to each other. Then, through the integration pairing, $\tilde{z}^{\beta-1}(1-\tilde{z})^{\gamma-\beta-1}(1-x\tilde{z})^{-\alpha}d\tilde{z}$ can be regarded as a generator of $H^1(X, \mathcal{S}^{-1})\cong\boldsymbol{C}$.

We proceed to hypergeometric functions in several complex variables. Consider the functions

$$F_A(\alpha, \beta_1, \cdots, \beta_n, \gamma_1, \cdots, \gamma_n, x_1, \cdots, x_n)$$
$$=\sum_{m_1,\cdots,m_n=0}^{\infty}\frac{(\alpha, m_1+\cdots+m_n)(\beta_1, m_1)\cdots(\beta_n, m_n)}{(\gamma_1, m_1)\cdots(\gamma_n, m_n)(1, m_1)\cdots(1, m_n)}x_1^{m_1}\cdots x_n^{m_n}\,,$$

$$F_B(\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n, \gamma, x_1, \cdots, x_n)$$
$$=\sum_{m_1,\cdots,m_n=0}^{\infty}\frac{(\alpha_1, m_1)\cdots(\alpha_n, m_n)(\beta_1, m_1)\cdots(\beta_n, m_n)}{(\gamma, m_1+\cdots+m_n)(1, m_1)\cdots(1, m_n)}x_1^{m_1}\cdots x_n^{m_n}$$

and

$$F_D(\alpha, \beta_1, \cdots, \beta_n, \gamma, x_1, \cdots, x_n)$$
$$=\sum_{m_1,\cdots,m_n=0}^{\infty}\frac{(\alpha, m_1+\cdots+m_n)(\beta_1, m_1)\cdots(\beta_n, m_n)}{(\gamma, m_1+\cdots+m_n)(1, m_1)\cdots(1, m_n)}x_1^{m_1}\cdots x_n^{m_n}\,.$$

It is known that these functions have the integral representations

$$(1.7)\quad \frac{\Gamma(\beta_1)\cdots\Gamma(\beta_n)\Gamma(\gamma_1-\beta_1)\cdots\Gamma(\gamma_n-\beta_n)}{\Gamma(\gamma_1)\cdots\Gamma(\gamma_n)}F_A(\alpha, \beta_1, \cdots, \beta_n, \gamma_1, \cdots, \gamma_n, x_1, \cdots, x_n)$$
$$=\int_0^1\cdots\int_0^1 z_1^{\beta_1-1}\cdots z_n^{\beta_n-1}(1-z_1)^{\gamma_1-\beta_1-1}\cdots$$
$$\cdots(1-z_n)^{\gamma_n-\beta_n-1}(1-x_1z_1-\cdots-x_nz_n)^{-\alpha}dz_1\cdots dz_n\,,$$

$$(1.8)\quad \frac{\Gamma(\beta_1)\cdots\Gamma(\beta_n)\Gamma(\gamma-\beta_1-\cdots-\beta_n)}{\Gamma(\gamma)}F_B(\alpha_1,\cdots,\alpha_n,\beta_1,\cdots,\beta_n,\gamma,x_1,\cdots,x_n)$$

$$=\int\cdots\int_{z_1,\cdots,z_n,1-z_1-\cdots-z_n\geqq 0} z_1^{\beta_1-1}\cdots z_n^{\beta_n-1}(1-z_1-\cdots-z_n)^{\gamma-\beta_1-\cdots-\beta_n-1}\times$$

$$(1-x_1z_1)^{-\alpha_1}\cdots(1-x_nz_n)^{-\alpha_n}dz_1\cdots dz_n\,,$$

$$(1.9)\quad \frac{\Gamma(\beta_1)\cdots\Gamma(\beta_n)\Gamma(\gamma-\beta_1-\cdots-\beta_n)}{\Gamma(\gamma)}F_D(\alpha,\beta_1,\cdots,\beta_n,\gamma,x,\cdots,x_n)$$

$$=\int\cdots\int_{\substack{z_1,\cdots,z_n\geqq 0\\ 1-z_1-\cdots-z_n\geqq 0}} z_1^{\beta_1-1}\cdots z_n^{\beta_n-1}(1-z_1-\cdots-z_n)^{\gamma-\beta_1-\cdots-\beta_n-1}\times$$

$$(1-x_1z_1-\cdots-x_nz_n)^{-\alpha}dz_1\cdots dz_n$$

under the restrictions similar to (1.3)

$$(1.10)\quad 0<\operatorname{Re}\beta_1<\operatorname{Re}\gamma_1,\cdots,0<\operatorname{Re}\beta_n<\operatorname{Re}\gamma_n \qquad \text{for (1.7)},$$

$$(1.11)\quad 0<\operatorname{Re}\beta_1,\cdots,0<\operatorname{Re}\beta_n,\ 0<\operatorname{Re}(\gamma-\beta_1-\cdots-\beta_n) \qquad \text{for (1.8) and (1.9)}.$$

For each of the multiple integrals, the integrand is a product of powers of linear functions in $z_1,\cdots,z_n$. Let $L_1,\cdots,L_p$ be the hyperplanes in $\boldsymbol{C}^n$ obtained from the linear functions by equating them to zero. The region of integration is bounded by the intersections of $\boldsymbol{R}^n$ with some of the hyperplanes, say $L_1,\cdots,L_q$. Suppose that the point $(x_1,\cdots,x_n)\in\boldsymbol{C}^n$ is situated so that none of the other hyperplanes $L_{q+1},\cdots,L_p$ intersects the region of integration. Then it is not hard to find a contractible domain $\boldsymbol{D}$ in $\boldsymbol{C}^n$ which contains the region of integration but intersects none of $L_{q+1},\cdots,L_p$. As before, from the integrand restricted to $\boldsymbol{X}=\boldsymbol{D}-\bigcup_{j=1}^{q}L_j$, we obtain a Riemann domain $\tilde{\boldsymbol{X}}$ spread over $\boldsymbol{X}$ and a holomorphic n-form $\tilde{\omega}$ in a natural way. By definition $\tilde{\boldsymbol{X}}$ is a covering space of $\boldsymbol{X}$. We denote the n-form $\tilde{\omega}$ symbolically by

$$(1.12)\quad \prod_{j=1}^{n}\tilde{z}_j^{\beta_j-1}\prod_{j=1}^{n}(1-\tilde{z}_j)^{\gamma_j-\beta_j-1}(1-x_1\tilde{z}_1-\cdots-x_n\tilde{z}_n)^{-\alpha}d\tilde{z}_1\wedge\cdots\wedge d\tilde{z}_n\,,$$

$$(1.13)\quad \prod_{j=1}^{n}\tilde{z}_j^{\beta_j-1}(1-\tilde{z}_1-\cdots-\tilde{z}_n)^{\gamma-\beta_1-\cdots-\beta_n-1}\prod_{j=1}^{n}(1-x_j\tilde{z}_j)^{-\alpha_j}d\tilde{z}_1\wedge\cdots\wedge d\tilde{z}_n\,,$$

or

$$(1.14)\quad \prod_{j=1}^{n}\tilde{z}_j^{\beta_j-1}(1-\tilde{z}_1-\cdots-\tilde{z}_n)^{\gamma-\beta_1-\cdots-\beta_n-1}(1-x_1\tilde{z}_1-\cdots-x_n\tilde{z}_n)^{-\alpha}d\tilde{z}_1\wedge\cdots\wedge d\tilde{z}_n\,.$$

As in the case of one variable, in order to relax restriction (1.10) or (1.11), it is natural to find a suitable n-cycle in $\tilde{\boldsymbol{X}}$ and to get a relationship between each of integrals (1.17), (1.18) and (1.19) and the integral of the corresponding form (1.12), (1.13) or (1.14) over the n-cycle. To find an n-cycle in $\tilde{\boldsymbol{X}}$ we need the study of the topology of $\tilde{\boldsymbol{X}}$.

The purpose of this paper is to carry out the above plan. It is easy for integral (1.17) and so this case will be treated briefly in Section 2. The cases (1.8) and (1.9) will be studied in Section 3. Our main results are Theorem 3.1, Propositions 3.4 and 3.7 which describe the homotopy type of $\boldsymbol{X}$ and the homology of $\tilde{\boldsymbol{X}}$. The homology of $\boldsymbol{X}$ with coefficients in a local system is given in Section 4. In Sections 2$\sim$4 we shall tacitly suppose that n is greater than one.

In subsequent papers we shall study the topology of $\boldsymbol{C}^n$ minus a finite collection of hyperplanes in general position and various integral representations of hypergeometric functions in several complex variables.

For a general background on hypergeometric functions in several variables we refer to [1] and [2].

§2. Case of the function F_A.

Consider integral (1.7). The region of integration is the n-dimensional cube in $\boldsymbol{R}^n$ bounded by the $2n$ hyperplanes given by

$$L_j^0:\ z_j=0\,, \qquad j=1, \cdots, n$$

and

$$L_j^1:\ 1-z_j=0\,, \qquad j=1, \cdots, n\,.$$

Suppose that $(x_1, \cdots, x_n)\in\boldsymbol{C}^n$ is situated so that the hyperplane

$$1-x_1z_1-\cdots-x_nz_n=0$$

does not meet the region of integration. We can take as $\boldsymbol{D}$ the cartesian product of n copies of D, where D is the rectangle in $\boldsymbol{C}$ given in Section 1. Then $\boldsymbol{X}=\boldsymbol{D}-\bigcup_{j=1}^{n}(L_j^0\cup L_j^1)$.

Suppose that $1, \beta_1, \cdots, \beta_n, \gamma_1-\beta_1, \cdots, \gamma_n-\beta_n$ are linearly independent over the rationals. Then the Riemann domain of the integrand $\tilde{\boldsymbol{X}}$ is the covering space of $\boldsymbol{X}$ corresponding to the commutator subgroup of the fundamental group $\pi_1(\boldsymbol{X})$. Since $\boldsymbol{X}$ is equal to the n-fold cartesian product of $X=D-\{0, 1\}\subset\boldsymbol{C}$, the fundamental group $\pi_1(\boldsymbol{X})$ and its commutator subgroup are isomorphic to the n-fold direct products of the fundamental group $\pi_1(X)$ and its commutator subgroup respectively. It follows from this that the Riemann domain $\tilde{\boldsymbol{X}}$ is biholomorphic to the cartesian product of n copies of the Riemann surface $\tilde{X}$.

This fact enables us to solve our problem. Indeed, from the Künneth formula, the homology groups $H_q(\tilde{X}, \boldsymbol{Z})$ are given by

$$H_q(\tilde{X}, \boldsymbol{Z}) = \begin{cases} 0 & \text{for } q > n \\ \displaystyle\sum_{\substack{i_s=0 \text{ or } 1 \\ i_1+\cdots+i_n=q}} H_{i_1}(\tilde{X}, \boldsymbol{Z}) \otimes \cdots \otimes H_{i_n}(\tilde{X}, \boldsymbol{Z}) & \text{for } q \leqq n . \end{cases}$$

In particular,

$$H_n(\tilde{X}, \boldsymbol{Z}) = H_1(\tilde{X}, \boldsymbol{Z}) \otimes \cdots \otimes H_1(\tilde{X}, \boldsymbol{Z}) .$$

We thus obtain the

PROPOSITION 2.1. *The n-th homology group $H_n(\tilde{X}, \boldsymbol{Z})$ is a free $\boldsymbol{Z}(H_1(X, \boldsymbol{Z}))$-module on one generator. As a generator one can take the homology class represented by the n-dimensional torus $\tilde{T} = \tilde{C} \times \cdots \times \tilde{C}$ in $\tilde{X}$, where $\tilde{C}$ is the loop in $\tilde{X}$ given in Section* 1.

Concerning the relation between integral (1.7) and the integral of n-form (1.12) over $\tilde{T}$, we obtain the

PROPOSITION 2.2. *Under restriction* (1.10) *we have*

$$\int_0^1 \cdots \int_0^1 \prod_{j=1}^n z_j^{\beta_j-1} \prod_{j=1}^n (1-z_j)^{\gamma_j-\beta_j-1} (1-x_1 z_1 - \cdots - x_n z_n)^{-\alpha} dz_1 \cdots dz_n$$
$$= (-1)^n \prod_{j=1}^n (1-e^{2\pi i \beta_j})^{-1} (1-e^{2\pi i (\gamma_j-\beta_j)})^{-1} \int_{\tilde{T}} \tilde{\omega} ,$$

where $\tilde{\omega}$ denotes n-form (1.12).

The right-hand side of the above formula is well-defined if none of β_j and $\gamma_j - \beta_j$ is an integer. On the other hand, the left-hand side is meaningful under restriction (1.10). It follows that

$$\prod_{j=1}^n (1-e^{2\pi i \beta_j})^{-1} (1-e^{2\pi i (\gamma_j-\beta_j)})^{-1} \int_{\tilde{T}} \tilde{\omega}$$

is holomorphic with respect to all the parameters α, β_j and γ_j except at β_j or $\gamma_j - \beta_j = 0, -1, -2, \cdots$.

§ 3. Case of the functions F_B and F_D.

We proceed to integrals (1.8) and (1.9). The regions of integration are both the set in $\boldsymbol{R}^n$ bounded by the hyperplanes

$$L_j^0 : z_j = 0 , \qquad j = 1, \cdots , n$$

and the hyperplane

$$L_{n+1} : 1 - z_1 - \cdots - z_n = 0 .$$

Suppose that $(x_1, \cdots , x_n) \in \boldsymbol{C}^n$ is situated so that none of the hyperplanes

$$L_{x_j} : x_j z_j = 1$$

meets the region of integration in case (1.8) and the hyperplane

$$L_x : 1 - x_1 z_1 - \cdots - x_n z_n = 1$$

does not meet the region of integration in case (1.9). We can then take as $\boldsymbol{D}$ the open set consisting of those points $(z_1, \cdots, z_n)$ for which

$$u_1 > -\rho, \cdots, u_n > -\rho,$$
$$u_1 + \cdots + u_n < 1 + \rho,$$
$$|v_1| < \rho, \cdots, |v_n| < \rho,$$

where $z_j = u_j + iv_j$, $j = 1, \cdots, n$, and ρ is a sufficiently small positive number.

Suppose that $1, \beta_1, \cdots, \beta_n, \gamma - \beta_1 - \cdots - \beta_n$ are linearly independent over the rationals. Then $\tilde{X}$ is the covering space of $\boldsymbol{X} = \boldsymbol{D} - L_1^0 \cup \cdots \cup L_n^0 \cup L_{n+1}$ corresponding to the commutator subgroup of the fundamental group $\pi_1(\boldsymbol{X})$.

It is not difficult to see that the inclusion map $\boldsymbol{X} \to \boldsymbol{C}^n - L_1^0 \cup \cdots \cup L_n^0 \cup L_{n+1}$ is a homotopy equivalence. Therefore, we can suppose that $\boldsymbol{X} = \boldsymbol{C}^n - L_0^1 \cup \cdots \cup L_n^0 \cup L_{n+1}$ to study the homology of $\tilde{X}$. Let $\boldsymbol{Y}$ be the subspace of $\boldsymbol{C}^{n+1}$ consisting of those points $w = (w_1, \cdots, w_{n+1})$ for which none of w_j is zero, that is to say,

$$\boldsymbol{Y} = \boldsymbol{C}^* \times \cdots \times \boldsymbol{C}^* \qquad ((n+1)\text{-fold product}),$$

where $\boldsymbol{C}^* = \boldsymbol{C} - \{0\}$. Then $\boldsymbol{X}$ is affinely mapped onto the intersection of $\boldsymbol{Y}$ and the hyperplane

$$L : w_1 + \cdots + w_{n+1} = 1$$

by the affine transformation

$$\text{(3.1)} \qquad \begin{cases} w_1 = z_1, \\ \vdots \\ w_n = z_n, \\ w_{n+1} = 1 - z_1 - \cdots - z_n. \end{cases}$$

Hereafter we identify $\boldsymbol{X}$ with $\boldsymbol{Y} \cap L$ through this transformation:

$$\boldsymbol{X} = \{w = (w_1, \cdots, w_{n+1}) \mid w \in \boldsymbol{Y}, \ w_1 + \cdots + w_{n+1} = 1\}.$$

Denote by $\boldsymbol{X}_0$ the subspace of the $(n+1)$-dimensional torus $T^{n+1} = S^1 \times \cdots \times S^1$ ($(n+1)$-fold product) consisting of points $(w_1, \cdots, w_{n+1})$ such that at least one of the coordinates equals 1. We also denote by $\tilde{\boldsymbol{X}}_0$ the subspace of $\boldsymbol{R}^{n+1}$ consisting of points $(t_1, \cdots, t_{n+1})$ such that at least one of the coordinates is an integer. Then the map p_0 given by

$$p_0(t_1, \cdots, t_{n+1}) = (e^{2\pi i t_1}, \cdots, e^{2\pi i t_{n+1}})$$

is a covering map $\tilde{\boldsymbol{X}}_0 \to \boldsymbol{X}_0$. We shall prove the

THEOREM 3.1. *The space $\boldsymbol{X}$ has the same homotopy type as $\boldsymbol{X}_0$.*

PROOF. The proof will be divided into three steps.

First step. We shall write an element $w=(w_1, \cdots, w_{n+1})$ in $\boldsymbol{C}^{n+1}$ as $w=(u, v)$, where $u=(u_1, \cdots, u_{n+1})$, $v=(v_1, \cdots, v_{n+1})$ and $w_j=u_j+iv_j$, $j=1, \cdots, n+1$. Putting

$$X^*=\{w=(u, v) \mid w\in Y,\ u_1+\cdots+u_{n+1}=1\}\ ,$$

we shall prove the

Assertion 1. X is a deformation retract of X^*.

Define a homotopy $f_t\colon X^*\to X^*$, $0\leqq t\leqq 1$, by

$$\text{(3.2)}\qquad \begin{cases} f_t(u, v)=(u, v')\,, \\ v'_j=v_j-t(v_1+\cdots+v_{n+1})u_j\,, \qquad j=1, \cdots, n+1\,. \end{cases}$$

We shall show that f_1 is a deformation retraction $X^*\to X$. Indeed, we have, for $t=1$, $\sum_{j=1}^{n+1} v'_j=\sum_{j=1}^{n+1} v_j-\sum_{j=1}^{n+1} v_j\cdot\sum_{j=1}^{n+1} u_j=0$, whence $f_1(X^*)\subset X$. It is obvious that f_0 is the identity map on X^* and that for $(u, v)\in X$ we have $f_t(u, v)=(u, v)$, $0\leqq t\leqq 1$. This proves Assertion 1.

Second step. We define X_0^* to be the subspace of X^* consisting of those points $w=(u, v)$ for which $v_j=0$ if $u_j=\max_k u_k$. Observe that $\max_k u_k>0$ since $\sum_{j=1}^{n+1} u_j=1$.

Assertion 2. The inclusion map $i_0\colon X_0^*\subset X^*$ is a homotopy equivalence.

To prove this, consider the homotopy $g_t\colon X^*\to X^*$, $0\leqq t\leqq 1$, defined by

$$\text{(3.3)}\qquad \begin{cases} g_t(u, v)=(u, v')\,, \\ v'_j=\Bigl(1-t\dfrac{u_j}{\max_k u_k}\Bigr)v_j\,, \qquad j=1, \cdots, n+1\,. \end{cases}$$

Since g_t keeps X_0^* invariant and $g_1(X^*)\subset X_0^*$, $g_t\circ i_0$ is a homotopy between the identity on X_0^* and $g_1\circ i_0$. Of course, g_t is a homotopy between the identity on X^* and $i_0\circ g_1=g_1$. Thus the inclusion i_0 is a homotopy equivalence.

Third step. Consider the usual projection

$$r=r_1\times\cdots\times r_{n+1}\colon Y\longrightarrow T^{n+1}\,,$$

where $r_j(w_j)=w_j/|w_j|$. It is clear that the image of X_0^* under r is the space X_0 defined before. We shall prove the

Assertion 3. The restriction of r to X_0^*, $r_0\colon X_0^*\to X_0$, is a homotopy equivalence.

To prove this, it is sufficient to construct a cross section $s\colon X_0\to X_0^*$ of r_0 and a homotopy between the identity on X_0^* and $s\circ r_0$. First define functions $h_1, h_2, \cdots, h_{n+1}\colon X_0\to \boldsymbol{R}$ by

$$h_j(w)=\begin{cases}\delta & \text{when } u_j \leqq 0\\ \delta+nu_j & \text{when } u_j>0\end{cases}\tag{3.4}$$

where $w=(u, v)$ as before and δ is a positive number not exceeding 1. Next define $h: \boldsymbol{X}_0 \to \boldsymbol{R}$ by

$$h(w)=\sum_{j=1}^{n+1} h_j(w)u_j\,.$$

The fact that some of w_j are equal to unity implies that $0<\delta \leqq h_j(w) \leqq \delta+n$, $j=1, \cdots, n+1$, and $0<\delta+n(1-\delta) \leqq h(w) \leqq (\delta+n)(n+1)$. Finally define a cross-section s by

$$s(w)=\Big(\frac{h_1(w)}{h(w)}w_1, \cdots, \frac{h_{n+1}(w)}{h(w)}w_{n+1}\Big)\,.\tag{3.5}$$

To see that s is really a cross section, it is sufficient to check that s is a map $X_0 \to \boldsymbol{X}_0^*$ because clearly $r \circ s$ is the identity. We have

$$\sum_{j=1}^{n+1} \mathrm{Re}\Big(\frac{h_j(w)}{h(w)}w_j\Big)=\sum_{j=1}^{n+1}\frac{h_j(w)u_j}{h(w)}=1\,.$$

If some of w_j, say $w_1, \cdots, w_k$, are equal to one, then $h_1(w)=\cdots=h_k(w)>h_j(w)$, $j=k+1, \cdots, n+1$, from which

$$\mathrm{Re}\Big(\frac{h_1(w)}{h(w)}w_1\Big)=\cdots=\mathrm{Re}\Big(\frac{h_k(w)}{h(w)}w_k\Big)>\mathrm{Re}\Big(\frac{h_j(w)}{h(w)}w_j\Big)\,, \qquad j=k+1, \cdots, n+1$$

and

$$\mathrm{Im}\Big(\frac{h_1(w)}{h(w)}w_1\Big)=\cdots=\mathrm{Im}\Big(\frac{h_k(w)}{h(w)}w_k\Big)=0\,.$$

Thus $s(w) \in \boldsymbol{X}_0^*$. As a homotopy between the identity and $s \circ r_0$ we can take

$$w \longmapsto (1-t)w+ts \circ r_0(w)\,.$$

This completes the proof of Assertion 3.

Combining Assertions 1, 2 and 3, we conclude that $\boldsymbol{X}$ is homotopically equivalent to $\boldsymbol{X}_0$, and moreover that $r_0 \circ g_1: \boldsymbol{X} \to \boldsymbol{X}_0$ and $f_1 \circ s: \boldsymbol{X}_0 \to \boldsymbol{X}$ are homotopy equivalences which are inverse of each other. (Cf. (3.2), (3.3) and (3.5).)

Theorem 3.1 is thus established.

The investigation of homological properties of $\tilde{\boldsymbol{X}}$ is, by Theorem 3.1, reduced to that of homological properties of $\tilde{\boldsymbol{X}}_0$, which are easy to describe. For this purpose we need the

PROPOSITION 3.2. *Let $p_0: \tilde{\boldsymbol{X}}_0 \to \boldsymbol{X}_0$ be the same as before. Then $\pi_1(\boldsymbol{X}_0)$ is a free abelian group on $n+1$ generators and the covering map p_0 is universal.*

PROOF. Consider the usual universal covering map $p_1: \boldsymbol{R}^{n+1} \to T^{n+1}$, for which we have $p_1|\tilde{\boldsymbol{X}}_0=p_0$. The fundamental group $\pi_1(T^{n+1})$ is a free abelian group of rank $n+1$ and operates on $\boldsymbol{R}^{n+1}$ as the group of covering transformations. It is easy to see that the homomorphism $\pi_1(X_0)\to\pi_1(T^{n+1})$ induced by the inclusion $X_0\subset T^{n+1}$ is surjective. It follows from the general theory of covering spaces that the covering space $\tilde{X}_0$ corresponds to the kernel K of the homomorphism $\pi_1(X_0)\to\pi_1(T^{n+1})$ and its group of covering transformations is identified with $\pi_1(X_0)/K=\pi_1(T^{n+1})$. It is easy to see that $\tilde{X}_0$ is simply connected since $n>1$ and hence the covering map $p_0: \tilde{X}_0\to X_0$ is universal.

This completes the proof.

As to the homology of $\tilde{X}_0$, the following proposition is immediate.

PROPOSITION 3.3. *The homology groups $H_q(\tilde{X}_0, \boldsymbol{Z})$ vanish for $q\neq 0$, n. $H_n(\tilde{X}_0, \boldsymbol{Z})$ is a free $\boldsymbol{Z}(H_1(X_0, \boldsymbol{Z}))$-module with one generator. As a generator we may take the class represented by the topological n-sphere Σ^n in $\tilde{X}_0$ given by*

$$\Sigma^n=\{(t_1, \cdots, t_{n+1})\in \boldsymbol{R}^{n+1} \mid 0\leqq t_i\leqq 1 \text{ and some } t_i \text{ equals } 0 \text{ or } 1\}\ .$$

From Theorems 3.1 and 3.3 we obtain

PROPOSITION 3.4. *Let X and $\tilde{X}$ be the same as before. Then the n-th homology group $H_n(\tilde{X}, \boldsymbol{Z})$ is a free $\boldsymbol{Z}(H_1(X, \boldsymbol{Z}))$-module with one generator which is represented by a topological n-sphere $\tilde{S}$ in $\tilde{X}$ corresponding to Σ^n.*

We notice that the method of calculating $H_n(\tilde{X}, \boldsymbol{Z})$ is modeled on that of calculating $H_1(\tilde{X}^1, \boldsymbol{Z})$, where $X^1=\boldsymbol{C}-\{0, 1\}$. The only difference consists in the fact that $\pi_1(X_0^1)$ is a free group with two generators and hence that K is not trivial.

REMARK 3.5. Consider the usual universal covering map $p: \boldsymbol{C}^{n+1}\to(\boldsymbol{C}^*)^{n+1}$ defined by $p(w_1, \cdots, w_{n+1})=(e^{2\pi i w_1}, \cdots, e^{2\pi i w_{n+1}})$, whose restriction to $\tilde{X}_0$ coincides with $p_0: \tilde{X}_0\to X_0$ defined above. It is easy to see that $p^{-1}(X)$ and $p^{-1}(X^*)$ are both universal covering spaces of X and X^* respectively and that we obtain the following commutative diagram

$$\begin{tikzcd}
\tilde{X}_0 \arrow[r, "\tilde{s}"] \arrow[d, "p_0"'] & p^{-1}(X^*) \arrow[r, "\tilde{f}_1"] \arrow[d] & p^{-1}(X) \arrow[d, "p"] \\
X_0 \arrow[r, "s"] & X^* \arrow[r, "f_1"] & X
\end{tikzcd}$$

where $\tilde{s}$ and $\tilde{f}_1$ are the homotopy equivalences determined by s and f_1 respectively in an obvious way. The above observation shows that *the covering space $\tilde{X}$ can be identified with $p^{-1}(X)$ and therefore $\tilde{X}$ is embeddable as a closed analytic submanifold of complex codimension* 1 *in $\boldsymbol{C}^{n+1}$.*

REMARK 3.6. The composite map $\tilde{\varphi}=\tilde{f}_1\circ\tilde{s}: \tilde{X}_0\to p^{-1}(X)$ is expressible in terms of the coordinates $(t_1, \cdots, t_{n+1})$ of a point $t\in\tilde{X}_0$ and those $(\tilde{w}_1, \cdots, \tilde{w}_{n+1})$ of a point $\tilde{w}\in p^{-1}(X)$ as

$$\tilde{w}_j=\frac{1}{2\pi i}\log\Big[\frac{h_j(p_0(t))}{h(p_0(t))}\Big(e^{2\pi i t_j}-i\cos 2\pi t_j\sum_{k=1}^{n+1}\frac{h_k(p_0(t))}{h(p_0(t))}\sin 2\pi t_k\Big)\Big],$$
$$j=1, 2, \cdots, n+1,$$

where we take the branch of log such that for $(0, \cdots, 0)$ every $2\pi i w_j$ assumes real numbers. We have similarly

$$w_j=\frac{h_j(p_0(t))}{h(p_0(t))}\Big(e^{2\pi i t_j}-i\cos 2\pi t_j\sum_{k=1}^{n+1}\frac{h_k(p_0(t))}{h(p_0(t))}\sin 2\pi t_k\Big)$$

for $\varphi=p\circ\tilde{\varphi}=(w_1, \cdots, w_{n+1})$.

We note that if all t_j are integers, then we have

$$\tilde{w}_j(t_1, \cdots, t_n)=\frac{1}{2\pi i}\log\frac{1}{n+1}+t_j, \qquad j=1, \cdots, n+1,$$

$$w_j(t_1, \cdots, t_n)=\frac{1}{n+1}, \qquad j=1, \cdots, n+1,$$

and that if we let t_j vary from 0 to 1 holding the other t_k fixed as 0 or 1, then $\varphi(t_1, \cdots, t_{n+1})$ describes a curve in C^{n+1} as indicated in Fig. 4.

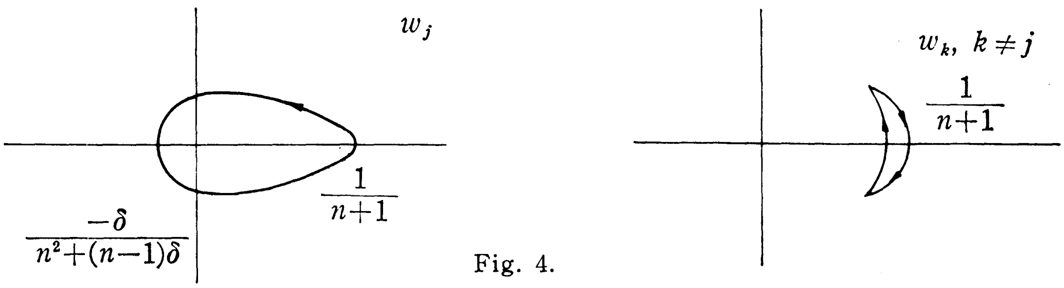

Fig. 4.

Finally we investigate the integral over the n-sphere $\tilde{S}=\tilde{\varphi}(\Sigma^n)$ representing the generator of $H_n(\tilde{X}, Z)$ considered as a free $Z(H_1(X, Z))$-module. Let $\varepsilon=(\varepsilon_1, \cdots, \varepsilon_{n+1})$, $\varepsilon_j=0$ or 1, be any one of 2^{n+1} vertices of Σ^n and for a real number ρ', $0\leqq\rho'\leqq 1/4$, denote by $\Delta_{\varepsilon,\rho'}$ the part of Σ^n defined by

$$\Delta_{\varepsilon,\rho'}=\{(t_1, \cdots, t_{n+1})\mid |t_j-\varepsilon_j|\leqq 1/4-\rho'\}\cap\Sigma^n.$$

Let ρ be a small positive number and define subsets A_ρ and B_ρ of X by

$$A_\rho=X\cap\{(w_1, \cdots, w_{n+1})\mid w_j \text{ are real and } w_j\geqq\rho \text{ for all } j\},$$

$$B_\rho=X\cap\{(w_1, \cdots, w_{n+1})\mid |w_j|\leqq\rho \text{ for some } j\}.$$

Then we shall find a piecewise smooth map: $\tilde{\psi}\colon \Sigma^n\to p^{-1}(X)$ with the following properties:

1) $\tilde{\psi}$ is homotopic to $\tilde{\varphi}$,
2) $p\circ\tilde{\psi}$ maps each $\Delta_{\varepsilon,\rho'}$ bijectively onto A_ρ and $\Sigma^n-\bigcup_{\varepsilon}\Delta_{\varepsilon,\rho'}$ into B_ρ,

where ρ' is a suitable small number which tends to 0 as $\rho \to 0$.

Let N be a positive number greater than

$$\max\{v_j \mid w_j = u_j + iv_j,\ (w_1, \cdots, w_{n+1}) \in \varphi(\Sigma^n)\}\ .$$

Let V be the vector subspace of $\boldsymbol{R}^{n+1}$ defined by the linear equation

$$v_1 + \cdots + v_{n+1} = 0$$

and let V_r be the subset of V defined by $|v_j| \leqq r$ for all j. Let $\dot{V}_r$ denote the subset of V_r defined by $|v_j| = r$ for some j. Note that, through the identification $w = (u; v)$, $\boldsymbol{X}$ is contained in $\boldsymbol{R}^{n+1} \times V$ and $\varphi(\Sigma^n)$ is contained in $\boldsymbol{R}^{n+1} \times V_N$. We denote by $\boldsymbol{R}^+$ the set of non-negative real numbers. Then it is not difficult to construct a map $\mu : \boldsymbol{R}^+ \times V \to V$ such that

1)′ μ is homotopic to the projection into the second factor,

2)′ putting $\mu_x(v) = \mu(x, v)$ we have

$$\begin{cases} \mu_x(V_N - V_{\rho-x}) = \dot{V}_{\rho-x}\,, \\ \mu_x | V_{\rho-x} = \text{identity} \end{cases}$$

for $x \leqq \rho$; $\mu_x(V_N) = 0$ for $x \geqq \rho$.

Next define a map d: $\boldsymbol{X} \to \boldsymbol{X}$ by

$$d(u, v) = (u, \mu(\min|u_j|, v))\,.$$

By the condition 1)′ d is homotopic to the identity. Moreover, if the number δ used in the definition of the cross-section s (see (3.4)) is sufficiently small, then, by the condition 2)′, $d \circ \varphi(\Sigma^n)$ is contained in $A_\rho \cup B_\rho$ and also in $\boldsymbol{D}$ as defined in the beginning of this section. It is not difficult to see that the set $(d \circ \varphi)^{-1}(A_\rho) \cap \Delta_{\varepsilon,0}$ is of the form $\Delta_{\varepsilon,\rho'}$ for some small ρ', and $d \circ \varphi$ maps $\Delta_{\varepsilon,\rho'}$ bijectively onto A_ρ.

We set $\psi = d \circ \varphi$. Since d is homotopic to the identity, by the covering homotopy property, there is a map $\tilde{\psi}$ satisfying 1), 2) and such that $\psi = p \circ \tilde{\psi}$.

Consider the integral of n-form (1.13) or (1.14) over $\tilde{\psi}(\Sigma^n)$:

$$\int_{\tilde{\psi}(\Sigma^n)} \tilde{\omega}\,.$$

We write it as

$$\sum_{\varepsilon} \int_{\tilde{\psi}(\Delta_{\varepsilon,\rho'})} \tilde{\omega} + \int_{\tilde{\psi}(\Sigma^n - \bigcup_{\varepsilon} \Delta_{\varepsilon,\rho'})} \tilde{\omega}\,.$$

Taking account of (3.1) and the orientation of $\Delta_{\varepsilon,\rho'}$ induced from that of Σ^n we see that the first sum of the above integrals is equal to

$$\pm(1-e^{2\pi i\beta_1}) \cdots (1-e^{2\pi i\beta_n})(1-e^{2\pi i(\gamma-\beta_1-\cdots-\beta_n)}) \int\cdots\int_{A_\rho} g(z_1, \cdots, z_n)\,dz_1 \cdots dz_n$$

where $g(z_1, \cdots, z_n)$ stands for the integrand in (1.8) or (1.9). On the other hand, we have

$$\int \cdots \int_{\widetilde{\varphi}(\Sigma^n - \underset{\varepsilon}{\cup} \Delta_{\varepsilon,\rho'})} g(z_1, \cdots, z_n) dz_1 \cdots dz_n \longrightarrow 0 \qquad (\rho \to 0)$$

under the restriction (1.11).

Thus we arrive at the

PROPOSITION 3.7. *Under* (1.11) *and with a suitable orientation of* $\tilde{S}$ *we have*

$$\begin{aligned}\int \cdots \int_{z_1, \cdots, z_n, 1-z_1-\cdots-z_n \geqq 0} & z_1^{\beta_1-1} \cdots z_n^{\beta_n-1}(1-z_1-\cdots-z_n)^{\gamma-\beta_1-\cdots-\beta_n-1}(1-x_1z_1)^{-\alpha_1} \\ & \cdots (1-x_nz_n)^{-\alpha_n} dz_1 \cdots dz_n \\ = \frac{1}{(1-e^{2\pi i(\gamma-\beta_1-\cdots-\beta_n)}) \prod_{j=1}^{n} (1-e^{2\pi i \beta_j})} & \int_{\tilde{S}} \tilde{z}_1^{\beta_1-1} \cdots \tilde{z}_n^{\beta_n-1} \\ \cdot (1-\tilde{z}_1-\cdots-\tilde{z}_n)^{\gamma-\beta_1-\cdots-\beta_n-1} & (1-x_1\tilde{z}_1)^{-\alpha_1} \cdots (1-x_n\tilde{z}_n)^{-\alpha_n} d\tilde{z}_1 \wedge \cdots \wedge d\tilde{z}_n\end{aligned}$$

and

$$\begin{aligned}\int \cdots \int_{z_1, \cdots, z_n, 1-z_1-\cdots-z_n \geqq 0} & z_1^{\beta_1-1} \cdots z_n^{\beta_n-1}(1-z_1-\cdots-z_n)^{\gamma-\beta_1-\cdots-\beta_n-1}(1-x_1z_1-\cdots-x_nz_n)^{-\alpha} dz_1 \cdots dz_n \\ = \frac{1}{(1-e^{2\pi i(\gamma-\beta_1-\cdots-\beta_n)}) \prod_{j=1}^{n} (1-e^{2\pi i \beta_j})} & \int_{\tilde{S}} \tilde{z}_1^{\beta_1-1} \cdots \tilde{z}_n^{\beta_n-1} \\ \cdot (1-\tilde{z}_1-\cdots-\tilde{z}_n)^{\gamma-\beta_1-\cdots-\beta_n-1} & (1-x_1\tilde{z}_1-\cdots-x_n\tilde{z}_n)^{-\alpha} d\tilde{z}_1 \wedge \cdots \wedge d\tilde{z}_n .\end{aligned}$$

§ 4. Homology with coefficients in local systems.

The case of F_A being easy we shall only treat the case of F_B and F_D. We assume for simplicity that n is greater than 1. Let $\beta_1, \cdots, \beta_n$ and γ be complex numbers. Let l_j, $1 \leqq j \leqq n+1$, be the loop in $\boldsymbol{X}_0$ defined by

$$l_j(t) = (1, \cdots, 1, e^{2\pi i t}, 1, \cdots, 1)$$

where $e^{2\pi i t}$ is placed at the j-th factor. The l_j define homotopy classes in $\pi_1(\boldsymbol{X}_0) = H_1(\boldsymbol{X}_0)$ which we shall also denote by the same letters. Then the l_j are generators of the free abelian group $\pi_1(\boldsymbol{X}_0)$. Consider the representation θ: $\pi_1(\boldsymbol{X}_0) \to \boldsymbol{C}^*$ given by

$$\theta(l_j) = e^{2\pi i \beta_j}, \qquad 1 \leqq j \leqq n,$$
$$\theta(l_{n+1}) = e^{2\pi i(\gamma-\beta_1-\cdots-\beta_n)}.$$

Let $\mathcal{S}_0$ be the local system with stalk $\boldsymbol{C}$ associated to θ. We shall also consider the local system $\mathcal{S} = (f_1 \circ s)^* \mathcal{S}_0$ and $\mathcal{S}^{-1}$ on $\boldsymbol{X}$ where $f_1 \circ s$ is the homotopy equivalence given in Section 3. Note that the loops $f_1 \circ s \circ l_j$ have the form as described in Fig. 4, Section 3. Just as in Introduction if follows that the differential form (1.13) or (1.14) can be viewed as an element of $H^n(X, \mathcal{S}^{-1})$.

PROPOSITION 4.1. *Suppose that at least one of $\beta_1, \cdots, \beta_n, \beta_{n+1}=\gamma-\beta_1-\cdots-\beta_n$ is not an integer, i. e. the local system $\mathcal{S}$ is not trivial. Then the homology groups $H_q(X, \mathcal{S})$ vanish for $q\neq n$. $H_n(X, \mathcal{S})$ has dimension* 1 *and is generated by the image of the homology class of $\tilde{S}=\tilde{\varphi}(\Sigma)$ in $\tilde{X}$.*

PROOF. Let $p_0: \tilde{X}_0\to X_0$ be the universal covering map as given in Section 3. We shall mean by a k-wall of $\tilde{X}_0$ an open k-cube in $\tilde{X}_0$ of the form

$$\{(t_1, \cdots, t_{n+1})\in\tilde{X}_0 \mid n_i<t_i<n_i+1 \text{ for } i\in K \text{ and } t_j\in \boldsymbol{Z} \text{ for } j\notin K\}$$

where K is a subset of $\{1, \cdots, n+1\}$ consisting of k elements and n_i's are integers. Let $C^*(\tilde{X}_0)=\sum C_k(\tilde{X}_0)$ be the chain complex in which $C_k(\tilde{X}_0)$ is the free abelian group generated by the oriented k-walls in $\tilde{X}_0$ for $0\leqq k\leqq n$ and $C_k(\tilde{X}_0)=0$ for $k<0$ or $n<k$ and the boundary is the usual one. The complex $C_*(\tilde{X}_0)$ gives rise to the homology of $\tilde{X}_0$ as is well known. Moreover the fundamental group $\pi_1(X_0)$ acts upon $C_*(\tilde{X}_0)$ on the right freely and if σ is a k-wall then $p_0\sigma$ is an open cell in X_0. It follows that the complex $C_*(\tilde{X}_0)\otimes_\theta \boldsymbol{C}$ gives rise to the homology $H_*(X_0, \mathcal{S})$. More precisely let $\pi_1(X_0)$ act upon $C_*(\tilde{X}_0)\otimes\boldsymbol{C}$ on the left by

$$\xi\cdot c\otimes\alpha=c\xi^{-1}\otimes\theta(\xi)\alpha$$

and let $Q_*=\sum_k Q_k$ be the subcomplex of $C_*(\tilde{X}_0)\otimes\boldsymbol{C}$ generated by the elements $(1-\xi)x$, $\xi\in\pi_1(X_0)$, $x\in C_*(\tilde{X}_0)\otimes\boldsymbol{C}$. The complex $C_*(\tilde{X}_0)\otimes_\theta\boldsymbol{C}$ is by definition $C_*(\tilde{X}_0)\otimes\boldsymbol{C}/Q_*$. It follows that $C_*(\tilde{X}_0)\otimes_\theta\boldsymbol{C}$ has a basis $\{\sigma_K\}$ where K ranges over a subset of $\{1, \cdots, n+1\}$ such that $0\leqq|K|<n+1$, $|K|$ being the number of the set K, and the boundary operator ∂ is given by

$$\partial\sigma_K=\sum_{j\in K}\varepsilon(K, j)(1-e^{2\pi i\beta_j})\sigma_{K-\{j\}}$$

where

$$\varepsilon(K, j)=(-1)^s$$

if $K=\{j_1, \cdots, j_s, \cdots, j_{|K|}\}$ with $j_1<\cdots<j_s<\cdots<j_{|K|}$ and $j=j_s$.

Now assume that one of β_j, say β_{n+1}, is not an integer. We define a homotopy operator $s: C_k(\tilde{X}_0)\otimes_\theta\boldsymbol{C}\to C_{k+1}(\tilde{X}_0)\otimes_\theta\boldsymbol{C}$ for $k<n$ by

$$s(\sigma_K)=\begin{cases}\dfrac{(-1)^{|K|+1}}{1-e^{2\pi i\beta_{n+1}}}\sigma_{K\cup\{n+1\}}, & \text{if } n+1\notin K,\\[2ex] 0, & \text{if } n+1\in K.\end{cases}$$

It is not hard to verify the relation $\partial s+s\partial=1$ on $C_k(\tilde{X}_0)\otimes_\theta\boldsymbol{C}$ for $0\leqq k<n$. It follows that $H_k(X_0, \mathcal{S}_0)=0$ for $k\neq n$.

Next we put $K_j=\{1, 2, \cdots, n+1\}-\{j\}$ for $1\leqq j\leqq n+1$ and $\sigma_j=\sigma_{K_j}$. $\{\sigma_j\}$ is a basis of $C_n(\tilde{X}_0)\underset{\theta}{\otimes}\boldsymbol{C}$. Assuming that β_{n+1} is not an integer it is easy to see that $\sum_{j=1}^{n+1}a_j\sigma_j$ is a cycle if and only if

$$a_j=(-1)^j(1-e^{2\pi i\beta_j})a_{n+1}/(-1)^{n+1}(1-e^{2\pi i\beta_{n+1}}).$$

It follows that $H_n(X_0, \mathcal{S}_0)$ has dimension 1. Moreover the image of Σ is $\pm\sum_{j=1}^{n+1}(-1)^j(1-e^{2\pi i\beta_j})\sigma_j$ and hence its homology class generates $H_n(X_0, \mathcal{S}_0)$. Applying the homotopy equivalence $f_1\circ s$ we obtain the results.

From Proposition 4.1 we obtain immediately the

COROLLARY 4.2. *If at least one of* $\beta_1, \cdots, \beta_n, \gamma-\beta_1-\cdots-\beta_n$ *is not an integer then the cohomology groups* $H^k(X, \mathcal{S}^{-1})$ *vanish except for* $k=n$ *while* $H^n(X, \mathcal{S}^{-1})$ *has dimension* 1. *If none of* $\beta_1, \cdots, \beta_n, \gamma-\beta_1-\cdots-\beta_n$ *is an integer then* $H^n(X, \mathcal{S}^{-1})$ *is generated by*

$$\tilde{z}_1^{\beta_1-1}\cdots\tilde{z}_n^{\beta_n-1}(1-\tilde{z}_1-\cdots-\tilde{z}_n)^{\gamma-\beta_1-\cdots-\beta_n-1}\prod_{j=1}^{n}(1-x_j\tilde{z}_j)^{-\alpha_j}d\tilde{z}_1\wedge\cdots\wedge d\tilde{z}_n$$

or

$$\tilde{z}_1^{\beta_1-1}\cdots\tilde{z}_n^{\beta_n-1}(1-\tilde{z}_1-\cdots-\tilde{z}_n)^{\gamma-\beta_1-\cdots-\beta_n-1}(1-x_1\tilde{z}_1-\cdots-x_n\tilde{z}_n)^{-\alpha}d\tilde{z}_1\wedge\cdots\wedge d\tilde{z}_n.$$

REMARK. In Section 3 we have assumed that $1, \beta_1, \cdots, \beta_n, \gamma-\beta_1-\cdots-\beta_n$ are linearly independent over the rationals for the sake of simplicity. Suppose that they are linearly dependent over the rational numbers. Then the kernel G of the homomorphism $\theta: \pi_1(X)\rightarrow\boldsymbol{C}^*$ is a non-trivial subgroup and the function $z_1^{\beta_1-1}\cdots z_n^{\beta_n-1}(1-z_1-\cdots-z_n)^{\gamma-\beta_1-\cdots-\beta_n-1}$ is uniformized by the covering space $\tilde{X}_G=\tilde{X}/G$, i.e. the manifold $\tilde{X}_G$ is the Riemann domain associated to that function. The homology $H_*(\tilde{X}_G, \boldsymbol{Z})$ can also be computed. We remark that the homomorphism $p_*: H_*(\tilde{X}, \boldsymbol{Z})\rightarrow H_*(\tilde{X}_G, \boldsymbol{Z})$ is not surjective and moreover, if at least one of $\beta_1, \cdots, \beta_n, \gamma-\beta_1-\cdots-\beta_n$ is not an integer then $p_*(\Sigma)$ does not vanish.

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