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On the Hecke operators for $\Gamma_0(N)$ and class fields over quadratic number fields

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Introduction.

It was shown by Shimura [3, § 7.7], [5] that the eigen-values of Hecke operators for the cusp forms of "Neben"-type (in Hecke's sense) are closely connected with the reciprocity law in certain abelian extensions of a real quadratic field, and such extensions can be generated by the coordinates of certain points of finite order on an abelian variety associated with the cusp forms. Especially, in [5], *some fundamental theorems* about a class-field-theoretical treatment of these extensions in the case of arbitrary levels, and various detailed examples in the case of square-free levels were given. As a continuation of this theory, we are naturally led to investigate the eigen-values of Hecke operators for the cusp forms of an arbitrary level, especially, the case in which the level is divisible by a prime power p^n $(n>1)$. Recently, H. Hijikata [1] has succeeded in extending the result of Eichler (the trace formula for Hecke operators) to arbitrary levels including both "Haupt" and "Neben"-types, and moreover, applying this Hijikata's formula, in [7], one of the authors of the present note has given an explicit trace formula for a certain restricted part of the space of cusp forms of "Haupt"-type for arbitrary levels. By means of these formulae, we can obtain some numerical eigen-values of Hecke operators for the "essential part" (see T. Miyake [2] and Shimura [5, p. 133]) of the spaces. Though Shimura [5] considered only the cases of "Neben"-type, looking at [3, Prop. 3.64] and [5, § 9] carefully, we can also expect to develop the idea of [5] in the case of "Haupt"-type if levels are divisible by a higher power of a prime. Actually, Shimura [6] indicates this possibility by giving *a twisting operator* and an abelian variety associated to the cusp forms of weight 2 of "Haupt"-type analogous to "Neben"-type (see text or [6]). Now one of the aims of the present note (§ 1 and § 2 below) is to investigate this abelian variety. More precisely, take a cusp form $f(z)$ (which is a common eigen-function of Hecke operators) of weight 2 with respect to $\Gamma_0(p^n)$, $n>1$ of "Haupt"-type. By applying the result of [6] to the group

$$\Gamma=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_0(p^n)\,\middle|\, a\equiv d\equiv 1 \pmod{p}\right\},$$

we obtain an abelian variety A associated to $f(z)$ as a factor of the jacobian variety of $\mathfrak{H}^*/\Gamma$, and the twisting endomorphism η of A. Here $\mathfrak{H}^*$ means the union of the complex upper half plane

$$\mathfrak{H}=\{z\in\boldsymbol{C}\mid \mathrm{Im}(z)>0\}$$

and the cusps of Γ. It can be observed that under a certain condition on $f(z)$, we can obtain A rational over $\boldsymbol{Q}$ and η rational over a quadratic extension k of $\boldsymbol{Q}$, which form a system similar to the type of abelian varieties discussed in [5, § 9]. (Note that in the present case, the field K generated over $\boldsymbol{Q}$ by the Fourier coefficients of $f(z)$ in question, is not a CM-field.) In § 2, by giving a few examples ($N=p^3$, $p=5$ and 7), we shall discuss some arithmetical properties of A corresponding to $f(z)$ and show that the coordinates of some specific points of finite order on such an A can generate an abelian extension of k. (The field k can be either real or imaginary.)

As an addition to the various examples in [5], we shall discuss in § 3 more examples for which the level is divisible by the smallest prime power 2^2. (As mentioned at the beginning, such a case is not included in the examples of [5].) Namely, we shall consider the space $S_2(4M,(\frac{M}{\cdot}))$ for several primes $M\equiv 1 \pmod 4$. Here we denote by $S_2(4M,(\frac{M}{\cdot}))$ the space of all cusp forms of "Neben"-type of weight 2 with respect to $\Gamma_0(4M)$, and $(\frac{M}{\cdot})$ the quadratic residue symbol. Repeating the same procedure as in [5], here we also obtain a certain abelian extension of the real quadratic field $\boldsymbol{Q}(\sqrt{M})$. One should notice that, in these cases, the conductor of such an extension is divisible by a prime factor of 2, and although the present observation deals with only the cases of level $=4M$, it seems that these are typical enough in extending the same investigation* for the levels containing a factor of another prime power p^n.

The authors would like to express their hearty thanks to Professor G. Shimura who indicated them the explicit way how to construct the class fields in the present paper by showing his manuscript of [6]; and to Dr. T. Miyake for his valuable discussions during the preparation of the paper.

§ 1. A few facts of A from [6] for $\Gamma_0(p^n)$.

For a positive integer N, put

* One can find a few more examples of the characteristic polynomials of Hecke operators for the level $=3^2\cdot M$ in H. Hijikata's article on the "Seminar on modern methods in Number theory", Tokyo, (1971).

$$\Gamma_0(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\boldsymbol{Z}) \,\middle|\, c \equiv 0 \pmod{N}\right\},$$

$$\Gamma_1(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \,\middle|\, a \equiv d \equiv 1 \pmod{N}\right\}.$$

We consider any group Γ such that $\Gamma_1(N) \subset \Gamma \subset \Gamma_0(N)$, and call it *a group of level N*. Let $J_\Gamma = J$ denote the jacobian variety of $\mathfrak{H}^*/\Gamma$ and $S_2(\Gamma)$ the vector space of all holomorphic cusp forms on $\mathfrak{H}$, of weight 2 with respect to Γ. Hereafter we restrict ourselves to the case where N is a prime power p^n, $n>1$ and

$$\Gamma=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p^n) \,\middle|\, a \equiv d \equiv 1 \pmod{p}\right\}.$$

Here we shall recall a few facts in [**6**]. It is known that J is defined over $\boldsymbol{Q}$. Let $f(z)=\sum_{m=1}^{\infty} a_m e^{2\pi i m z}$, with $a_1=1$, be an element of $S_2(\Gamma_0(p^n))$, that is a common eigen-function of Hecke operators T_m for all m. Then $f(z)$ is also an eigen-function as an element of $S_2(\Gamma)$ (see [**3**, Prop. 3.36]). Let K be the subfield of $\boldsymbol{C}$ generated over $\boldsymbol{Q}$ by a_m for all m. (Note that K is a totally real algebraic number field.) Then we can apply the argument of [**6**, § 1] for these Γ, J, f and K. By [**6**, Th. 1], we know that

(1.1) *There exists a triple (A, ν, θ) formed by the objects satisfying the following conditions.*

- (i) *(A, ν) is a quotient of J by an abelian subvariety rational over $\boldsymbol{Q}$. (ν is a natural map $J \to A$.)*
- (ii) *θ is an isomorphism of K into $\mathrm{End}(A) \otimes \boldsymbol{Q}$ such that $\nu \circ \xi_m = \theta(a_m) \cdot \nu$ for all m. (ξ_m is an element of $\mathrm{End}(J)$, associated with T_m.)*
- (iii) $\dim(A) = [K : \boldsymbol{Q}]$.

It is this abelian variety A which we shall investigate in § 2, in the framework of [**5**, § 9]. For this purpose, we shall recall here a few more properties of A in [**6**, § 4]. Let χ be a real primitive character of $(\boldsymbol{Z}/p\boldsymbol{Z})^\times$ of order 2. Now we take $N(=M)=p^n$ $(n>1)$, $r=p$ and $s=N$ in the notation of [**6**, § 4], then our Γ (this is, of course, a group of level N) satisfies the set of conditions (4.8) in [**6**, § 4]. Therefore, as (4.9) in [**6**, § 4], we suppose that the following condition is satisfied:

(*) *There is an automorphism ρ of K, other than the identity map, such that $\chi(m) \cdot a_m = a_m^\rho$ for all m. (This implies especially that $\rho^2 = 1$ and $a_m = 0$ if $(m, p) \neq 1$.)*

Then, by [**6**, Prop. 8 and Prop. 9], we know that

(1.2) *Under the assumption* (*), A *has an endomorphism* η *defined over the quadratic extension* k *of* $\boldsymbol{Q}$ *corresponding to* χ, *which satisfies*

(i) $\eta^{\varepsilon} = -\eta$ *if* ε *is the generator of* $\mathrm{Gal}(k/\boldsymbol{Q})$,
(ii) $\eta^2 = \chi(-1)p \cdot \mathrm{id}_A$,
(iii) $\eta \circ \theta(a) = \theta(a^{\rho}) \circ \eta$ *for every* $a \in K$.

Thus, if an eigen-function $f(z)$ in $S_2(\Gamma_0(p^n))$, satisfying (*) exists, we can obtain the couple (A, θ) having the properties (1.1) and (1.2). Let F be the invariant subfield of K under ρ in (*). (Note that $[K : F] = 2$.) Now we observe that F, K and the couple (A, θ) thus obtained satisfy, under an obvious modification, the conditions (9.1-5) in [**5**, § 9]. (In the notation of [**5**, § 9], take as $\theta(d) = \chi(-1) \cdot p \cdot \mathrm{id}_A$ and $\theta(h) = \eta$. The field K in the present case is not a CM-field as assumed there.)

Let $\mathfrak{o}_K$ and $\mathfrak{o}_F$ denote the ring of all algebraic integers in K and F, respectively. Let $\mathfrak{b}_0$ denote the ideal of $\mathfrak{o}_K$ generated by all x in $\mathfrak{o}_K$ such that $x^{\rho} = -x$. Also define the ideals $\mathfrak{b}$ and $\mathfrak{c}$, exactly in the same manner as in [**5**, § 2], for the present F and K. We put

$$\mathfrak{x} = \{t \in A \mid \theta(\mathfrak{b})t = 0\} .$$

By means of the same reasoning as in [**5**, § 9], $\mathfrak{x}$ is $\mathfrak{o}_K$-isomorphic to $(\mathfrak{o}_K/\mathfrak{b})^2$, and η acts on $\mathfrak{x}$ as an endomorphism. Now let us assume, as (9.8) in [**5**, § 9];

(**) $\chi(-1) \cdot p \equiv e^2 \bmod \mathfrak{c}$ *for some element* e *of* $\mathfrak{o}_F$ *prime to* $\mathfrak{c}$.

With such an e, put

$$\mathfrak{y} = \{t \in \mathfrak{x} \mid (\eta - \theta(e))t = 0\} ,$$

$$\mathfrak{z} = \{t \in \mathfrak{x} \mid (\eta + \theta(e))t = 0\} .$$

Then, as in [**5**, Prop. 9.2], we can easily verify

(1.3) *The submodules* $\mathfrak{y}$ *and* $\mathfrak{z}$ *are* $\mathfrak{o}_F$*-isomorphic to* $\mathfrak{o}_F/\mathfrak{c}$, *and* $\mathfrak{x} = \mathfrak{y} \oplus \mathfrak{z}$.

Let $k(\mathfrak{x})$ (resp. $k(\mathfrak{y})$ and $k(\mathfrak{z})$) denote the smallest extension of k over which the points of $\mathfrak{x}$ (resp. $\mathfrak{y}$ and $\mathfrak{z}$) are rational. For the same reason as in [**3**, Th. 7.30], [**5**, § 2], $k(\mathfrak{x})$ is an abelian extension of k and letting $\mathrm{Gal}(k(\mathfrak{x})/k)$ act on $\mathfrak{y}$ and $\mathfrak{z}$, we obtain an injective homomorphism

$$\mathrm{Gal}(k(\mathfrak{x})/k) \longrightarrow (\mathfrak{o}_F/\mathfrak{c})^{\times} \times (\mathfrak{o}_F/\mathfrak{c})^{\times} .$$

The class-field-theoretical investigation for the $(\mathfrak{o}_F/\mathfrak{c})^{\times}$-valued "characters" as in [**5**, § 2] for the abelian extension $k(\mathfrak{y})$ (resp. $k(\mathfrak{z})$) over k, will be discussed in the next § 2.

§ 2. The case $N=5^3$ and 7^3.

Firstly, as mentioned in the Introduction, let us recall the property [**3**, Prop. 3.64] for our case where $N=p^n$, $n>1$.

(2.1) *If* $f(z)=\sum_{m=1}^{\infty} a_m e^{2\pi imz} \in S_2(\Gamma_0(p^n))$, *then* $f_\chi(z)=\sum_{m=1}^{\infty} \chi(m)a_m e^{2\pi imz} \in S_2(\Gamma_0(p^n))$ *for a primitive character* χ *of* $(\boldsymbol{Z}/p\boldsymbol{Z})^\times$ *of order* 2.

Let $S_2^0(\Gamma_0(p^n))$ denote the "essential part" (see [**2**, p. 176] or [**5**, § 1]) of $S_2(\Gamma_0(p^n))$. For an obvious reason, it is necessary and natural to investigate the common eigen-functions $f(z)=\sum_{m=1}^{\infty} a_m e^{2\pi imz}$ of Hecke operators T_m (for all m) in $S_2^0(\Gamma_0(p^n))$. In the following tables we shall give the characteristic polynomials of T_m for a several primes m in $S_2^0(\Gamma_0(p^n))$, with $N=p^3$, $p=5$ and 7. (Of course each of the characteristic roots of these polynomials gives the m-th Fourier coefficients a_m for some $f(z)$.) Let K denote, for a fixed f, the subfield of $\boldsymbol{C}$ generated over $\boldsymbol{Q}$ by the coefficients a_m for all m.

(a) $N=5^3$

m	$\chi(m)$	characteristic polynomial of T_m		
		I	I_χ	II
2	−1	X^2+X-1	X^2-X-1	X^4-8X^2+11
3	−1	X^2+3X+1	X^2-3X+1	X^4-7X^2+11
11	+1	$(X+3)^2$	$(X+3)^2$	$(X-2)^4$
19	+1	X^2+5X+5	X^2+5X+5	$(X^2-10X+20)^2$
29	+1	X^2-45	X^2-45	$(X^2+5X-5)^2$

(b_1) $N=7^3$

m	$\chi(m)$	characteristic polynomial of T_m	
		I	II
2	+1	X^3+4X^2+3X-1	$X^6+2X^5-6X^4-10X^3+10X^2+11X-1$
3	−1	X^3	$X^6+5X^5-X^4-34X^3-28X^2+49X+49$
5	−1	X^3	
11	+1	$X^3+9X^2+20X+13$	
29	+1	$X^3+15X^2+26X-211$	

(b_2) $N=7^3$ (continued)

m	$\chi(m)$	characteristic polynomial of T_m		
		II_χ	III	IV
2	+1	$X^6+2X^5-6X^4-10X^3+10X^2+11X-1$	$(X^3-2X^2-X+1)^2$	$X^3-3X^2-4X+13$
3	−1	$X^6-5X^5-X^4+34X^3-28X^2-49X+49$	$X^6-20X^4+124X^2-232$	X^3
5	−1		$X^6-24X^4+164X^2-232$	X^3
11	+1		$(X^3-X^2-2X+1)^2$	$X^3-5X^2-36X+167$
29	+1		$(X^3+X^2-16X-29)^2$	$X^3-13X^2-30X+601$

For an explanation of the table, let us introduce the group $\Gamma^*(N)$ which is generated by all the elements of $\Gamma_0(N)$ and $\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$.

Let us now consider the special case $N=5^3$. We have $\dim S_2(\Gamma_0(5^3)) = \dim S_2^0(\Gamma_0(5^3))=8$, $\dim S_2(\Gamma^*(5^3))=\dim S_2^0(\Gamma^*(5^3))=2$ in this case. ($S_2^0(\Gamma^*(N))$ denotes also the "essential part" of $S_2(\Gamma^*(N))$.) In the table (a), the columns I and I_χ contain the characteristic polynomials of T_m corresponding to a basis $\{f\}$ of $S_2^0(\Gamma^*(5^3))$ and $\{f_\chi\}$ on account of (2.1). The remaining 4-dimensional part II is most interesting because, as the table shows, the field K and the eigen-functions $f(z)$ corresponding to this part satisfy the assumption (*) in § 1. Therefore, hereafter we restrict our discussion to the part II. To be more precise, let us fix an eigen-function $f(z)=\sum_{m=1}^{\infty} a_m e^{2\pi i m z}$ corresponding to the part II, i. e. fix the field K generated over $\boldsymbol{Q}$ by one of the roots $X^4-8X^2+11=0$, e. g., we take $K=\boldsymbol{Q}(\sqrt{4+\sqrt{5}})$. Let $\mathfrak{Z}$ denote the set of all isomorphisms of K into $\boldsymbol{C}$. Then by [5, Prop. 1.2], f and all its "companions" $f_\sigma=\sum_{m=1}^{\infty} a_m^\sigma e^{2\pi i m z}$ ($\sigma\in\mathfrak{Z}$) form a basis for the 4-dimensional part II. The table (a) tells us that K has a non-trivial automorphism ρ of order 2 and the fixed subfield F under ρ is $F=\boldsymbol{Q}(\sqrt{5})$. Moreover, considering f as an element in $S_2(\Gamma)$ in § 1, the condition (*) is satisfied. Thus we obtain a system $\{(A,\theta), \eta, k, K/F\}$ in § 1, with $k=\boldsymbol{Q}(\sqrt{5})$ corresponding to χ. Let $\mathfrak{o}_k$ denote the ring of all algebraic integers in k. In the present case we see that $\mathfrak{b}_0=\mathfrak{b}=(\sqrt{4+\sqrt{5}})$, $\mathfrak{c}=(4+\sqrt{5})$. Therefore the condition (**) in § 1

$$\chi(-1)\cdot 5\equiv e^2 \pmod{(4+\sqrt{5})}$$

is satisfied with e. g. $e=4$.

REMARK 2.1. It was indicated by Shimura [3, § 7.7], [5, p. 148] that in the "Neben"-type case, $N(\mathfrak{c})$ and $Tr_{k/\boldsymbol{Q}}(u)$ have a non-trivial common factor, where k is a corresponding real quadratic field and u the fundamental unit of k. Here we remark that in the present case there is still some relation among these, namely, take the fundamental unit $u=\frac{1+\sqrt{5}}{2}$ in $k=\boldsymbol{Q}(\sqrt{5})$, then the table of part II shows that $N(\mathfrak{c})$ and $Tr_{k/\boldsymbol{Q}}(u^5)$ consist of the same prime factor 11. This fact will be used in the later discussion.

Now let us consider the structure of the extension $k(\mathfrak{x})/k$, especially, $k(\mathfrak{y})/k$ in § 1, for the present (A, θ), $\mathfrak{b}=(\sqrt{4+\sqrt{5}})$, $\mathfrak{c}=(4+\sqrt{5})$ and

$$\mathfrak{x}=\{t\in A \mid \theta(\mathfrak{b})t=0\}$$

$$\mathfrak{y}=\{t\in \mathfrak{x} \mid (\eta-\theta(e))t=0\}\ .$$

We use the same notation as that of [5] in the following discussion, except for a minor obvious change. As mentioned at the end of § 1, from the action of $\mathrm{Gal}(k(\mathfrak{y})/k)$ on y we obtain an injective homomorphism

$$r'\colon\ \mathrm{Gal}(k(\mathfrak{y})/k)\longrightarrow(\mathfrak{o}_F/\mathfrak{c})^\times\cong(\boldsymbol{Z}/11\boldsymbol{Z})^\times\ ,$$

and put $r(\mathfrak{a})=r'\left(\left(\frac{k(\mathfrak{y})/k}{\mathfrak{a}}\right)\right)$.

PROPOSITION 2.2. *The field $k(\mathfrak{y})$ is the maximal ray class field over k of conductor $5\cdot\mathfrak{q}\mathfrak{p}_\infty$ with a prime factor $\mathfrak{q}$ of 11 in k. The archimedean prime $\mathfrak{p}_\infty$ of k is uniquely determined for $\mathfrak{q}$ by the condition that $v<0$ at $\mathfrak{p}_\infty$ for every $v\in\mathfrak{o}_k^\times$ such that $N_{k/\boldsymbol{Q}}(v)=-1$ and $v\equiv 1 \pmod{\mathfrak{q}}$. Moreover, one has*

$$r((\alpha))=\left(\frac{\alpha}{\mathfrak{p}_\infty}\right)\cdot\varphi(\alpha)\cdot\mu(\alpha \bmod \mathfrak{q})\ ,$$

for every α in k prime to $5\cdot\mathfrak{q}$ where μ is the isomorphism of $\mathfrak{o}_k/(\mathfrak{q})$ onto $\mathfrak{o}_F/\mathfrak{c}$ and φ is a homomorphism (character) of $(\mathfrak{o}_k/(5))^\times$ onto $(\mathfrak{o}_F/\mathfrak{c})^\times$ of order 10 such that

$$\varphi(m)=\chi(m)$$

for $m\in\boldsymbol{Z}$.

It should be noted that in the present case, the invariant subfield F under ρ coincides with the field k corresponding to χ.

PROOF. It is known that every finite prime factor $\mathfrak{p}$ of the conductor $\mathfrak{f}$ of $k(\mathfrak{y})/k$ divides $N(\mathfrak{c})\cdot N$ (see [3, § 7.5, p. 181 and Prop. 7.23]). Put $\mathfrak{f}$ in the following form

$$\mathfrak{f}=\mathfrak{p}_\infty^a\mathfrak{p}_\infty^{\varepsilon b}\cdot\prod_{\mathfrak{p}}\mathfrak{p}^{f_\mathfrak{p}}\ ,\qquad 0\leqq a\ ,\ b\leqq 1\ ,$$

with the prime factors $\mathfrak{p}$ of $5\cdot 11$ in k. Applying [3, (7.5.1)] for the present Γ and by the same argument as in the proof of [5, Th. 2.3], we first obtain

(2.2) $r((m))=\left(\frac{m}{p_\infty}\right)\cdot\chi(m)\cdot(m \bmod \mathfrak{q})$ *for every* $m\in\boldsymbol{Z}$ *prime to* $5\cdot\mathfrak{q}$, *where* p_∞ *is the archimedean prime of* $\boldsymbol{Q}$.

Therefore $[k(\mathfrak{y}):k]=5$ or 10, and so $\mathfrak{q}$-exponent $f_\mathfrak{q}$ (in $\mathfrak{f}$)$=1$ by [**3**, Lemma 7.32]. For the determination of $f_{\mathfrak{q}^\varepsilon}$, we can use the following fact which is nothing but [**5**, Th. 2.8] for the present case.

(2.3) *Let* $\mathfrak{f}_0$ *be the finite part of* $\mathfrak{f}$. *Let* q *be a rational prime which divides* $N(\mathfrak{f}_0)$ *but not* $N=5^3$. *Suppose that* $\chi(q)=1$, *and* a_q *is prime to* $\mathfrak{c}=(4+\sqrt{5})$. *Then* $\mathfrak{f}_0$ *is divisible by only one of the two prime factors of* q *in* k. *Moreover, if* $\mathfrak{q}$ *denotes that factor of* q, *then*

$$r(\mathfrak{q}^\varepsilon)\equiv a_q \pmod{\mathfrak{c}}.$$

Take $q=11$. From the table (a), $a_{11}=2$. Therefore one has $f_{\mathfrak{q}^\varepsilon}=0$. Hence $\mathfrak{f}_0$ is of the form $\sqrt{5}^m\cdot\mathfrak{q}$. By means of Hasse's conductor ramification theorem and (2.2) we know that $1\leqq m\leqq 2$. On the other hand, we can easily check that the smallest exponent n which satisfies $u^n\equiv 1 \bmod \mathfrak{q}$ is a multiple of 5, where $u=(1+\sqrt{5})/2$ is the fundamental unit of $k=\boldsymbol{Q}(\sqrt{5})$. Then if one consider the degree of the maximal ray class field over k mod $\mathfrak{f}$ (as mentioned before, this must be divisible by 5), one can not have $m=1$. Hence $\mathfrak{f}_0=5\cdot\mathfrak{q}$. By the same argument as in [**5**, Prop. 2.5], we know that $\mathfrak{f}$ is divisible by exactly one of the two archimedean primes of k, say $\mathfrak{p}_\infty$. Thus $\mathfrak{f}=5\mathfrak{q}\cdot\mathfrak{p}_\infty$. Therefore one has

$$r((\alpha))=\left(\frac{\alpha}{\mathfrak{p}_\infty}\right)\cdot\varphi(\alpha)\cdot\mu(\alpha \bmod \mathfrak{q})$$

for every α in k prime to $5\cdot\mathfrak{q}$, with a homomorphism φ of $(\mathfrak{o}_k/(5))^\times$ into $(\mathfrak{o}_F/\mathfrak{c})^\times$ and the isomorphism μ of $\mathfrak{o}_k/\mathfrak{q}$ onto $\mathfrak{o}_F/\mathfrak{c}$. Our next task is to determine the order of φ. Let us take again $q=11$. By (2.3), one has $r(\mathfrak{q})^\varepsilon\equiv a_{11}=2$ mod $(4+\sqrt{5})$. Hence $\varphi(\mathfrak{q}^\varepsilon)\equiv 3$ mod $(4+\sqrt{5})$. Hence $\varphi(\mathfrak{q}^\varepsilon)^5=1$. On the other hand, $\varphi(m)=\chi(m)$ for $m\in\boldsymbol{Z}$. Therefore the order of φ is 10. If $v\in\mathfrak{o}_k^\times$, $N_{k/\boldsymbol{Q}}(v)=-1$ and $v\equiv 1 \bmod \mathfrak{q}$ then $v=\pm u^{5n}$ with an odd integer n. Obviously one has $\varphi(v)=\varphi\left(\pm\left(\frac{11+5\sqrt{5}}{2}\right)\right)^n=\varphi(\pm 3)^n=(-1)^n=-1$. We have $\varphi(v)\cdot\left(\frac{v}{\mathfrak{p}_\infty}\right)=1$, so that $v<0$ at $\mathfrak{p}_\infty$. Note that $r(\mathfrak{q}^\varepsilon)\equiv a_{11}=2$ mod $(4+\sqrt{5})$. Therefore the order of r is 10. Thus one sees that the extension $k(\mathfrak{y})/k$ is the maximal ray class field of conductor $\mathfrak{f}=5\cdot\mathfrak{q}\cdot\mathfrak{p}_\infty$ by considering its degree. This completes the proof.

REMARK 2.3. As a direct consequence of this proposition and [**5**, Th. 2.3], we have the following fact. Let p be a rational prime such that $\chi(p)=1$, i. e., p decomposes into two distinct primes $\mathfrak{p}=\gamma\mathfrak{o}_k$ and $\mathfrak{p}^\varepsilon=\gamma^\varepsilon\mathfrak{o}_k$ in $k=\boldsymbol{Q}(\sqrt{5})$.

It is easily seen that γ can be so chosen as $p=\gamma\cdot\gamma^{\varepsilon}$, γ is totally positive and $\gamma\equiv\pm1 \bmod (5)$, if $p\equiv1 \bmod 5$ or $\gamma\equiv\pm2 \bmod (5)$ if $p\equiv-1 \bmod 5$. Then $a_p\equiv\gamma+\gamma^{\varepsilon} \bmod \mathfrak{c}$, if $p\equiv1 \bmod 5$ or $a_p\equiv-(\gamma+\gamma^{\varepsilon}) \bmod \mathfrak{c}$, if $p\equiv-1 \bmod 5$.

Now let us consider the endomorphism algebra $\mathrm{End}_{\boldsymbol{Q}}(A)$ of the abelian variety A. Define an abelian subvariety B of A by

$$B=(\theta(\sqrt{5})+\eta)A\,.$$

Then B is rational over k, $A=B+B^{\varepsilon}$, $B^{\varepsilon}=(\theta(\sqrt{5})-\eta)A$, and $B\cap B^{\varepsilon}$ is a finite group annihilated by $\theta(2\sqrt{5})$. Denote by $\theta_F(a)$ the restriction of $\theta(a)$ to B for every $a\in F$. Then θ_F is an isomorphism of F into $\mathrm{End}_{\boldsymbol{Q}}(B)$. We can also define an isomorphism θ_F^{ε} of F into $\mathrm{End}_{\boldsymbol{Q}}(B^{\varepsilon})$ by $\theta_F^{\varepsilon}(a)=\theta_F(a)^{\varepsilon}$.

PROPOSITION 2.4. *The abelian variety B is simple, and* $\mathrm{End}_{\boldsymbol{Q}}(B)=\theta_F(F)$.

PROOF. We consider the p-th power Frobenius endomorphism φ_p of B modulo $\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal in $k=\boldsymbol{Q}(\sqrt{5})$ such that $N\mathfrak{p}=p$, $\chi(p)=1$. Take $p=19$ and 29. Then by the table (a), one knows $F(\varphi_{19})=F(\sqrt{-46-10\sqrt{5}})$ and $F(\varphi_{29})=F\left(\sqrt{\frac{-197+15\sqrt{5}}{2}}\right)$. One can easily check $F(\varphi_{19})\ncong F(\varphi_{29})$. This completes the proof by the same argument as that of [**3**, Th. 7.39].

As the second example, let us consider the case $N=7^3$. We have $\dim S_2^0(\Gamma_0(7^3))=24$ and $\dim S_2^0(\Gamma^*(7^3))=9$. Each eigen-function which belongs to the columns I in $(\mathfrak{b}_1)$ and IV in $(\mathfrak{b}_2)$ corresponds to the zeta-function of $\boldsymbol{Q}(\sqrt{-7})$ with a Grössen-character as was shown in [**4**]. The remaining 6-dimensional part III (in $(\mathfrak{b}_2)$) also satisfies (*) in § 1 as the part II in the case $N=5^3$. We fix our attention to this part III. In the present case the field k which corresponds to χ is $\boldsymbol{Q}(\sqrt{-7})$. Let F_7 be the maximal real subfield of $\boldsymbol{Q}(e^{\frac{2\pi i}{7}})$. Put $\alpha_0=e^{\frac{2\pi i}{7}}+e^{\frac{-2\pi i}{7}}$. Let us fix an eigen-function $f(z)=\sum_{m=1}^{\infty}a_m e^{2\pi i m z}$, and fix the field K generated over $\boldsymbol{Q}$ by one of the roots $X^6-20X^4+124X^2-232=0$, e. g. we take $a_3=\sqrt{2(3-\alpha_0)}$, and $K=F_7(\sqrt{2(3-\alpha_0)})$. The field K has the non-trivial automorphism ρ of order 2 with the fixed subfield $F=F_7$. In the present case we have $\mathfrak{b}_0=(\sqrt{2(3-\alpha_0)})$, the "odd part" $\mathfrak{b}=(3-\alpha_0, \sqrt{2(3-\alpha_0)})$ and $\mathfrak{c}=(3-\alpha_0)$, $N(\mathfrak{c})=29$. The condition (**) is satisfied with e. g., $e=14$. As was shown in the examples of M. Yamauchi [**7**], we can make the following observation, although we do not know to what extent this is true in general. Observe that α_0 is one of the fundamental unit of F_7. We have $N_{F_7/\boldsymbol{Q}}(\alpha_0^7-1)=8\cdot29$. Therefore $N\mathfrak{c}$ and $N_{F_7/\boldsymbol{Q}}(\alpha_0^7-1)$ have the common prime factor 29.

Using the same notation as the preceding case, let us consider the extension $k(\mathfrak{y})/k$.

PROPOSITION 2.5. *The field $k(\mathfrak{y})$ is a ray class field over k of conductor*

$7\cdot\mathfrak{q}$ with a prime factor $\mathfrak{q}$ of 29 in k, and one has

$$r((\alpha))=\varphi(\alpha)\cdot\mu(\alpha \bmod \mathfrak{q})$$

for every α in k prime to $7\cdot\mathfrak{q}$ where μ is the isomorphism of $\mathfrak{o}_k/\mathfrak{q}$ onto $\mathfrak{o}_F/(3-\alpha_0)$ and φ is a homomorphism of $(\mathfrak{o}_k/(7))^\times$ into $(\mathfrak{o}_F/(3-\alpha_0))^\times$ of order 14 such that

$$\varphi(m)=\chi(m)$$

for $m\in\boldsymbol{Z}$.

PROOF. By the same reasoning as in the preceding case, we have

(2.4) $\quad r((m))=\chi(m)\cdot(m \bmod \mathfrak{q})$ *for every $m\in\boldsymbol{Z}$ prime to $7\cdot\mathfrak{q}$.*

From the table (b_2) and the numerical observation about several a_m (not given here), one has $a_{29}=\alpha_0(3-\beta_0)$ with $\beta_0=e^{\frac{4\pi i}{7}}+e^{\frac{-4\pi i}{7}}$. Hence $(a_{29}, 3-\alpha_0)=1$. Therefore the same assumption as in the preceding (2.3) is satisfied. Thus the conductor $\mathfrak{f}$ is of the form $\sqrt{-7}^m\mathfrak{q}$. By (2.4), if we consider $r((10))$, then we know the order of r is 28. Hence $m=2$ (by the same argument as in the proof of Proposition 2.2). This proves the first assertion. Using the same fact as (2.3) in the present case, one has $r(\mathfrak{q}^\varepsilon)\equiv a_{29} \bmod (3-\alpha_0)$, hence $\varphi(\mathfrak{q}^\varepsilon)\equiv 23 \bmod (3-\alpha_0)$. Therefore $\varphi(\mathfrak{q}^\varepsilon)^7=1$. Thus, on account of (2.4) we know that the order of φ is 14. This completes the proof of our proposition.

REMARK 2.6. Let p be a rational prime such that $\chi(p)=1$, i.e., p decomposes into two distinct primes $\mathfrak{p}=\gamma\mathfrak{o}_k$ and $\mathfrak{p}^\varepsilon=\gamma^\varepsilon\mathfrak{o}_k$ in $k=\boldsymbol{Q}(\sqrt{-7})$. γ can be so chosen as $p=\gamma\cdot\gamma^\varepsilon$, $\chi(\gamma+\gamma^\varepsilon)=1$. Observe that $\varphi(\gamma^7)=\chi(\gamma+\gamma^\varepsilon)=1$. Let π and π' be the solutions of $X^2-a_pX+p\equiv 0 \bmod \mathfrak{c}$. Then again by [**5**, Th. 2.3] we have $\pi^7+\pi'^7\equiv\gamma^7+\gamma^{\varepsilon 7} \bmod \mathfrak{c}$.

Let us consider the endomorphism algebra $\mathrm{End}_{\boldsymbol{Q}}(A)$ in the present case $N=7^3$. Put $\delta=\sqrt{2(3-\alpha_0)}$. It can easily verified that η (where $\eta^2=(-7)\cdot\mathrm{id}_A$) and $\theta(\delta)$ generate an indefinite quaternion subalgebra $\mathfrak{A}$ of $\mathrm{End}_{\boldsymbol{Q}}(A)$ over $F=F_7$. Observe that $\delta^2=2(3-\alpha_0)\in N_{F(\sqrt{-7})/F}(F(\sqrt{-7}))$. Therefore $\mathfrak{A}$ is isomorphic to $M_2(F)$. Denote by $'$ the quaternion conjugation of $\mathfrak{A}$. Then we can find an element ξ of $\mathfrak{A}$ such that $\xi\cdot\xi'=0$ and $\xi^2=e\xi$ with $e\in\theta(\mathfrak{o}_F)\cap\mathrm{End}(A)$. Moreover we can put ξ in the form $\xi=a+b\cdot\theta(\delta)+c\eta+d\theta(\delta)\cdot\eta$ with $a, b, c, d\in\theta(\mathfrak{o}_F)\cap\mathrm{End}(A)$. Define an abelian subvariety B of A by

$$B=\xi A\,.$$

Then B is rational over k, $A=B+B^\varepsilon$, $B^\varepsilon=\xi^\varepsilon A$. Applying (1.2) to ξ^ε, one has $\xi^\varepsilon=a+b\theta(\delta)-c\eta-d\theta(\delta)\eta$ and $\xi\cdot\xi'=\xi^\varepsilon\cdot\xi'^\varepsilon=0$. Take an element $\xi t=\xi^\varepsilon t'\in B\cap B^\varepsilon$. Then

$$0=(\xi^\varepsilon)'\cdot(\xi^\varepsilon)t'=(\xi^\varepsilon)'\cdot\xi t=(\xi-2b\theta(\delta))\xi t=(e-2b\theta(\delta))\xi t\,.$$

Therefore $B\cap B^\varepsilon$ is a finite group annihilated by $(e-2b\theta(\delta))$. Here we can

also define an isomorphism θ_F of F into $\mathrm{End}_{\boldsymbol{Q}}(B)$ as in Proposition 2.4.

PROPOSITION 2.7. *The notation being as above, the abelian variety B is simple and* $\mathrm{End}_{\boldsymbol{Q}}(B)=\theta_F(F)$.

PROOF. Here again, we consider the p-th power Frobenius endomorphism φ_p of B modulo $\mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal in $k=\boldsymbol{Q}(\sqrt{-7})$ such that $N\mathfrak{p}=p$, $\chi(p)=1$. Take $p=2$ and 11. By the table (b_2), we know that $F(\varphi_2)\cong F(\varphi_{11})$ by considering each discriminant of $F(\varphi_p)$. Therefore, by the same argument of Proposition 2.4, $\mathrm{End}_{\boldsymbol{Q}}(B)$ is isomorphic to F, so that B is simple.

§ 3. The case of "Neben"-type of level $4M$, $M\equiv 1 \pmod 4$.

Throughout this section, we assume that M is a prime and $\equiv 1 \pmod 4$. We shall use freely the same notation and terminology in [**5**, § 2]. Let $S_2^0(4M, (\frac{M}{\ }))$ denote the "essential part" of $S_2(4M, (\frac{M}{\ }))$ (see [**2**], [**5**, § 1]). Now consider (A, θ), B, K, F, $\mathfrak{b}$, $\mathfrak{c}$, r, etc. in [**5**, § 2] for a fixed eigen-function $f(z)=\sum_{m=1}^{\infty} a_m e^{2\pi i m z}$ of $S_2^0(4M, (\frac{M}{\ }))$. The table (c) (resp. (d)) gives the Fourier coefficients a_p for $M=29$ (resp. $M=53, 61, 101$). Let k denote the quadratic extension of $\boldsymbol{Q}$ corresponding to $(\frac{M}{\ })$, namely $k=\boldsymbol{Q}(\sqrt{M})$. Then the fundamental unit u of $k=\boldsymbol{Q}(\sqrt{M})$ is given by $u=\frac{5+\sqrt{29}}{2}, \frac{7+\sqrt{53}}{2}, \frac{39+5\sqrt{61}}{2}, 10+\sqrt{101}$ for $M=29, 53, 61, 101$, respectively. Here we can make an empirical observation that $N(\mathfrak{c})$ and $N_{k/\boldsymbol{Q}}(u^3-1)$ consist of the same prime factors, if we disregard 2 and 3 except for the case $M=101$. (In the case $M=101$, $N(\mathfrak{c})$ and $N_{k/\boldsymbol{Q}}(u-1)$ have the same prime factor 5). From these data, it seems that $N(\mathfrak{c})$ and $N_{k/\boldsymbol{Q}}(u^l-1)$ (l depends on the square factor of the level) have a non-trivial common factor. Let $\mathfrak{o}_k$ and $\mathfrak{o}_F$ denote the rings of all algebraic integers in k and in F. Now we are interested in the field $k(\mathfrak{y})$ generated over k by the coordinates of the points of $\mathfrak{y}$ (see [**5**, (2.7), (2.8)]). Let us now discuss a special case $M=29$ which seems to be typical.

(c) level $=4\cdot 29$

p	$\left(\frac{M}{p}\right)$	a_p	p	$\left(\frac{M}{p}\right)$	a_p
3	$-$	$\sqrt{-7}$	17	$-$	$-2\sqrt{-7}$
5	$+$	1	19	$-$	$-2\sqrt{-7}$
7	$+$	-2	23	$+$	6
11	$-$	$\sqrt{-7}$	31	$-$	$-\sqrt{-7}$
13	$+$	5	37	$-$	$-4\sqrt{-7}$

(c) level $=4\cdot29$ (continued)

p	$\left(\frac{M}{p}\right)$	a_p	p	$\left(\frac{M}{p}\right)$	a_p
41	$-$	$2\sqrt{-7}$	71	$+$	-8
43	$-$	$-3\sqrt{-7}$	73	$-$	$4\sqrt{-7}$
47	$-$	$3\sqrt{-7}$	79	$-$	$-\sqrt{-7}$
53	$+$	5	83	$+$	2
59	$+$	-14	89	$-$	$2\sqrt{-7}$
61	$-$	$2\sqrt{-7}$	97	$-$	$-2\sqrt{-7}$
67	$+$	-4			

In this case, we have $\dim S_2^0(4\cdot29, (\frac{29}{\cdot}))=2$, $K=\boldsymbol{Q}(\sqrt{-7})$, $F=\boldsymbol{Q}$, $k=\boldsymbol{Q}(\sqrt{29})$ and $\mathfrak{c}=(7)$.

PROPOSITION 3.1. *The field $k(\mathfrak{y})$ is the maximal ray class field over k of conductor $2\mathfrak{q}\mathfrak{p}_\infty$ with a prime factor $\mathfrak{q}$ of 7 in k. The archimedean prime $\mathfrak{p}_\infty$ of k is uniquely determined for $\mathfrak{q}$ by the condition that $v>0$ at $\mathfrak{p}_\infty$ for every $v\in\mathfrak{o}_k^\times$ such that $N_{k/\boldsymbol{Q}}(v)=-1$ and $v\equiv1 \pmod{\mathfrak{q}}$. Moreover one has*

$$r((\alpha))=\left(\frac{\alpha}{\mathfrak{p}_\infty}\right)\cdot\lambda(\alpha \bmod (2))\cdot\mu(\alpha \bmod \mathfrak{q})$$

for every α in k prime to $2\mathfrak{q}$, where λ is a homomorphism of $(\mathfrak{o}_k/(2))^\times$ into $(\boldsymbol{Z}/(7))^\times$, and μ is the isomorphism of $\mathfrak{o}_k/\mathfrak{q}$ onto $\boldsymbol{Z}/(7)$.

The following method of proof is the same as that of [**5**, Prop. 7.1].

PROOF. First we observe that the finite part of $\mathfrak{f}(r)$ is divisible by the prime factors of $2\cdot7$. As the table (c) shows, $a_7=-2$ is prime to $\mathfrak{c}=7$. Therefore, on account of [**5**, Th. 2.8, Prop. 2.4 and Prop. 2.5] $\mathfrak{f}(r)$ is of the form $2^c\cdot\mathfrak{q}\cdot\mathfrak{p}_\infty$ where $\mathfrak{q}$ is a prime factor of 7 in k, and one has

$$r((\alpha))=\left(\frac{\alpha}{\mathfrak{p}_\infty}\right)\cdot\lambda(\alpha \bmod 2^c)\cdot\mu(\alpha \bmod \mathfrak{q})$$

with characters λ of $(\mathfrak{o}_k/(2)^c)^\times$ and μ of $(\mathfrak{o}_k/\mathfrak{q})^\times$ (both with values in $(\boldsymbol{Z}/(7))^\times$. Let U be the group of all (2)-units in the (2)-completion of k, and let $U_n=\{u\in U \mid u\equiv1 \bmod (2)^n\}$ for every integer $n\geqq0$. Define a $(\boldsymbol{Z}/(7))^\times$-valued character π of U by $\pi(\alpha)=\lambda(\alpha \bmod (2)^c)$. Then c is the smallest integer n such that $U_n\subset\mathrm{Ker}(\pi)$. Since $(\boldsymbol{Z}/(7))^\times$ is of order 6, we have $\pi^6=1$. Now it can easily be verified that

$$U_3\subset\{u^6 \mid u\in U\}\,.$$

Therefore $c\leqq3$. Observe that for any element $z\in U_1$, $z^4\in U_3$, so that $\pi(z^4)=1$. Since $\pi(z^6)=1$, we have $\pi(z^2)=1$, so that $U_1^2=\{u^2 \mid u\in U\}\subset\mathrm{Ker}(\pi)$,

Now U_2/U_1^2 is generated by a positive integer $m \equiv 5 \pmod 8$. Take m so that $m \equiv 1 \pmod 7$. By [**5**, Th. 2.2], we have $1 = r((m)) = \pi(m)$. This proves that $U_2 \subset \mathrm{Ker}(\pi)$. Observe that $71 = N_{k/\boldsymbol{Q}}(\alpha_0)$ with $\alpha_0 = 10 \pm \sqrt{29} \in U_1$. By [**5**, Th. 2.2], $r((\alpha_0))$ satisfies the congruence

$$X^2 - a_{71}X + 71 \equiv 0 \pmod 7.$$

Since $a_{71} = -8$, we see easily that $r((\alpha_0)^3) = 1$. On the other hand, we have $\alpha_0^3 \equiv 1 \bmod \mathfrak{q}$ with a suitable choice of $\alpha_0 = 10 + \sqrt{29}$ or $10 - \sqrt{29}$. Therefore $1 = r((\alpha_0)^3) = \pi(\alpha_0^3)$. Now it can easily be verified that U_1/U_2 is generated by α_0^3 and a positive integer $m \equiv 3 \bmod 4$. Take m so that $m \equiv 1 \bmod 7$. We have $1 = r((m)) = \pi(m)$, which shows that $U_1 \subset \mathrm{Ker}(\pi)$. Now observe that U/U_1 is isomorphic to $(\mathfrak{o}_k/(2))^\times$. Take $13 = N_{k/\boldsymbol{Q}}(\beta_0)$ with $\beta_0 = \dfrac{49 \pm 9 \cdot \sqrt{29}}{2}$. Then $r((\beta_0))$ satisfies

$$X^2 - a_{13}X + 13 \equiv 0 \bmod 7.$$

Since $a_{13} = 5$, we see that the order of $r((\beta_0))$ is 3 or 6. We have, on the other hand, $\beta_0 \equiv 1 \bmod \mathfrak{q}$ with $\beta_0 = \dfrac{49 + 9\sqrt{29}}{2}$ or $\dfrac{49 - 9\sqrt{29}}{2}$. Therefore $\pi(\beta_0) = r((\beta_0)) \neq 1$. Clearly, $\beta_0 \not\equiv 1 \bmod 2$. Hence U/U_1 is generated by β_0 and so $U \not\subset \mathrm{Ker}(\pi)$. Thus we have $c = 1$. If $v \in \mathfrak{o}_k^\times$, $N_{k/\boldsymbol{Q}}(v) = -1$ and $v \equiv 1 \bmod \mathfrak{q}$, then $v = \pm u^{3n}$ with an odd integer n, where $u = \dfrac{5 + \sqrt{29}}{2}$ is the fundamental unit of k. Obviously $v \equiv 1 \bmod 2$ so that $1 = r(v) = \left(\dfrac{v}{\mathfrak{p}_\infty}\right)$. Let E be the group of all units of $\mathfrak{o}_k$ and E_0 the subgroup of E consisting of all elements $u_0 \in E$ such that $u_0 \equiv 1 \bmod 2 \cdot \mathfrak{q} \cdot \mathfrak{p}_\infty$. It can easily be verified that $[E : E_0] = 6$. Thus one sees that the extension $k(\mathfrak{y})/k$ is the maximal ray class field of conductor $2 \cdot \mathfrak{q} \cdot \mathfrak{p}_\infty$ by considering its degree. This completes the proof of our proposition.

REMARK 3.2. As a direct consequence of Proposition 3.1, we have the following fact. Let p be a rational prime such that $\left(\dfrac{29}{p}\right) = 1$, $p = N_{k/\boldsymbol{Q}}(\gamma)$, $\gamma \in \mathfrak{o}_k$. It is easily seen that γ can be so chosen as $\gamma \equiv 1 \bmod (2)$ and γ is totally positive. Then $a_p \equiv \gamma + \gamma^\varepsilon \bmod (7)$.

By the same argument as above and referring to the following table (d), one can obtain the maximal ray class field $k(\mathfrak{y})$ over $k = \boldsymbol{Q}(\sqrt{M})$ for each $M = 53$, 61 and 101, of conductor $2 \cdot \mathfrak{q} \cdot \mathfrak{p}_\infty$ with $N_{k/\boldsymbol{Q}}(\mathfrak{q}) = 13$, 127 and 5, respectively. The verification for these cases is left to the reader as an exercise. The authors have also found a few more examples of the same nature for $S_2(4M, \psi)$ with a positive prime integer $M \equiv 3 \bmod (4)$ and a non-trivial real character ψ of $(\boldsymbol{Z}/4M)^\times$, which will not be discussed here.

(d) The table for $S_2^0(4M, (\frac{M}{\cdot}))$

$4\cdot M$	dim.	p	$\left(\frac{M}{p}\right)$	a_p
$4\cdot 53$	4	3	$-$	$\sqrt{-4+\sqrt{3}}$
		5	$-$	$\sqrt{-10-4\sqrt{3}}$
		7	$+$	$1+\sqrt{3}$
		11	$+$	$-1+\sqrt{3}$
		13	$+$	$-1-2\sqrt{3}$
		17	$+$	$3+2\sqrt{3}$
		19	$-$	$\sqrt{-16-9\sqrt{3}}$
		23	$-$	$\sqrt{-12+3\sqrt{3}}$
		29	$+$	$2-3\sqrt{3}$
		37	$+$	$2-\sqrt{3}$
		47	$+$	-6
$4\cdot 61$	4	3	$+$	$-\sqrt{2}$
		5	$+$	$1-\sqrt{2}$
		7	$-$	$\sqrt{-15+7\sqrt{2}}$
		11	$-$	$\sqrt{-17-9\sqrt{2}}$
		13	$+$	$-1+2\sqrt{2}$
		17	$-$	$\sqrt{-30+14\sqrt{2}}$
		19	$+$	$2-3\sqrt{2}$
		23	$-$	$\sqrt{-17-9\sqrt{2}}$
		41	$+$	$3+\sqrt{2}$
$4\cdot 101$	8	3	$-$	$(X^2+4)(X^6+13X^4+40X^2+25)=0$
		5	$+$	$(X-3)^2(X^3+3X^2-2X-3)^2=0$
		13	$+$	$(X+1)^2(X^3-X^2-10X+1)^2=0$
		17	$+$	$(X-5)^2(X^3+X^2-8X-11)^2=0$
		19	$+$	$(X+7)^2(X-2)^6=0$
		23	$+$	$(X-1)^2(X^3+2X^2-32X-88)^2=0$
		31	$+$	$(X+3)^2(X^3-10X^2-8X+120)^2=0$

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