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Semi-linear second-order elliptic equations in L^1

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Let L be a linear second-order elliptic differential operator in R^N, for instance, $L=-\Delta$. We consider the equation

$$(1) \qquad Lu+\beta(x, u(x))=f(x)$$

where $f(x)$ is integrable on $\Omega\subset R^N$ and $u(x)$ vanishes on the boundary of Ω. In case $\beta(x, u)$ is monotone increasing in u, possibly multi-valued, we prove the existence and uniqueness of solutions. Actually, L can be any abstract accretive operator in $L^1(\Omega)$ which satisfies a "maximum principle". In case $\beta(x, u)$ is not monotone but has the same sign as u, we prove the existence of solutions when $f(x)$ belongs to an Orlicz class arbitrarily close to $L^1(\Omega)$. We also consider equation (1) with a nonlinear boundary condition.

The linear case is considered by Stampacchia [**18**]. Our basic technique is to multiply the equation by various monotone functions of u. This method was used by Moser [**14**] in his proof of the DeGiorgi-Nash regularity theorem. The standard variational approach [**12, 17**] cannot be applied to our problem for two reasons. Firstly, $\beta(x, u)$ may be rapidly increasing in u and may even have vertical asymptotes. We can handle rapidly increasing non-monotone $\beta(x, u)$ by a lemma from [**20**]. We can handle (multi-valued) monotone graphs by techniques from [**6**]. Secondly, the merely integrable function $f(x)$ need not belong to the dual space of the space where an energy estimate holds.

While this work was in progress, we learned of four other related works. (i) Browder [**4**] allows rapidly increasing non-linear lower-order terms of high-order elliptic operators. Because his approach is variational, those of his f's which are functions must belong to a smaller space than $L^1(\Omega)$. (ii) Da Prato [**7**] considers equation (1) with $L=-\Delta$, $\beta(u)$ a monotone continuous function, and $f\in L^p(\Omega)$ for $p>1$. (iii) Konishi [**10**] has a result similar to part of our Theorem 1 in the case when $\beta(u)$ is a monotone continuous function and Ω is bounded. (His "sub-Markov" assumption is equivalent to our assumption (II).) His methods are entirely different from ours. (iv) Crandall's Theorem 4.12 in [**5**] is closely related to our Theorem 1 in case $L=-\Delta$. In

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a paper which is yet to be completed, we have obtained some analogous results for parabolic equations.

§ 1. An abstract formulation of the monotone case.

Let β be a maximal monotone graph in $R\times R$ which contains the origin. If the pair $(s, t)\in\beta$, we write $t\in\beta(s)$.

Let Ω be any measure space. We denote by $\| \|_p$ the norm in $L^p(\Omega)$. Let A be an unbounded linear operator on $L^1(\Omega)$ which satisfies the following conditions.

(I) It is a (closed) operator with dense domain $D(A)$ in $L^1(\Omega)$; for any $\lambda>0$, $I+\lambda A$ maps $D(A)$ one-one onto $L^1(\Omega)$ and $(I+\lambda A)^{-1}$ is a contraction in $L^1(\Omega)$(*).

(II) For any $\lambda>0$ and $f\in L^1(\Omega)$,

$$\sup_{\Omega}(I+\lambda A)^{-1}f\leqq\max\{0, \sup_{\Omega}f\}\ .$$

(By "sup" we mean the essential supremum. If $\sup f=\infty$, assumption (II) is empty.)

(III) There exists $\alpha>0$ such that

$$\alpha\|u\|_1\leqq\|Au\|_1 \qquad \text{for all} \quad u\in D(A)\,.$$

THEOREM 1. *For every $f\in L^1(\Omega)$, there exists a unique $u\in D(A)$ such that*

$$Au(x)+\beta(u(x))\ni f(x) \qquad a.\,e. \tag{2}$$

Moreover, if $f, \hat{f}\in L^1(\Omega)$ and u, $\hat{u}$ are the corresponding solutions of (2), *then*

$$\|(f-Au)-(\hat{f}-A\hat{u})\|_1\leqq\|f-\hat{f}\|_1\,. \tag{3}$$

In particular,

$$\|A(u-\hat{u})\|_1\leqq 2\|f-\hat{f}\|_1\,. \tag{4}$$

LEMMA 2. *Let γ be a maximal monotone graph in $R\times R$ which contains the origin. Assume that A satisfies* (I) *and* (II). *Let $1\leqq p\leqq\infty$ and $p'=p/(p-1)$, $p'=\infty$ if $p=1$. Let $u\in D(A)\cap L^p(\Omega)$ with $Au\in L^p(\Omega)$. Let $g\in L^{p'}(\Omega)$ be such that $g(x)\in\gamma(u(x))$ a. e. Then*

$$\int_{\Omega}Au(x)g(x)dx\geqq 0\,.$$

PROOF OF THEOREM 1. We denote, for u and $f\in L^1(\Omega)$, $f\in Bu$ whenever $f(x)\in\beta(u(x))$ a. e. We first establish (3) which implies (4) and the uniqueness. Let $g=f-Au\in Bu$ and $\hat{g}=\hat{f}-A\hat{u}\in B\hat{u}$. We multiply the equation

* (I) is equivalent to $-A$ generating a linear contraction semi-group in $L^1(\Omega)$.

$$A(u-\hat{u})+g-\hat{g}=f-\hat{f}$$

by

$$h(x)=\begin{cases} +1 & \text{on} \quad \{u>\hat{u}\}\cup\{g>\hat{g}\} \\ 0 & \text{on} \quad \{u=\hat{u}\}\cap\{g=\hat{g}\} \\ -1 & \text{on} \quad \{u<\hat{u}\}\cup\{g<\hat{g}\}\,. \end{cases}$$

Note that $h(x)$ is defined a.e. and is measurable. Clearly $h(x)\in \operatorname{sign}[u(x)-\hat{u}(x)] \cap \operatorname{sign}[g(x)-\hat{g}(x)]$, where $\operatorname{sign}[r]=+1$ for $r>0$, $\operatorname{sign}[r]=-1$ for $r<0$, and $\operatorname{sign}[0]=$ the interval $[-1, 1]$. Applying Lemma 2 in the case $p=1$, $\gamma=\operatorname{sign}$, we have $(A(u-\hat{u}), h)\geqq 0$. (This special case of Lemma 2 is a consequence of (I) since sign is the subdifferential of the norm.) Therefore we obtain

$$\|g-\hat{g}\|_1=(g-\hat{g}, h)\leqq(f-\hat{f}, h)\leqq\|f-\hat{f}\|_1\,.$$

This is estimate (3).

It follows that $A+B$ has closed range. Indeed, let $u_n\in D(A)$, $Au_n+Bu_n\ni f_n$ and $f_n\to f$ in $L^1(\Omega)$. By (4) we have

$$\|A(u_n-u_m)\|_1\leqq 2\|f_n-f_m\|_1\,.$$

Hence by (III) and the closedness of the operator A, $u_n\to u$ and $Au_n\to Au$ in $L^1(\Omega)$. Since β is maximal, $f-Au\in Bu$.

It remains to show that $A+B$ has dense range. (To accomplish this, we use some arguments from [**6**].) Let us approximate β by the Lipschitz functions

$$\beta_\lambda=\frac{1}{\lambda}(I-(I+\lambda\beta)^{-1})\,, \qquad \lambda>0\,.$$

First we solve the equation

$$\varepsilon u+Au+\beta_\lambda u=f \tag{5}$$

for any $\varepsilon>0$, $\lambda>0$, $f\in L^1(\Omega)$. Indeed, (5) can be rewritten as

$$\lambda\varepsilon u+\lambda Au+u=\lambda f+(I+\lambda\beta)^{-1}u\,,$$

$$u=\frac{1}{1+\lambda\varepsilon}\left\{I+\frac{\lambda}{1+\lambda\varepsilon}A\right\}^{-1}\{\lambda f+(I+\lambda\beta)^{-1}u\}\,. \tag{6}$$

The operator on the right side of (6) is the product of two contractions in $L^1(\Omega)$ and a number less than 1, hence is a strict contraction, so that it has a fixed point $u\in D(A)$. If in addition $f\in L^\infty(\Omega)$, then u and Au belong to $L^\infty(\Omega)$. For, from (II) we have

$$\|(I+\lambda A)^{-1}g\|_\infty\leqq\|g\|_\infty \qquad \text{for} \quad g\in L^1(\Omega)\cap L^\infty(\Omega)\,.$$

The same fixed point argument in the space $L^1(\Omega)\cap L^\infty(\Omega)$ shows that u and

Au are essentially bounded.

Now we let $\lambda\to 0$, keeping fixed $\varepsilon>0$ and $f\in L^1(\Omega)\cap L^\infty(\Omega)$. We denote the solution of (5) by u_λ. From (6),

$$\|u_\lambda\|\leqq(1+\lambda\varepsilon)^{-1}\{\lambda\|f\|+\|u_\lambda\|\}$$

whence

$$\|u_\lambda\|\leqq\frac{1}{\varepsilon}\|f\|$$

where $\|\ \|$ denotes the norm in $L^1(\Omega)\cap L^\infty(\Omega)$. If we multiply equation (5) by $\operatorname{sign}(u_\lambda)$ and by $|\beta_\lambda(u_\lambda)|^{p-2}\beta_\lambda(u_\lambda)$, make use of Lemma 2, and let $p\to\infty$, we also find $\{\beta_\lambda(u_\lambda)\}$ to be bounded in $L^1(\Omega)\cap L^\infty(\Omega)$ by the norm of f in that space.

Next, $\{u_\lambda\}$ and $\{\beta_\lambda(u_\lambda)\}$ are Cauchy sequences in $L^2(\Omega)$. To prove this, we subtract the equations for u_λ and u_μ and multiply the difference by $u_\lambda-u_\mu$. Using Lemma 2 with $\gamma=$ identity, we obtain

$$\varepsilon\|u_\lambda-u_\mu\|_2^2+(\beta_\lambda(u_\lambda)-\beta_\mu(u_\mu),\ u_\lambda-u_\mu)\leqq 0\,.$$

The last factor may be rewritten as

$$\begin{aligned} u_\lambda-u_\mu=&\ \{u_\lambda-(I+\lambda\beta)^{-1}u_\lambda\}\\ &+\{(I+\lambda\beta)^{-1}u_\lambda-(I+\mu\beta)^{-1}u_\mu\}+\{(I+\mu\beta)^{-1}u_\mu-u_\mu\}\,. \end{aligned}$$

The middle term makes a non-negative contribution because β is monotone. Hence

$$\varepsilon\|u_\lambda-u_\mu\|_2^2+(\beta_\lambda(u_\lambda)-\beta_\mu(u_\mu),\ \lambda\beta_\lambda(u_\lambda)-\mu\beta_\mu(u_\mu))\leqq 0\,.$$

Thus $\{u_\lambda\}$ is Cauchy in $L^2(\Omega)$ as $\lambda\to 0$. The limit u belongs to $L^1(\Omega)\cap L^\infty(\Omega)$. By Lemma 2.4 of [**6**], $\{\beta_\lambda u_\lambda\}$ is also Cauchy in $L^2(\Omega)$; its limit g belongs to $L^1(\Omega)\cap L^\infty(\Omega)$ and $g(x)\in\beta(u(x))$ a.e. since β is maximal. So we have $(\varepsilon I+A)u_\lambda\to f-g$ in $L^2(\Omega)$ as $\lambda\to 0$. Let $v=(\varepsilon I+A)^{-1}(f-g)$. By (I) and (II), v belongs to $D(A)\cap L^\infty(\Omega)$. Clearly $(\varepsilon I+A)(u_\lambda-v)\to 0$ in $L^2(\Omega)$. Multiplying this expression by $u_\lambda-v$ and using Lemma 2 again, we obtain $u_\lambda\to v$ in $L^2(\Omega)$. Hence $u=v$. By definition of v, $\varepsilon u+Au+g=f$.

Finally let $f\in L^1(\Omega)$ and let $f^\varepsilon\to f$ in $L^1(\Omega)$ where $f^\varepsilon\in L^1(\Omega)\cap L^\infty(\Omega)$. Let u^ε be the solution of

$$\varepsilon u^\varepsilon+Au^\varepsilon+Bu^\varepsilon\ni f^\varepsilon\,.$$

Using (III), together with estimate (4) with $\hat{u}=0$ and β replaced by $\beta+\varepsilon I$, we have

$$\alpha\|u^\varepsilon\|_1\leqq\|Au^\varepsilon\|_1\leqq 2\|f^\varepsilon\|_1\,.$$

Hence $\varepsilon u^\varepsilon\to 0$ in $L^1(\Omega)$. Hence

$$f=\lim(f^\varepsilon-\varepsilon u^\varepsilon)\in\lim(Au^\varepsilon+Bu^\varepsilon)$$

belongs to the closure of the range of $A+B$.

The proof of Lemma 2 depends on Lemma 3.

LEMMA 3. *Let T be a mapping from $L^1(\Omega)$ into $L^1(\Omega)$ such that (for $u, v \in L^1(\Omega)$):*

$$\|Tu-Tv\|_1 \leqq \|u-v\|_1 . \tag{7}$$

$$\min\{0, \inf_\Omega u\} \leqq Tu(x) \leqq \max\{0, \sup_\Omega u\} \qquad a.e. \tag{8}$$

Let j be a convex lower semi-continuous function from $(-\infty, \infty)$ to $[0, +\infty]$ such that $\min j = j(0) = 0$. Then

$$\int_\Omega j(Tu(x))dx \leqq \int_\Omega j(u(x))dx$$

for all $u \in L^1(\Omega)$ such that $j \circ u \in L^1(\Omega)$.

PROOF OF LEMMA 3. First we consider the particular convex functions (cf. [8]):

$$j_1(r) = (r-t)^+ \quad \text{and} \quad j_2(r) = (-r-t)^+$$

where t is some non-negative number. Let $y(x) = \min\{u(x), t\}$. Note that $y \in L^1(\Omega)$. By (8) we have $Ty(x) \leqq t$ a.e. and thus

$$(Tu(x)-t)^+ \leqq (Tu(x)-Ty(x))^+ \leqq |Tu(x)-Ty(x)| \qquad \text{a. e.}$$

Integrating this inequality over Ω and using (7), we obtain

$$\int_\Omega (Tu(x)-t)^+ dx \leqq \int_\Omega |u(x)-y(x)|dx = \int_\Omega (u(x)-t)^+ dx .$$

Note that the operator $u \to -T(-u)$ also satisfies (7) and (8). So we can apply the result just proved to this operator to obtain

$$\int_\Omega (-T(-u(x))-t)^+ dx \leqq \int_\Omega (u(x)-t)^+ dx .$$

If we let $v = -u$, then the lemma follows for $j_2(r)$. Combining the results for both j_1 and j_2, we have

$$\int_\Omega [t(Tu(x)-t)]^+ dx \leqq \int_\Omega [t(u(x)-t)]^+ dx \tag{9}$$

for all real t.

The general case follows by taking convex combinations of j_1 and j_2. In fact, let j be any convex C^1 function on R with uniformly Lipschitz derivative such that $\min j = j(0) = 0$. Then

$$j(r) = \int_{-\infty}^{\infty} \frac{j''(t)}{|t|}[t(r-t)]^+ dt \tag{10}$$

as can be easily checked by considering the cases $r \geqq 0$ and $r < 0$ separately.

Multiplying (9) by $j''(t)/|t|$ and integrating, we have

$$\int_{-\infty}^{\infty}\int_{\Omega}\frac{j''(t)}{|t|}[t(Tu(x)-t)]^{+}dxdt \leqq \int_{-\infty}^{\infty}\int_{\Omega}\frac{j''(t)}{t}[t(u(x)-t)]^{+}dxdt .$$

By Fubini's Theorem and (10), we obtain

$$\int_{\Omega} j(Tu(x))dx \leqq \int_{\Omega} j(u(x))dx .$$

If j is an arbitrary convex l. s. c. function, there is a sequence $\{j_\lambda\}$ of functions as above which converges to j monotonically from below. For instance, we can define (cf. [**2**])

$$j_\lambda(r)=\inf_t \left\{\frac{1}{2\lambda}|r-t|^2+j(t)\right\} .$$

By the preceding result we have

$$\int j_\lambda(Tu)dx \leqq \int j_\lambda(u)dx \leqq \int j(u)dx .$$

This implies $j(Tu)\in L^1(\Omega)$ and the desired inequality.

REMARK. This lemma can be regarded as an interpolation lemma in Orlicz classes. As was kindly pointed out to us by Jodeit, in the linear case it is essentially due to O'Neil [**15**] by a different proof. In the nonlinear case with $j(u)=|u|^p$, the lemma follows from Peetre [**16**] and Lions [**13**]. We thank L. Tartar for some helpful discussions on this subject.

PROOF OF LEMMA 2. Let j be the indefinite integral of γ satisfying $j(0)=0$. It is a convex l. s. c. function from $(-\infty, \infty)$ into $[0, +\infty]$ such that $\min j=0$. Its subdifferential ∂j equals γ. Since $g(x)\in \partial j(u(x))$ a. e., we have, from the definition of subdifferential,

$$\begin{aligned} j(T_\lambda u(x))-j(u(x)) &\geqq g(x)[T_\lambda u(x)-u(x)] \\ &= -\lambda g(x)(T_\lambda Au)(x) \qquad \text{a. e.} \end{aligned}$$

where $T_\lambda=(I+\lambda A)^{-1}$. Using the subdifferential property again,

$$j(0)-j(u(x))\geqq g(x)(0-u(x)) \qquad \text{a. e.}$$

Since gu is integrable, so is $j\circ u$. Applying Lemma 3 to the mapping T_λ, we have

$$\int_{\Omega} j(T_\lambda u(x))dx \leqq \int_{\Omega} j(u(x))dx .$$

Therefore, $(g, T_\lambda Au)\geqq 0$.

As is well-known, $T_\lambda Au \to Au$ in $L^1(\Omega)$ as $\lambda\to 0$. So we obtain the desired result in case $p=1$. If $1<p\leqq\infty$,

$$\|T_\lambda Au\|_p \leqq \|Au\|_p$$

(by Lemma 3 in case $p<\infty$, and by (II) in case $p=\infty$). If $1<p<\infty$, $T_\lambda Au \to Au$ weakly in $L^p(\Omega)$, which leads to the conclusion. In case $p=\infty$, a subsequence of $T_\lambda Au$ converges a.e. so that Lebesgue's dominated convergence theorem is applicable.

PROPOSITION 4. *Let Φ be a convex l.s.c. function from R into $[0, +\infty]$ such that* $\min \Phi = \Phi(0) = 0$. *Let $f \in L^1(\Omega)$ be such that $\Phi \circ f \in L^1(\Omega)$. Under the assumptions of Theorem* 1, *let u be the solution of* (2). *Then*

$$\int_\Omega \Phi(f - Au)dx \leqq \int_\Omega \Phi(f)dx\,.$$

In particular,

$$\|f - Au\|_p \leqq \|f\|_p \tag{11}$$

if $f \in L^1(\Omega) \cap L^p(\Omega)$, $1 \leqq p \leqq \infty$.

FIRST PROOF. Let T be the mapping $f \to f - Au$. By Theorem 1, T is a contraction in $L^1(\Omega)$. Therefore the proposition follows from Lemma 3 as soon as we prove that T satisfies (8). To do this, let $k = \max\{0, \sup_\Omega f\}$ and let $g = f - Au$. We must show that $g(x) \leqq k$ a.e. in Ω. If k does not belong to the range of β, this is obvious since $g(x) \in \beta(u(x))$ a.e. Suppose, on the other hand, that $k \in \beta(l)$ where $l \geqq 0$. Let $h(x)$ be the characteristic function of $\{u(x) > l\} \cup \{g(x) > k\}$. Clearly $h(x) \in \gamma(u(x))$ a.e. where $\gamma(r) = [\operatorname{sign}(r - l)]^+$. By Lemma 2,

$$0 \leqq (Au, h) = (f - g, h) \leqq (k - g, h)\,.$$

But $g(x) \geqq k$ a.e. on the set $\{u(x) > l\}$. Hence $g(x) \leqq k$ a.e. in Ω. Similarly $g(x) \geqq \min\{0, \inf_\Omega f\}$.

SECOND PROOF. We shall give a proof only in the case when: $u \in D(A) \cap L^\infty(\Omega)$, $Au \in L^\infty(\Omega)$, $f \in L^1(\Omega) \cap L^\infty(\Omega)$, β is a continuous monotone function, and Φ is a C^1 function with uniformly Lipschitz derivative. The general case is obtained by a passage to the limit. We multiply the equation by $\phi(\beta(u))$ where $\phi = \Phi'$.

$$(Au, \phi(\beta(u))) + (\beta(u), \phi(\beta(u))) = (f, \phi(\beta(u)))\,.$$

The first term is non-negative by Lemma 2. We apply Young's equality to the second term and Young's inequality to the third term (see Appendix). Thus

$$\int_\Omega [\Phi(\beta(u)) + \Psi(\phi(\beta(u)))]dx \leqq \int_\Omega [\Phi(f) + \Psi(\phi(\beta(u)))]dx\,.$$

Two of these terms are identical, which leaves us with the desired inequality.

PROPOSITION 5. *Let $f, \hat{f} \in L^1(\Omega)$. Let $u, \hat{u}$ be the corresponding solutions. Then*

$$\|[(\hat{f}-A\hat{u})-(f-Au)]^+\|_1 \leqq \|[\hat{f}-f]^+\|_1{}^{(*)}.$$

If $\hat{f} \leqq f$ *a.e. then* $\hat{f}-A\hat{u} \leqq f-Au$ *a.e. and* $\hat{u} \leqq u$ *a.e.*

PROOF. Let $g=f-Au$, $\hat{g}=\hat{f}-A\hat{u}$ so that $g \in B(u)$, $\hat{g} \in B(\hat{u})$. We multiply the equation $A(\hat{u}-u)+\hat{g}-g=\hat{f}-f$ by $h(x)$, where $h(x)$ is the characteristic function of $E=\{\hat{u}(x)>u(x)\} \cup \{\hat{g}(x)>g(x)\}$. Note that $h(x) \in \gamma(\hat{u}(x)-u(x))$ a. e. where $\gamma(r)=[\operatorname{sign}(r)]^+$. By Lemma 2, $(A(\hat{u}-u), h) \geqq 0$. Therefore,

$$\int_E (\hat{g}-g)dx \leqq \int_E (\hat{f}-f)dx \leqq \int_\Omega (\hat{f}-f)^+ dx .$$

By the monotonicity of β, we have $\hat{g} \geqq g$ on E and $\hat{g} \leqq g$ on the complement of E. Thus

$$\int_E (\hat{g}-g)dx = \int_\Omega (\hat{g}-g)^+ dx$$

and the desired estimate follows.

Now let $\hat{f} \leqq f$ a.e. Obviously $\hat{f}-A\hat{u} \leqq f-Au$ a.e. from what we have just proved. Let u_ε, $\hat{u}_\varepsilon$ be the solutions of

$$\varepsilon u_\varepsilon + Au_\varepsilon + \beta(u_\varepsilon) \ni f, \qquad \varepsilon \hat{u}_\varepsilon + A\hat{u}_\varepsilon + \beta(\hat{u}_\varepsilon) \ni \hat{f}.$$

We have just proved that

$$\hat{g}_\varepsilon = \hat{f} - A\hat{u}_\varepsilon \leqq f - Au_\varepsilon = g_\varepsilon \qquad \text{a. e.}$$

where $g_\varepsilon \in \varepsilon u_\varepsilon + \beta(u_\varepsilon)$, $\hat{g}_\varepsilon \in \varepsilon \hat{u}_\varepsilon + \beta(\hat{u}_\varepsilon)$. By the monotonicity of β, we get $\hat{u}_\varepsilon \leqq u_\varepsilon$ a.e. Letting $\varepsilon \to 0$, $\hat{u} \leqq u$ a.e.

PROPOSITION 6. *Let* $f, \hat{f} \in L^1(\Omega)$ *be such that* $A^{-1}(f-\hat{f}) \in L^\infty(\Omega)$. *If* u, $\hat{u}$ *are the corresponding solutions, then*

$$\|u-\hat{u}\|_\infty \leqq 2\|A^{-1}(f-\hat{f})\|_\infty .$$

PROOF. We approximate u by the solution of

$$\varepsilon u_\varepsilon + Au_\varepsilon + \beta(u_\varepsilon) \ni f + \varepsilon A^{-1} f .$$

Letting $v_\varepsilon = u_\varepsilon - A^{-1}f$, we have

$$\varepsilon v_\varepsilon + Av_\varepsilon + g_\varepsilon = 0 \qquad \text{where} \qquad g_\varepsilon \in \beta(v_\varepsilon + A^{-1}f).$$

Similarly $\hat{f}$ determines $\hat{u}_\varepsilon$, $\hat{v}_\varepsilon$ and $\hat{g}_\varepsilon$. Thus

$$\varepsilon(\hat{v}_\varepsilon - v_\varepsilon) + A(\hat{v}_\varepsilon - v_\varepsilon) + \hat{g}_\varepsilon - g_\varepsilon = 0 .$$

We multiply by $h(x)=$ the characteristic function of F where $F=\{\hat{v}_\varepsilon(x) - v_\varepsilon(x) > k\}$ and $k=\|A^{-1}(\hat{f}-f)\|_\infty \geqq 0$. By Lemma 2,

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* We thank M. Crandall for pointing out the quantitative aspect of this ordering property.

$$\varepsilon\int_F(\hat{v}_\varepsilon-v_\varepsilon)dx+\int_F(\hat{g}_\varepsilon-g_\varepsilon)dx\leqq 0\,.$$

But on F we have $\hat{v}_\varepsilon+A^{-1}\hat{f}>v_\varepsilon+A^{-1}f$, so that $\hat{g}_\varepsilon\geqq g_\varepsilon$ by the monotonicity of β. Therefore F has zero measure; that is, $\hat{v}_\varepsilon-v_\varepsilon\leqq k$ a.e. in Ω. So $\hat{u}_\varepsilon-u_\varepsilon\leqq 2k$ a.e. in Ω. Finally, we pass to the limit as $\varepsilon\to 0$.

The following lemma is a variation of Lemma 3. A related result may be found in [3].

LEMMA 3*. *Let T be a mapping from $L^1(\Omega)$ to $L^1(\Omega)$ such that*

(7) $\|Tu-Tv\|_1\leqq\|u-v\|_1$,

(8*) $Tu(x)-Tv(x)\leqq\max\{0,\sup_\Omega(u-v)\}$ a.e.

Let j be a convex function as in Lemma 3. Then

$$\int_\Omega j(Tu(x)-Tv(x))dx\leqq\int_\Omega j(u(x)-v(x))dx$$

for all $u, v\in L^1(\Omega)$ with $j\circ(u-v)\in L^1(\Omega)$.

PROOF. We define $y(x)=\min\{u(x), v(x)+t\}$ where $t\geqq 0$. Note that $y(x)$ is integrable and $y(x)\leqq v(x)+t$ a.e. By assumption (8*), $Ty(x)\leqq Tv(x)+t$ a.e. Thus

$$Tu(x)-Tv(x)-t\leqq Tu(x)-Ty(x)\qquad\text{a. e.}$$

Taking the positive part of each side of this inequality and integrating over Ω, we obtain

$$\begin{aligned}\int_\Omega[Tu(x)-Tv(x)-t]^+dx&\leqq\int[Tu(x)-Ty(x)]^+dx\\&\leqq\int|Tu(x)-Ty(x)|dx\\&\leqq\int|u(x)-y(x)|dx\\&=\int[u(x)-v(x)-t]^+dx\,,\end{aligned}$$

using assumption (7) and the definition of $y(x)$. Now switch the roles of u and v and let $t'=-t\leqq 0$. Then

$$\int[Tv(x)-Tu(x)+t']^+dx\leqq\int[v(x)-u(x)+t']^+dx\,.$$

Thus we have proved Lemma 3* for the particular cases $j_1(r)=(r-t)^+$ and $j_2(r)=(-r-t)^+$. The proof is concluded exactly as in Lemma 3, provided u is replaced by $u-v$ and Tu by $Tu-Tv$.

REMARK. In an effort to unify some of the preceding work, we make the following definition. We shall say that a (possibly nonlinear and multivalued) operator A *belongs to class* $\mathcal{A}$ if its domain and range are included in $L^1(\Omega)$

and, for all $\lambda>0$, $I+\lambda A$ is one-one and onto and $(I+\lambda A)^{-1}$ satisfies (7) and (8*).

Some of the preceding results, including Theorem 1, can be extended to nonlinear operators of class $\mathcal{A}$ under various additional assumptions. Still within the context of linear A, we can state

PROPOSITION 7. *If A is a linear, densely-defined operator which is of class $\mathcal{A}$, then $A+B$ is also of class $\mathcal{A}$.*

PROOF. Applying the results of Theorem 1 to the operators $I+\lambda A$ and λB, we see that the resolvent $[I+\lambda(A+B)]^{-1}$ exists. The beginning of the proof of Theorem 1 shows that the resolvent is a contraction in $L^1(\Omega)$. To prove (8*) for the resolvent, let

$$t=\max\{0, \sup_{\Omega}(\hat{f}-f)\}$$

where $f, \hat{f}\in L^1(\Omega)$. It is necessary to show that

$$T\hat{f}(x)-Tf(x)\leqq t \qquad \text{a. e.}$$

In case $t=0$, this is the last conclusion of Proposition 5. In case $t>0$, a slight change in the proof of Proposition 5 suffices.

The rest of this section is devoted to studying different assumptions on β and A.

REMARK. Let A satisfy (I) and (II) but not (III). Let β be as above but also onto. Let Ω be of finite measure. Then, for every $f\in L^\infty(\Omega)$, there exists $u\in D(A)\cap L^\infty(\Omega)$ such that $Au\in L^\infty(\Omega)$ and equation (2) is satisfied. (The solution may not be unique.)

PROOF. The operator $A_\varepsilon=A+\varepsilon I$, $\varepsilon>0$, satisfies (I), (II) and $\|A_\varepsilon u\|_1\geqq\varepsilon\|u\|_1$. By Theorem 1, there is a unique solution $u_\varepsilon\in D(A)$ of

$$\varepsilon u_\varepsilon+Au_\varepsilon+g_\varepsilon=f\,, \qquad g_\varepsilon\in B(u_\varepsilon)\,.$$

By (11), $\|g_\varepsilon\|_\infty\leqq\|f\|_\infty$. Since β is onto, $\{u_\varepsilon\}$ is also bounded in $L^\infty(\Omega)$. From the equation, the same is true of $\{Au_\varepsilon\}$. There is a sequence $\varepsilon_n\to 0$ so that

$$u_{\varepsilon_n}\to u\,, \quad Au_{\varepsilon_n}\to Au \qquad \text{weakly* in } L^\infty(\Omega)\,,$$

since A is weakly closed in $L^1(\Omega)$. Multiplying the equation by $u_\varepsilon-u$ and using the monotonicity of A, we have

$$(g_\varepsilon, u_\varepsilon-u)=(f-\varepsilon u_\varepsilon-Au_\varepsilon, u_\varepsilon-u)\leqq(f-\varepsilon u_\varepsilon-Au, u_\varepsilon-u)\,,$$
$$\limsup\,(g_{\varepsilon_n}, u_{\varepsilon_n}-u)\leqq 0\,.$$

Also, $g_{\varepsilon_n}\to f-Au$ weakly* in $L^\infty(\Omega)$. The maximal monotonicity of B now implies that $f-Au\in Bu$.

REMARK. Suppose that β depends on x as well as u. We assume that A satisfies (I), (II) and (III) and that Ω is of finite measure. For a. e. x, let $u\to\beta(x,u)$ be a maximal monotone graph with domain $(-\infty,\infty)$ passing through the origin. For each real u, let $x\to(I+\lambda\beta(x,\cdot))^{-1}(u)$ be measurable on Ω. Also let

$$M_C(x)=\sup\{v \mid v\in\beta(x,u),\ |u|\leqq C\}$$

belong to $L^2(\Omega)$ for all C. Then there is a unique solution

$$Au(x)+\beta(x,u(x))\ni f(x)\qquad \text{a. e.}$$

just as in Theorem 1. The only difficulty occurs in the proof that $A+B$ has dense range in $L^1(\Omega)$. As in the earlier proof, we let $\beta_\lambda(x,u)$ be the Yosida approximation of $\beta(x,u)$, we solve the equation

$$\varepsilon u_\lambda(x)+Au_\lambda(x)+\beta_\lambda(x,u_\lambda(x))=f(x)\qquad \text{a. e.}$$

where $f\in L^\infty(\Omega)$, and we have the bound for $\{u_\lambda\}$ in $L^\infty(\Omega)$. The novelty is that we are not allowed anymore to multiply the equation by $\beta_\lambda(x,u_\lambda(x))$. Instead we note that

$$|\beta_\lambda(x,u_\lambda(x))|\leqq|\beta^0(x,u_\lambda(x))|\leqq M(x)\qquad \text{a. e.},$$

so that $\{\beta_\lambda(x,u_\lambda(x))\}$ is bounded in $L^2(\Omega)$. The rest of the proof of Theorem 1 is unchanged.

§2. Some remarks on second-order elliptic operators in L^1.

We consider in R^N an open bounded set Ω with smooth boundary. On Ω we consider the differential operator

$$Lu=-\sum_{i,j}\frac{\partial}{\partial x_j}\Big(a_{ij}\frac{\partial u}{\partial x_i}\Big)+\sum_i\frac{\partial}{\partial x_i}(a_iu)+au$$

where $a_{ij},\ a_i\in C^1(\bar{\Omega})$; $a\in L^\infty(\Omega)$;

$$a\geqq 0,\qquad a+\sum_i\frac{\partial a_i}{\partial x_i}\geqq 0\qquad \text{a. e.};$$

and, for some positive constant α,

$$\sum_{i,j}a_{ij}\xi_i\xi_j\geqq\alpha|\xi|^2\quad \text{a. e.},\qquad \xi\in R^N.$$

The Sobolev space $W^{k,p}(\Omega)$ is defined as the Banach space of all functions in $L^p(\Omega)$ all of whose derivatives up to order k also belong to $L^p(\Omega)$. $W_0^{k,p}(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in this space; $1\leqq p\leqq\infty$, k is a positive integer. The norm is denoted by $\|\ \|_{k,p}$. The usual $L^p(\Omega)$ norm is denoted by $\|\ \|_p$. We write $H^k(\Omega)=W^{k,2}(\Omega)$.

If $1<p<\infty$, the natural realization of L on $L^p(\Omega)$ is denoted by A_p. Its domain is $D(A_p)=W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$. It is a closed operator which generates a contraction semigroup in $L^p(\Omega)$ (cf. [**1**]).

The case $p=1$ is different. We define

$$D(A)=\{u\in W_0^{1,1}(\Omega)\mid Lu\in L^1(\Omega)\}$$

where Lu is understood in the sense of distributions, and we define $Au=Lu$ for $u\in D(A)$. Equivalently, we may say that $u\in D(A)$ and $Au=f$ if and only if $u\in W_0^{1,1}(\Omega)$ and

$$\Sigma\Big(a_{ij}\frac{\partial u}{\partial x_i},\frac{\partial v}{\partial x_j}\Big)-\Sigma\Big(a_i u,\frac{\partial v}{\partial x_i}\Big)+(au,v)=(f,v) \tag{12}$$

for all $v\in W_0^{1,\infty}(\Omega)$.

THEOREM 8(*). *The operator A satisfies the following properties.*

(I) *In the space $L^1(\Omega)$, $D(A)$ is dense, A is closed, $I+\lambda A$ is onto and $(I+\lambda A)^{-1}$ is a contraction for $\lambda>0$.*

(II) $\sup_\Omega (I+\lambda A)^{-1}f\leqq \max\{0,\sup_\Omega f\}$ *for $\lambda>0$ and $f\in L^1(\Omega)$.*

$$D(A)\subset W_0^{1,q}(\Omega)\ \textit{for}\ 1\leqq q<N/(N-1)\,;\ \textit{for some}\ \alpha=\alpha(q)>0\,, \tag{13}$$

$$\alpha\|u\|_{1,q}\leqq\|Au\|_1\qquad \textit{for}\quad u\in D(A)\,.$$

$$A\ \textit{is the closure in}\ L^1(\Omega)\ \textit{of the operator}\ A_2\,. \tag{14}$$

LEMMA 9. *A contains the closure $\bar{A}_2$ in $L^1(\Omega)$ of A_2. Also $\bar{A}_2$ satisfies the estimate in condition* (13).

PROOF. We have $D(A_2)=H^2(\Omega)\cap H_0^1(\Omega)\subset W_0^{1,q}(\Omega)$. Consider the adjoint equation

$$-\Sigma\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial v}{\partial x_j}\Big)-\Sigma a_i\frac{\partial v}{\partial x_i}+av=-\Sigma\frac{\partial h_i}{\partial x_i}\,.$$

Stampacchia [17] has proven that there exists a solution $v\in H_0^1(\Omega)\cap L^\infty(\Omega)$ whenever $h_1,\cdots,h_N\in L^p(\Omega)$ with $p>N$. More precisely

$$\Sigma\int a_{ij}\frac{\partial v}{\partial x_j}\frac{\partial w}{\partial x_i}-\Sigma\int a_i\frac{\partial v}{\partial x_i}w+\int avw=\Sigma\int h_i\frac{\partial w}{\partial x_i}$$

for all $w\in H_0^1(\Omega)$. In addition, $\|v\|_\infty\leqq C\Sigma\|h_i\|_p$. We simply choose $w=u$ where $u\in D(A_2)$. Then

$$(v,A_2u)=\Sigma(h_i,\partial u/\partial x_i)\,,$$

$$|\Sigma(h_i,\partial u/\partial x_i)|\leqq C(\Sigma\|h_i\|_p)\|A_2u\|_1\,.$$

Therefore $\|\partial u/\partial x_i\|_q\leqq C\|A_2u\|_1$, where $q=p/(p-1)$, so that A_2 satisfies the estimate in (13).

LEMMA 10. *$\bar{A}_2$ satisfies the conditions of* (I).

PROOF. Let $f\in L^2(\Omega)$ and $u=(I+\lambda A_2)^{-1}f$. That is, $u+\lambda Lu=f$, $u\in D(A_2)$. We multiply this equation by $\phi(u)=\varepsilon^{-1}[u-(I+\varepsilon\,\mathrm{sign})^{-1}u]$; that is, $\phi(u)=\mathrm{sign}\,u$

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* This theorem is not new but we could find no explicit reference to it in the literature.

for $|u|\geqq\varepsilon$ and $\phi(u)=u/\varepsilon$ for $|u|\leqq\varepsilon$. We assert that $(Lu, \phi(u))\geqq 0$. Assuming this for the moment, we have

$$(u, \phi(u))\leqq(f, \phi(u))\leqq\|f\|_1 .$$

Letting $\varepsilon\to 0$, $\|u\|_1\leqq\|f\|_1$. To prove $I+\lambda\bar{A}_2$ is onto, let $f\in L^1(\Omega)$. Approximate f by L^2-functions f_n. Let $u_n=(I+\lambda A_2)^{-1}f_n$. By what we have just shown, $\{u_n\}$ converges in $L^1(\Omega)$. The limit u belongs to $D(\bar{A}_2)$ and $u+\lambda\bar{A}_2u=f$. Finally, we prove the above assertion. (Actually it is a special case of Lemma 17 below.) We have

$$(-\sum(a_{ij}u_{x_i})_{x_j}, \phi(u))=(\sum a_{ij}u_{x_i}u_{x_j}, \phi'(u))\geqq 0$$

since ϕ is monotone. If $\zeta(s)$ is defined by $\zeta'(s)=s\phi'(s)$, $\zeta(0)=0$, then $0\leqq\zeta(s)\leqq s\phi(s)$ and

$$((a_iu)_{x_i}, \phi(u))=-(a_iu, \phi'(u)u_{x_i})=-(a_i, \zeta(u)_{x_i})=((a_i)_{x_i}, \zeta(u)) .$$

Hence

$$((a_iu)_{x_i}+au, \phi(u))\geqq((a_i)_{x_i}+a, \zeta(u))\geqq 0 .$$

LEMMA 11. *$I+\lambda A$ is one-one for all $\lambda>0$.*

PROOF. We will prove the slightly stronger assertion that A itself is one-one. Suppose $Au=0$ where $u\in W_0^{1,1}(\Omega)$. That is, equation (12) holds with $f=0$. We shall choose the test function v in (12) as the solution in $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ of $L'v=g$ where L' is the formal adjoint of L, $g\in L^p(\Omega)$ and $p>N$; cf. [**1**]. The equation for v is understood in the sense that

$$\sum(w_{x_i}, a_{ij}v_{x_j})-\sum(w, a_iv_{x_i})+(w, av)=(w, g)$$

for all $w\in\mathcal{D}(\Omega)$. We may take any $w\in W_0^{1,1}(\Omega)$. Taking $w=u$, we have simply $0=(u, g)$. Since g is arbitrary, u must vanish.

PROOF OF THEOREM 8. Since $I+\bar{A}_2$ is onto and its extension $I+A$ is one-one, the two operators must coincide. This proves (14), (13) and (I). To prove the maximum principle (II), let $f\in L^1(\Omega)$ and $u=(I+\lambda A)^{-1}f$. We may assume $k=\sup f$ is finite. Let $f_n(x)=\max\{f(x), -n\}$. Then $f_n\in L^\infty(\Omega)$ and $f_n\to f$ in $L^1(\Omega)$. Let $u_n=(I+\lambda A)^{-1}f_n$. By a known maximum principle [**17**],

$$u_n(x)\leqq\max\{0, \sup f_n\}\leqq k \qquad \text{a. e.}$$

Since $u_n\to u$ in $L^1(\Omega)$ by Lemma 10, $u(x)\leqq k$ a. e.

Some of the preceding results may be summarized in the following form.

COROLLARY 12. *Let Ω be a bounded domain in R^n with smooth boundary. Let L be the elliptic operator defined at the beginning of this section. Let β be a maximal monotone graph in $R\times R$ containing the origin. Let $f\in L^1(\Omega)$. Then there exists a unique solution in $W_0^{1,1}(\Omega)$ of $Lu+\beta(u)\ni f$ with $Lu\in L^1(\Omega)$. Furthermore, $u\in W_0^{1,q}(\Omega)$ for all $q<N/(N-1)$. In case $f\in L^p(\Omega)$ where*

$1 < p < \infty$, *we have* $u \in W^{2,p}(\Omega)$.

REMARK. In the important case $f \in L \log L$, we may apply Proposition 4 with $\Phi(s) = |s| \log^+ |s|$. We obtain $Lu \in L \log L$. Under certain conditions, of which only special cases have appeared in the literature (cf. [**19**]), this implies that $u \in W^{2,1}(\Omega)$.

§ 3. The non-monotone case.

Let Ω be a bounded open set in R^N. Let L be the elliptic operator defined at the beginning of section 2, except that all the coefficients are assumed to be merely in $L^\infty(\Omega)$.

The (Orlicz) space in which we will work is determined by a C^2 function $\phi(s)$ which is odd, non-decreasing, $\phi(0) = 0$, $\phi(s) \to +\infty$ as $s \to +\infty$. We define

$$\Phi(s) = \int_0^s \phi(t)dt, \qquad \theta(s) = \int_0^s [\phi'(t)]^{1/2} dt.$$

We define $\Psi(r)$ as the conjugate convex function of $\Phi(s)$. See the Appendix for relations among these functions.

We denote by $\lambda_1 > 0$ the smallest eigenvalue of the operator $-\sum \partial/\partial x_j (a_{ij} \partial/\partial x_i)$ in Ω with Dirichlet boundary conditions. We postulate a *nonlinear term* $\beta(x, s)$ with the following properties.

(15) $\beta(x, s)$ is measurable in $x \in \Omega$, continuous in $s \in R$ and

$$\int_\Omega \sup_{|s| \leq \mu} |\beta(x, s)| dx < \infty \qquad \text{for all} \quad \mu.$$

(16) There exists $g \in L^1(\Omega)$ such that(*)

$$\liminf_{|s| \to \infty} \inf_{x \in \Omega} \frac{\beta(x, s)\phi(s) + g(x)}{(\theta(s))^2} > -\lambda_1.$$

We denote $Bu(x) = \beta(x, u(x))$.

THEOREM 13. *There exists a solution of* $Lu + Bu = 0$ *a.e. in* Ω *with the following properties:*

$$u \in W_0^{1,q}(\Omega) \qquad \text{for} \quad 1 \leq q < N/(N-1),$$

$$B(u) \in L^1(\Omega), \qquad B(u)\phi(u) \in L^1(\Omega),$$

$$\theta(u) \in H_0^1(\Omega).$$

We postpone the proof in favor of some applications.

COROLLARY 14 (*inhomogeneous equation*). *Let* $\beta(x, s)$ *satisfy* (15) *and* $s\beta(x, s) \geq 0$ *for a.e.* $x \in \Omega$ *and sufficiently large* $|s|$. (*The latter condition implies* (16)

* This kind of condition also exists in the regularity theory [**17**].

with $g=0$, which is actually sufficient.) *Assume that*

(17) $$s\phi''(s) \leqq \text{const}\, \phi'(s) \qquad \textit{for } |s| \textit{ sufficiently large.}$$

Then there exists a solution of

$$Lu(x)+\beta(x, u(x))=f(x) \qquad a.e. \quad in \ \Omega$$

with the properties of Theorem 13, *for each measurable function $f(x)$ for which $\Phi(f(x))$ is integrable.*

PROOF. We verify the assumptions of Theorem 13 for the nonlinear term $\beta(x, s)-f(x)$. Obviously (15) is satisfied. We choose $g(x)=\Phi(kf(x))$ where $k>1$ will be chosen below. Since $\Phi(ks)/\Phi(s)$ is bounded as $|s|\to\infty$ (see Appendix), $g\in L^1(\Omega)$. Note that $\beta(x, s)\phi(s)\geqq 0$ by assumption. By Young's inequality,

$$|f(x)\phi(s)| \leqq \Phi(kf(x))+\Psi(k^{-1}\phi(s)).$$

Since Ψ is convex and $\Psi(0)=0$, we have $\Psi(k^{-1}\phi(s))\leqq k^{-1}\Psi(\phi(s))$. Thus for $|s|$ large

$$\frac{[\beta(x, s)-f(x)]\phi(s)+g(x)}{\theta^2(s)} \geqq -k^{-1}\frac{\Psi(\phi(s))}{\theta^2(s)}.$$

Since the last quotient is bounded as $|s|\to\infty$ (see Appendix), k may be chosen so large that (16) holds.

EXAMPLES. In this and the following example we assume $s\beta(x, s)\geqq 0$ for large $|s|$. Choosing $\phi(s)=|s|^{p-1}\operatorname{sign} s$, where $1<p<\infty$, we obtain from Corollary 14 a solution of $Lu+Bu=f$ for any $f\in L^p(\Omega)$. For a sharper result, see Theorem 18 below. As a second example, we choose $\phi(s)=\log(1+|s|)\operatorname{sign} s$ so that $\theta(s)=2[(1+|s|)^{1/2}-1]\operatorname{sign} s$. Then we have

COROLLARY 15. *For each $f\in L\log L$, there is a solution of $Lu+Bu=f$ satisfying*

$$u\in L^{N/(N-2)}(\Omega)\cap W_0^{1,q}(\Omega), \qquad \textit{for } q<N/(N-1), \quad \textit{if } N>1^{(*)},$$
$$B(u)\log(1+|u|)\in L^1(\Omega),$$
$$[(1+|u|^{1/2})-1]\operatorname{sign} u\in H_0^1(\Omega).$$

REMARK. While Corollary 14 does not permit a solution for arbitrary *integrable $f(x)$*, it does allow $f(x)$ to belong to an "arbitrarily" smaller class. There are two restrictions on $\phi(s)$: that it goes to infinity with s, and condition (17). It is only the first of these which restricts the size of the class of f's and excludes the L^1 case. Thus another example of an allowable class is $L\log\log L$.

LEMMA 16. *Suppose that $\beta(x, s)$ is uniformly bounded on $\Omega\times R$, as well as measurable in x and continuous in s. Then there exists a solution $u\in H_0^1(\Omega)\cap L^\infty(\Omega)$ of the equation $Lu+B(u)=0$.*

* $u\in W^{2,1}(\Omega)\cap W_0^{1,1}(\Omega)$ if $N=1$.

PROOF. For any $v \in L^2$, let u be the solution in $H_0^1(\Omega)$ of the *linear* equation $Lu+B(v)=0$. Upon multiplying this equation by u and integrating over Ω, we find

$$C\|u\|_{1,2} \leqq \|Bv\|_2 \leqq |\Omega|^{1/2}\|Bv\|_\infty$$

where C is a constant. So the mapping $v \to u$ takes $L^2(\Omega)$ into a compact set in $L^2(\Omega)$. Since this mapping is continuous, it has a fixed point belonging to $H_0^1(\Omega)$, by Schauder's theorem. By the maximum principle [17], the solution is essentially bounded.

LEMMA 17. *For any $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$, we have the inequality*

$$(Lv, \phi(v)) \geqq \max\{\alpha\|\theta(v)\|_{1,2}^2, \lambda_1\|\theta(v)\|_2^2\} \geqq 0 .$$

PROOF. We first consider the lower-order terms

$$\begin{aligned} ((a_i v)_{x_i}, \phi(v)) &= -(a_i v, \phi'(v) v_{x_i}) \\ &= -(a_i, \Psi(\phi(v))_{x_i}) \\ &= ((a_i)_{x_i}, \Psi(\phi(v))) \end{aligned}$$

since

$$s\phi'(s) = (s\phi(s) - \Phi(s))' = [\Psi(\phi(s))]' .$$

Also, since $a \geqq 0$,

$$(av, \phi(v)) \geqq \int a\Psi(\phi(v))dx .$$

Adding, we find

$$((a_i v)_{x_i} + av, \phi(v)) \geqq \int ((a_i)_{x_i} + a)\Psi(\phi(v))dx \geqq 0 .$$

We write the second-order terms as

$$\begin{aligned} -((a_{ij} v_{x_i})_{x_j}, \phi(v)) &= (a_{ij} v_{x_i}, \phi'(v) v_{x_j}) \\ &= (a_{ij}\theta(v)_{x_i}, \theta(v)_{x_j}) \end{aligned}$$

since $\theta'(v)^2 = \phi'(v)$. The proof is completed by using the ellipticity and the meaning of λ_1.

PROOF OF THEOREM 13. We truncate $\beta(x, s)$ as follows:

$$\beta_n = \begin{cases} \beta & \text{where } |\beta| \leqq n \\ n \operatorname{sign} \beta & \text{where } |\beta| \geqq n . \end{cases}$$

For each (x, s), $\beta_n(x, s)$ has the same sign as $\beta(x, s)$ but decreased magnitude. Denote the operator $u(\cdot) \to \beta_n(\cdot, u(\cdot))$ by B_n. By Lemma 16, there exists $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that $Lu_n + B_n(u_n) = 0$.

Hypothesis (16) may be expressed as follows: There exists $\varepsilon > 0$ and $\mu > 0$ such that

$$\beta(x, s)\phi(s) > (\varepsilon-\lambda_1)\theta^2(s)-g(x) \qquad \text{a.e.}$$

for $|s| \geqq \mu$. Hypothesis (15) gives an estimate for $|s| \leqq \mu$. Hence

$$\beta_n(x, s)\phi(s) \geqq \min\{0, \beta(x, s)\phi(s)\},$$

$$\beta_n(x, s)\phi(s) \geqq (\varepsilon-\lambda_1)\theta^2(s)-h(x) \qquad \text{a.e.}$$

where $h(x)$ is also integrable. Taking $s=u_n(x)$ and integrating, we find

$$(B_n(u_n), \phi(u_n)) \geqq (\varepsilon-\lambda_1)\|\theta(u_n)\|_2^2-\|h\|_1.$$

On the other hand, by Lemma 17,

$$(Lu_n, \phi(u_n)) \geqq \lambda_1\|\theta(u_n)\|_2^2.$$

Adding, we conclude that $\{\theta(u_n)\}$ is bounded in $L^2(\Omega)$ and that both $(B_n(u_n), \phi(u_n))$ and $(Lu_n, \phi(u_n))$ are bounded. By Lemma 17, $\{\theta(u_n)\}$ is also bounded in $H_0^1(\Omega)$.

We can also make the following estimates.

$$\int |B_n(u_n)\phi(u_n)|\,dx = (B_n(u_n), \phi(u_n)) - 2\int_{B_n(u_n)\phi(u_n)\leqq 0} B_n(u_n)\phi(u_n)\,dx$$

$$\leqq (B_n(u_n), \phi(u_n)) + 2(\lambda_1-\varepsilon)\|\theta(u_n)\|_2^2 + 2\|h\|_1$$

and

$$\int |B_n(u_n)|\,dx \leqq \int |B_n(u_n)\phi(u_n)|\,dx + \int_{|\phi(u_n)|<1} |B(u_n)|\,dx.$$

The last term is bounded because the integration occurs over the set $\{|u_n| \leqq \phi^{-1}(1)\}$ and $\beta(x, u_n)$ is bounded by an integrable function there. Thus $\{B_n(u_n)\phi(u_n)\}$ and $\{B_n(u_n)\}$ are both bounded in $L^1(\Omega)$. Therefore, $\{Lu_n\} = \{-B_n(u_n)\}$ is also bounded in $L^1(\Omega)$. So by (13), $\{u_n\}$ is bounded in $W_0^{1,q}(\Omega)$ for any $q < N/(N-1)$.

By weak compactness, we can choose a subsequence (for which we do not bother changing notation) so that $\{u_n\}$ converges weakly in $W_0^{1,q}(\Omega)$ and $\{\theta(u_n)\}$ converges weakly in $H_0^1(\Omega)$. Let $u = \lim u_n$. By strong compactness, we may assume that $u_n \to u$ a.e., and hence that $\theta(u_n) \to \theta(u)$ a.e. as well as weakly in $H_0^1(\Omega)$. It also follows that $B_n(u_n) \to B(u)$ a.e. and $B_n(u_n)\phi(u_n) \to B(u)\phi(u)$ a.e. By Fatou's lemma, $B(u)$ and $B(u)\phi(u)$ are integrable on Ω. Since $\phi(s) \to \infty$ as $s \to \infty$, the argument of [**20**] shows that $B_n(u_n) \to B(u)$ strongly in $L^1(\Omega)$. Thus each term in the approximate equation converges in $L^1(\Omega)$, and $Lu + B(u) = 0$.

Further regularity can be obtained with the use of Sobolev's inequality. We state it only in the L^p-case.

THEOREM 18. *Let $f \in L^p(\Omega)$ for some $1 < q < N/2$. Let*

$$p=\frac{N-2}{N-2q}q>q.$$

Let $s\beta(x,s)\geqq 0$ *for large* s. *Then there exists a solution of* $Lu+B(u)=f$ *in* Ω *satisfying:*

$$u\in L^{Np/(N-2)}(\Omega)=L^{Nq/(N-2q)}(\Omega),$$

$$|u|^{p/2}\operatorname{sign}u\in H_0^1(\Omega),$$

$$|u|^{p-1}B(u)\in L^1(\Omega).$$

[Note that this is stronger than the example following Corollary 14 mainly in that f need not belong to $L^p(\Omega)$.]

PROOF. We choose the multiplier $\phi(s)=|s|^{p-1}\operatorname{sign}s$. By Hölder's inequality,

$$(f,\phi(u))\leqq\|f\|_q\|u^{p-1}\|_{q'}=\|f\|_q\|u\|_{Np/(N-2)}^{p-1}.$$

On the other hand, by Lemma 17, there is a positive number C such that

$$(Lu,\phi(u))\geqq C\|\theta(u)\|_{2N/(N-2)}^2=C\|u\|_{Np/(N-2)}^p.$$

Thus the solution of the truncated problem $Lu_n+B_n(u_n)=f_n$ remains bounded in $L^{Np/(N-2)}$, and $(f_n,\phi(u_n))$ is bounded. The proof continues as before.

THEOREM 19. (a) *If* $f\in L^{N/2}(\Omega)$, *then for some* $\lambda>0$ *the solution satisfies* $\exp(\lambda|u|)\in L^{N/(N-2)}(\Omega)$,

$$\left[\exp\left(\frac{\lambda}{2}|u|-1\right)\right]\operatorname{sign}(u)\in H_0^1(\Omega),\qquad B(u)\exp\left(\frac{\lambda}{2}|u|\right)\in L^1(\Omega).$$

(b) *If* $f\in L^q(\Omega)$ *with* $q>N/2$, *then the solution is uniformly Hölder-continuous.* (*Assume* a_{ij}, $a_i\in C^1(\bar{\Omega})$ *and* $\sup_{|s|\leqq\mu}|\beta(x,s)|\in L^q(\Omega)$.)

PROOF. Part (a). We choose $\phi(s)=[\exp\lambda|s|-1]\operatorname{sign}(s)$ so that $\theta(s)=2\lambda^{-1/2}[\exp(\lambda|s|/2)-1]\operatorname{sign}(s)$. From the proof of Theorem 13, together with Sobolev's inequality, the approximate solutions satisfy

$$c\lambda^{-1}\|e^{\lambda|u_n|/2}-1\|_{2N/(N-2)}^2+(B_n(u_n),\phi(u_n))\leqq\|f_n\|_{N/2}\|e^{\lambda|u_n|}-1\|_{N/(N-2)}$$

with c independent of u, λ, N, n. We need only choose λ less than $c/\|f\|_{N/2}$ to balance the right side. Part (a) follows.

Part (b) is essentially contained in Stampacchia [17], but for completeness we present a proof. We recall from Theorem 18 that $u\in L^p(\Omega)$ for all finite p and that

$$c_1\|\theta(u)\|_{2N/(N-2)}^2\leqq\|f\|_q\|u^{p-1}\|_{q'}\leqq c_2\|u\|_{pq'}^{p-1}.$$

(The constants are independent of p.) Hence

$$\|u\|_{pN/(N-2)}^p\leqq C_3p\|u\|_{pq'}^{p-1}.$$

If $\|u\|_p\leqq 1$ for an infinite sequence of p's going to $+\infty$, then $\|u\|_\infty\leqq 1$. In the contrary case, we may assume $\|u\|_p > 1$ for all $p \geqq p_0$. So we may replace the exponent $(p-1)$ above by p. Defining

$$\theta = \frac{N}{N-2}\frac{1}{q'} > 1, \qquad p_\nu = \theta^\nu p_0,$$

we have

$$\|u\|_{p_{\nu+1}} \leqq (c_4 p_\nu)^{q'/p_\nu} \|u\|_{p_\nu} \qquad (\nu = 0, 1, 2, \cdots).$$

Iterating as in Moser [**14**], we have

$$\|u\|_{p_\nu} \leqq c_5 \|u\|_{p_0}.$$

So the solution is bounded. Hence $Au = f - B(u) \in L^q(\Omega)$. So $u \in W^{2,q}(\Omega)$ and it is Hölder-continuous.

§ 4. A nonlinear boundary condition.

In this section we will solve an equation of the same form as in section 3 with the boundary condition

$$\frac{\partial u}{\partial n_L} + \gamma(x, u) = 0 \qquad \text{on} \quad \Gamma.$$

Let Ω be a bounded open set in R^N with smooth boundary Γ. We define L as in section 2 with $a_{ij}, a_i \in C^1(\bar{\Omega})$ and $a \in L^\infty(\Omega)$. In addition we assume

$$a \geqq \alpha', \qquad a + \sum_i \frac{\partial a_i}{\partial x_i} \geqq \alpha' \qquad \text{a.e. in } \Omega$$

for some positive constant α', and

$$\sum_i a_i n_i \geqq 0 \qquad \text{on} \quad \Gamma$$

where $n = (n_i)$ is the unit outward normal on Γ. We denote

$$\frac{\partial}{\partial n_L} = \sum_{i,j} a_{ij} n_j \frac{\partial}{\partial x_i}.$$

DEFINITION. Let $u \in W^{1,1}(\Omega)$, $f \in L^1(\Omega)$, $g \in L^1(\Gamma)$. We say that u is a *weak solution* of the Neumann problem

$$Lu = f \quad \text{in} \quad \Omega, \qquad \frac{\partial u}{\partial n_L} = g \quad \text{on} \quad \Gamma \tag{18}$$

provided the following identity holds for all $v \in C^1(\bar{\Omega})$.

$$\begin{aligned} a(u, v) &= \int_\Omega \Big\{ \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_i \frac{\partial}{\partial x_i}(a_i u) v + auv \Big\} dx \\ &= \int_\Omega fv \, dx + \int_\Gamma gv \, d\Gamma. \end{aligned} \tag{19}$$

Let λ be a positive constant so small that

$$\alpha\int_{\Omega}\sum_{i}\left|\frac{\partial u}{\partial x_i}\right|^2 dx+\alpha'\int_{\Omega}|u|^2dx \geqq \lambda\int_{\Omega}|u|^2dx+\lambda\int_{\Gamma}|u|^2d\Gamma$$

for all $u\in H^1(\Omega)$.

Let ϕ be a function as in Theorem 13. Let $\beta(x, s)$ and $\gamma(x, s)$ be nonlinear terms satisfying:

$\beta(x, s)$ is measurable in $x\in\Omega$, continuous in $s\in R$.

$\gamma(x, s)$ is measurable in $x\in\Gamma$, continuous in $s\in R$.

(20) $\int_{\Omega}\sup_{|s|\leqq\mu}|\beta(x, s)|dx<+\infty, \qquad \int_{\Omega}\sup_{|s|\leqq\mu}|\gamma(x, s)|dx<+\infty \qquad$ for all μ.

(21) There exist $h_1\in L^1(\Omega)$ such that

$$\liminf_{|s|\to+\infty}\inf_{x\in\Omega}\frac{\beta(x, s)\phi(s)+h_1(x)}{(\theta(s))^2}>-\lambda$$

and $h_2\in L^1(\Gamma)$

$$\liminf_{|s|\to+\infty}\inf_{x\in\Gamma}\frac{\gamma(x, s)\phi(s)+h_2(x)}{(\theta(s))^2}>-\lambda\,.$$

We denote $Bu(x)=\beta(x, u(x))$ for a. e. $x\in\Omega$ and $Cu(x)=\gamma(x, \tau u(x))$ for a. e. $x\in\Gamma$, where τ is the trace operator mapping $W^{1,p}(\Omega)$ onto $W^{1-1/p,p}(\Gamma)$.

THEOREM 20. *There exists a weak solution of*

$$Lu+Bu=0 \quad in \quad \Omega\,, \qquad \frac{\partial u}{\partial n_L}+Cu=0 \quad on \quad \Gamma \tag{22}$$

with the following properties:

$$u\in W^{1,q}(\Omega) \qquad for \quad 1\leqq q<N/(N-1)\,,$$
$$B(u) \quad and \quad B(u)\phi(u)\in L^1(\Omega)\,,$$
$$C(u) \quad and \quad C(u)\phi(u)\in L^1(\Gamma)\,,$$
$$\theta(u)\in H^1(\Omega)\,.$$

COROLLARY 21. *Let β and γ satisfy* (20) *and* (21) *and* $s\beta(x, s)\geqq 0$ *a. e. in* Ω, $s\gamma(x, s)\geqq 0$ *a. e. in* Γ *for sufficiently large* $|s|$. *Let* $\phi(s)$ *satisfy* (17). *Then there exists a weak solution of*

$$Lu+Bu=f \quad in \quad \Omega\,, \qquad \frac{\partial u}{\partial n_L}+Cu=g \quad on \quad \Gamma \tag{23}$$

with the same properties as in Theorem 20, *so long as* $\Phi(f)\in L^1(\Omega)$ *and* $\Phi(g)\in L^1(\Gamma)$.

THEOREM 22. *Let* $\beta(x, s)$ *and* $\gamma(x, s)$ *be monotone functions of* s, *in addition to satisfying* (20) *and* (21). *For any* $f\in L^1(\Omega)$ *and* $g\in L^1(\Gamma)$ *there exists a unique weak solution of* (23) *satisfying* $u\in W^{1,q}(\Omega)$ *for* $1\leqq q<N/(N-1)$,

$$B(u)\in L^1(\Omega), \qquad C(u)\in L^1(\Gamma).$$

Corollary 21 can be applied to the L^p $(1<p<+\infty)$ and $L\log L$ cases as in section 3.

We begin the proofs with an estimate for the linear problem (18). This is the analogue for the Neumann problem of Lemma 9.

LEMMA 23. *Let u be a weak solution of* (18). *Then we have $u\in W^{1,q}(\Omega)$ for all $1\leqq q<N/(N-1)$ and*

$$\|u\|_{1,q}\leqq C_q(\|f\|_{L^1(\Omega)}+\|g\|_{L^1(\Gamma)}).$$

In order to prove Lemma 23, we first note the following.

LEMMA 24. *Given $h_0, h_1, \cdots, h_N\in \mathcal{D}(\Omega)$, there exists a unique $v\in C^1(\bar{\Omega})$ satisfying*

$$\text{(24)}\qquad a^*(v,\zeta)=\int_\Omega\Big\{\sum_{i,j}a_{ij}\frac{\partial v}{\partial x_j}\frac{\partial \zeta}{\partial x_i}-\sum_i a_i\frac{\partial v}{\partial x_i}\zeta+av\zeta\Big\}dx+\int_\Gamma\sum a_in_iv\zeta\,d\Gamma$$
$$=\int_\Omega\Big\{h_0\zeta+\sum_i h_i\frac{\partial\zeta}{\partial x_i}\Big\}dx$$

for all $\zeta\in W^{1,1}(\Omega)$. In addition,

$$\|v\|_\infty\leqq C_p\sum_{i=0}^N\|h_i\|_p \qquad \textit{for all}\quad p>N.$$

PROOF OF LEMMA 24. For all $\zeta\in H^1(\Omega)$ we have

$$a^*(\zeta,\zeta)\geqq\alpha\int_\Omega|\nabla\zeta|^2dx+\int_\Omega\Big(a+\frac{1}{2}\sum_i\frac{\partial a_i}{\partial x_i}\Big)|\zeta|^2dx+\frac{1}{2}\int_\Gamma\sum_i a_in_i|\zeta|^2d\Gamma$$
$$\geqq\int(\alpha|\nabla\zeta|^2+\alpha'|\zeta|^2)dx.$$

By the Lax-Milgram lemma, there exists $v\in H^1(\Omega)$ satisfying (24) for all $\zeta\in H^1(\Omega)$. From the results of [**1**] we know that $v\in C^1(\bar{\Omega})$. Hence (24) holds for all $\zeta\in W^{1,1}(\Omega)$. In order to establish the L^∞ bound on v, we use Stampacchia's method. If in (24) we choose $\zeta(x)=\max\{v(x)-k, 0\}$ where $k\geqq 0$, we obtain

$$a^*(v,\zeta)=a^*(\zeta,\zeta)+k\int_\Omega a\zeta\,dx+k\int_\Gamma(\sum_i a_in_i)\zeta\,d\Gamma$$
$$=\int_\Omega h_0\zeta\,dx+\int_\Omega\sum_i h_i\frac{\partial\zeta}{\partial x_i}dx.$$

Consequently (since $k\geqq 0$, $\zeta\geqq 0$, $a\geqq 0$, $\sum_i a_in_i\geqq 0$)

$$\alpha\int_\Omega\sum_i\Big|\frac{\partial\zeta}{\partial x_i}\Big|^2dx+\alpha'\int_\Omega|\zeta|^2dx\leqq\int_\Omega h_0\zeta\,dx+\int_\Omega\sum_i h_i\frac{\partial\zeta}{\partial x_i}dx.$$

We conclude, for example as in [**9**] (proof of Lemma 7.3), that the stated bound on v is valid.

PROOF OF LEMMA 23. Let $h_0, h_1, \cdots, h_N \in \mathcal{D}(\Omega)$ and let $v \in C^1(\bar{\Omega})$ be given by Lemma 24. We take $\zeta = u$ in (24). On the other hand, we know from (19) that

$$a^*(v, u) = \int_\Omega fv dx + \int_\Gamma gv d\Gamma .$$

Therefore

$$\int_\Omega \left\{h_0 u + \sum_i h_i \frac{\partial u}{\partial x_i}\right\} dx \leqq (\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\Gamma)}) \|v\|_{L^\infty(\Omega)}$$
$$\leqq C_p(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\Gamma)}) \sum_{i=0}^N \|h_i\|_p$$

for all $p > N$, which implies the conclusion of Lemma 23.

LEMMA 25 (analogue of Lemma 16). *Suppose that $\beta(x, s)$ and $\gamma(x, s)$ are uniformly bounded on $\Omega \times R$ and on $\Gamma \times R$, respectively, as well as measurable in x and continuous in s. Then there exists a weak solution $u \in H^1(\Omega) \cap L^\infty(\Omega)$ of the problem*

$$Lu + Bu = 0 \quad in\ \Omega, \qquad \frac{\partial u}{\partial n_L} + Cu = 0 \quad on\ \Gamma .$$

PROOF. For any $v \in H^{3/4}(\Omega)$, let u be the weak solution in $H^1(\Omega)$ of the problem

$$Lu + Bv = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n_L} + Cv = 0 \quad \text{on } \Gamma,$$

which exists by Lax-Milgram lemma since $a(u, u)$ is coercive. Also we have $\|u\|_{1,2} \leqq C(\|Bv\|_{L^2(\Omega)} + \|Cv\|_{L^2(\Gamma)})$. So the mapping $v \to u$ has a fixed point. As in the proof of Lemma 24, we can check easily that $u \in L^\infty(\Omega)$.

LEMMA 26 (analogue of Lemma 17). *Let $u \in H^1(\Omega) \cap L^\infty(\Omega)$ be a solution of* (18) *with f and g essentially bounded. Then*

$$\alpha \int_\Omega \sum_i \left|\frac{\partial}{\partial x_i} \theta(u)\right|^2 dx + \alpha' \int_\Omega u\phi(u) dx \leqq \int_\Omega f\phi(u) dx + \int_\Gamma g\phi(u) d\Gamma .$$

PROOF. We simply put $v = \phi(u)$ in (19). As in Lemma 17, we have

$$\int_\Omega \frac{\partial}{\partial x_i}(a_i u)\phi(u) dx = \int_\Gamma a_i n_i \{u\phi(u) - \Psi(\phi(u))\} d\Gamma + \int_\Omega \frac{\partial a_i}{\partial x_i} \Psi(\phi(u)) dx .$$

Therefore

$$\alpha \int_\Omega \sum_i \left|\frac{\partial}{\partial x_i} \theta(u)\right|^2 dx + \int_\Gamma \sum_i a_i n_i \{u\phi(u) - \Psi(\phi(u))\} d\Gamma$$
$$+ \int_\Omega \sum_i \frac{\partial a_i}{\partial x_i} \Psi(\phi(u)) dx + \int_\Omega au\phi(u) dx$$
$$\leqq \int_\Omega f\phi(u) dx + \int_\Gamma g\phi(u) d\Gamma .$$

Noting that $r\phi(r)-\Psi(\phi(r))\geqq 0$, the desired estimate follows.

PROOF OF THEOREM 20. The proof is very similar to that of Theorem 13. We truncate $\beta(x, s)$ exactly as before. We truncate $\gamma(x, s)$ in the same way. By Lemma 25 we solve

$$Lu_n+B_nu_n=0 \quad \text{in } \Omega, \qquad \frac{\partial u_n}{\partial n_L}+C_nu_n=0 \quad \text{on } \Gamma.$$

As before we have

$$\int_\Omega B_nu_n\phi(u_n)dx \geqq (\varepsilon-\lambda)\int_\Omega |\theta(u_n)|^2dx-\text{const}.$$

and

$$\int_\Gamma C_nu_n\phi(u_n)d\Gamma \geqq (\varepsilon-\lambda)\int_\Gamma |\theta(u_n)|^2d\Gamma-\text{const}.$$

Therefore

$$\begin{aligned}\alpha\int_\Omega \sum_i \left|\frac{\partial}{\partial x_i}\theta(u_n)\right|^2dx+\alpha'\int_\Omega u_n\phi(u_n)dx& \\ +(\varepsilon-\lambda)\int_\Omega |\theta(u_n)|^2dx+(\varepsilon-\lambda)\int_\Gamma |\theta(u_n)|^2d\Gamma& \\ \leqq \text{const}.&\end{aligned}$$

Since $r\phi(r)\geqq|\theta(r)|^2$, we deduce from the choice of λ, that $\{\theta(u_n)\}$ is bounded in $H^1(\Omega)$, $\int_\Omega B_nu_n\phi(u_n)dx$ and $\int_\Gamma C_nu_n\phi(u_n)d\Gamma$ are bounded. Hence $\{B_nu_n\}$ is bounded in $L^1(\Omega)$ and $\{C_nu_n\}$ in $L^1(\Gamma)$. By Lemma 23, $\{u_n\}$ is bounded in $W^{1,q}(\Omega)$ for $q<N/(N-1)$. The proof is completed as before. It should be noted that $\{\tau u_n\}$ is bounded in $W^{1-1/q,q}(\Gamma)$ and hence may be assumed to converge a.e. on Γ. By Fatou's lemma, $C(u)$ and $C(u)\phi(u)$ are integrable on Γ. Since $\phi(s)\to+\infty$ as $s\to+\infty$, $C_n(u_n)\to C(u)$ in $L^1(\Gamma)$. It follows that u is a weak solution of (22).

PROOF OF COROLLARY 21. As in Corollary 14, we verify that $Bu-f$ and $Cu-g$ satisfy the conditions of Bu and Cu, respectively, in Theorem 20.

PROOF OF THEOREM 22. We may assume that $\beta(x, 0)=0$ and $\gamma(x, 0)=0$, for otherwise, they may be absorbed in $f(x)$ and $g(x)$, respectively. Let f_n, g_n be sequences of square-integrable functions tending to f, g (respectively) in $L^1(\Omega)$, $L^1(\Gamma)$. Let u_n be a solution of

$$Lu_n+Bu_n=f_n \quad \text{in } \Omega, \qquad \frac{\partial u_n}{\partial n_L}+Cu_n=g_n \quad \text{on } \Gamma,$$

according to Corollary 21. We subtract the equations for u_n and u_m and multiply by $\phi(u_n-u_m)$ where $\phi(s)$ is a smooth, bounded, monotone approximation to sign$[s]$. Estimating as in Lemma 26, we get

$$\alpha'\int_\Omega (u_n-u_m)\phi(u_n-u_m)dx$$

$$+\int_{\Omega}(B(u_n)-B(u_m))\phi(u_n-u_m)dx+\int_{\Gamma}(Cu_n-Cu_m)\phi(u_n-u_m)d\Gamma$$

$$\leqq\int_{\Omega}(f_n-f_m)\phi(u_n-u_m)dx+\int_{\Gamma}(g_n-g_m)\phi(u_n-u_m)d\Gamma\,.$$

Letting $\phi\rightarrow$sign, we see that u_n, Bu_n and Cu_n all converge in L^1. Using Lemma 23, the existence part of Theorem 22 follows.

To prove the uniqueness, let u be any solution and let h_n, k_n be L^2 functions tending to $f-Bu$, $g-Cu$ in $L^1(\Omega)$, $L^1(\Gamma)$, respectively. Let $w_n\in H^1(\Omega)$ be the solution of the linear problem

$$Lw_n=h_n \quad \text{in } \Omega\,, \qquad \frac{\partial w_n}{\partial n_L}=k_n \quad \text{on } \Gamma\,.$$

By Lemma 23 we have

$$\|w_n-u\|_{1,q}\leqq C_q(\|h_n-f+Bu\|_{L^1(\Omega)}+\|k_n-g+Cu\|_{L^1(\Gamma)})$$

and therefore $w_n\rightarrow u$ in $W^{1,q}(\Omega)$. Now suppose $\hat{u}$ is another solution as in Theorem 22 and let $\hat{h}_n$, $\hat{k}_n$, $\hat{w}_n$ be constructed as above. Multiplying the equation $L(w_n-\hat{w}_n)=h_n-\hat{h}_n$ by $\phi(w_n-\hat{w}_n)$, we obtain

$$\alpha'\int_{\Omega}(w_n-\hat{w}_n)\phi(w_n-\hat{w}_n)dx\leqq\int_{\Omega}(h_n-\hat{h}_n)\phi(w_n-\hat{w}_n)dx$$

$$+\int_{\Gamma}(k_n-\hat{k}_n)\phi(w_n-\hat{w}_n)d\Gamma\,.$$

Letting $n\rightarrow+\infty$ we have

$$\alpha'\int_{\Omega}(u-\hat{u})\phi(u-\hat{u})dx\leqq\int_{\Omega}(-Bu+B\hat{u})\phi(u-\hat{u})dx$$

$$+\int_{\Gamma}(-Cu+C\hat{u})\phi(u-\hat{u})d\Gamma\,.$$

(We can always assume that h_n, $\hat{h}_n$, k_n, $\hat{k}_n$ are bounded by a fixed integrable function, and then apply Lebesgue convergence theorem.) As $\phi\rightarrow$sign, we get $u=\hat{u}$.

Appendix.

The conjugate convex function of $\Phi(s)$ is defined by

$$\Psi(r)=\sup_s\,[rs-\Phi(s)]\,.$$

The supremum is attained if and only if $r=\phi(s)$ where $\phi=\Phi'$. See [11]. Our assumption that $\phi(s)\rightarrow+\infty$ as $s\rightarrow+\infty$ implies that $\theta(s)$ is bounded away from zero for large s.

LEMMA. *Assume that there is a constant c_1 so that $s\phi''(s)\leqq c_1\phi'(s)$ for* *large s. Then*

(i) *$\Phi(ks)/\Phi(s)$ is bounded for large s,*

(ii) *$\Psi(\phi(s))[\theta(s)]^{-2}$ is bounded for large s.*

PROOF. By assumption, $s^{-c_1}\phi'(s)$ is non-increasing for large s. Hence for $k>1$, $\phi'(ks)\leqq k^{c_1}\phi'(s)$. Integrating this inequality twice and using the assumption that $\phi(s)\to+\infty$ as $s\to+\infty$, we get (i).

By definition of θ, we have $(\theta')^2=\phi'$. Hence $2\theta'\theta''=\phi''$. By assumption,

$$\phi'(c_1\theta'-2s\theta'')=s\theta'(c_1\phi'/s-2\theta'\theta'')\geqq s\theta'(\phi''-2\theta'\theta'')=0$$

for large s. Arguing separately over the intervals where ϕ is constant and where ϕ is increasing, we see that $(c_1+2)\theta-2s\theta'$ is non-decreasing for large s. Thus $s\theta'\leqq 2c_3\theta$ for large s. Multiplying by θ', we have $s\phi'\leqq 2c_3\theta\theta'$ for large s. Integrating once more,

$$\Psi(\phi(s))=s\phi(s)-\Phi(s)\leqq c_4\theta^2(s) \qquad \text{for large } s.$$

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