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A supplement to my paper "On zeta-theta functions"

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The purpose of the present paper is to supplement our previous paper [1] in showing that the zeta-theta function introduced there is essentially equal to the non-holomorphic Eisenstein series.

By Theorem 4, in [1], the zeta-theta functions $\zeta_j(\omega, s)$ for $j=0, 1$, have the following form:

$$\zeta_j(\omega, s)=\theta_j(\omega)\mathcal{Z}(\omega, s),$$

$$\mathcal{Z}(\omega, s)=\Gamma((1/2)s)\zeta(s)(\pi\eta)^{-(1/2)s}+\Gamma((1/2)(s+1))\eta^{(1/2)s}\zeta(s+1)\pi^{-(1/2)(s+1)}$$

$$+\sum_{\substack{m\neq 0\\ n\neq 0}} e^{2\pi i\xi mn}\left|\frac{m}{n}\right|^{(1/2)s}K_{(1/2)s}(2\pi\eta|mn|),$$

where

$$\theta_0(\omega)=\sum_{m\in \boldsymbol{Z}} e^{-2\pi i m^2\bar{\omega}}$$

$$\theta_1(\omega)=\sum_{m\in \boldsymbol{Z}} e^{-2\pi i(m+1/2)^2\bar{\omega}}$$

$$\omega=\xi+i\eta,\ \eta>0$$

and

$K_u(z)$ is the modified Bessel function.

Then we can write $\mathcal{Z}(\omega, s)$ in the following form:

$$\mathcal{Z}(\omega, s)=\Gamma((1/2)s)\zeta(s)(\pi\eta)^{-(1/2)s}+\Gamma((1/2)(s+1))\eta^{(1/2)s}\zeta(s+1)\pi^{-(1/2)(s+1)}$$

$$+2\sum_{\substack{m=1\\ n=1}} (e^{2\pi i\xi mn}+e^{-2\pi i\xi mn})\left(\frac{m}{n}\right)^{(1/2)s}K_{(1/2)s}(2\pi\eta mn).$$

On the other hand, it is known that the non-holomorphic Eisenstein series

$$Q(\omega, s)=\sum{}'\frac{\eta^s}{|m+n\omega|^{2s}},$$

where the sum is over all $(m, n)\in \boldsymbol{Z}^2$ except for $(0, 0)$, has the following expansion (see, for example, C. L. Siegel [2], p. 290):

$$Q(\omega, s)=2\eta^s\zeta(2s)+2\pi^{1/2}\eta^{1-s}\Gamma(s-1/2)\zeta(2s-1)\Gamma(s)^{-1}$$

$$+2\pi^s\eta^{1/2}\Gamma(s)^{-1}\sum_{\substack{m=1\\n=1}}^{\infty}(e^{2\pi i\xi mn}+e^{-2\pi i\xi mn})\int_0^{\infty}u^{s-1/2}e^{-\pi\eta(m^2u^{-1}+n^2u)}\frac{du}{u},$$

where the last integral equals

$$2\left(\frac{m}{n}\right)^{s-1/2}K_{s-1/2}(2\pi\eta mn).$$

Comparing the infinite series expressions of $\mathcal{Z}(\omega, s)$ and $Q(\omega, s)$, we get the following

THEOREM.

$$\zeta_j(\omega, s)=(1/2)\theta_j(\omega)\Gamma((1/2)(s+1))\pi^{-(1/2)(s+1)}\eta^{-1/2}Q(\omega, (1/2)(s+1))$$

for $j=0, 1$.

REMARK.

1) From this theorem, we see that the series expression of $\mathcal{Z}(\omega, s)$ converges for $\mathrm{Re}\, s>1$.

2) Since the Eisenstein series $Q(\omega, s)$ is invariant by the modular substitutions, we can derive, from this theorem, the transformation formula of $\zeta_j(\omega, s)$, independent of [**1**].

3) We know that $\zeta_j(\omega, s)$ satisfies the functional equation

$$\zeta_j(\omega, s)=\zeta_j(\omega, -s)$$

(see [**1**], 4.1). Therefore we can derive, from this theorem, the functional equation of the Eisenstein series $Q(\omega, s)$.

References

[1] K. Katayama, On zeta-theta functions, J. Math. Soc. Japan, **24** (1972), 307-332.
[2] C. L. Siegel, Analytische Zahlentheorie II, Göttingen, 1963.

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