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Quasi-permutation modules over finite groups

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Let Π be a finite group. A finitely generated Z-free Π-module is briefly called a Π-module. A Π-module is called a permutation Π-module if it is expressible as a direct sum of some $\{Z\Pi/\Pi_i\}$ where each Π_i is a subgroup of Π. Further a Π-module M is called a quasi-permutation Π-module if there exists an exact sequence: $0\to M\to S\to S'\to 0$ where S and S' are permutation Π-modules.

In [2] we have studied the properties of quasi-permutation modules in relation with a problem in invariant theory. In this paper we will give some basic results on quasi-permutation modules as a continuation to [2].

First we will consider projective quasi-permutation Π-modules.

Let R be a Dedekind domain and K be the quotient field of R. Let Σ be a separable K-algebra and Λ be an R-order in Σ. Denote by $\boldsymbol{P}(\Lambda)$ the set of all isomorphism types of finitely generated projective (left) Λ-modules and put $\boldsymbol{P}_0(\Lambda)=\{[P]\in \boldsymbol{P}(\Lambda)\,|\,P$ is locally free$\}$. Let $P_0(\Lambda)$ be the Grothendieck group of $\boldsymbol{P}_0(\Lambda)$. We define an epimorphism $\mu_\Lambda: P_0(\Lambda)\to Z$ by $\mu_\Lambda([P_1]-[P_2])=\mathrm{rank}_\Sigma KP_1-\mathrm{rank}_\Sigma KP_2$. Now we put $C(\Lambda)=\mathrm{Ker}\,\mu_\Lambda$ and call this the (reduced) projective class group of Λ (cf. [5], [11]). Especially, if Λ is commutative, then $C(\Lambda)$ is isomorphic to the Picard group of Λ. Further let Ω be a maximal R-order in Σ which contains Λ. We define a homomorphism: $\nu_{\Omega/\Lambda}: C(\Lambda)\to C(\Omega)$ by $\nu_{\Omega/\Lambda}([P_1]-[P_2])=[\Omega\otimes_\Lambda P_1]-[\Omega\otimes_\Lambda P_2]$. Then it is known that $\nu_{\Omega/\Lambda}$ is an epimorphism but not always a monomorphism. Hence putting $\tilde{C}(\Lambda)=\mathrm{Ker}\,\nu_{\Omega/\Lambda}$, we have an exact sequence:

$$0\longrightarrow \tilde{C}(\Lambda)\longrightarrow C(\Lambda)\longrightarrow C(\Omega)\longrightarrow 0\,.$$

Especially let $\Lambda=Z\Pi$ and let Ω_Π be a maximal order in $Q\Pi$ which contains $Z\Pi$. Then, by the Swan's theorem ([11]), we have $\boldsymbol{P}_0(Z\Pi)=\boldsymbol{P}(Z\Pi)$ and $\tilde{C}(Z\Pi)=\{[\mathfrak{a}]-[Z\Pi]\in C(Z\Pi)\,|\,\mathfrak{a}$ is a projective (left) ideal of $Z\Pi$ such that $\Omega_\Pi\mathfrak{a}\oplus\Omega_\Pi\cong\Omega_\Pi\oplus\Omega_\Pi$ as Ω_Π-modules$\}$. It is noted that $\tilde{C}(Z\Pi)$ does not depend on the choice of Ω_Π (cf. [3]). On the other hand, we put $C^q(Z\Pi)=\{[\mathfrak{a}]-[Z\Pi]\in C(Z\Pi)\,|\,\mathfrak{a}$ is a quasi-permutation projective (left) ideal of $Z\Pi\}$. Then it is easily seen that $C^q(Z\Pi)$ is also a subgroup of $C(Z\Pi)$.

Let Π be a cyclic group of order n and σ be a generator of Π. We denote by $\Phi_m(T)$ the m-th cyclotomic polynomial and by ζ_m a primitive m-th root of unity. Let M be a Π-module and put $M^{\Phi_m}=\{u\in M \mid \Phi_m(\sigma)u=0\}$ for any $m|n$. Then $Z\Pi/(\Phi_m(\sigma))\cong Z[\zeta_m]$ and M^{Φ_m} can be regarded as a $Z[\zeta_m]$-module for any $m|n$. In [**2**], (1.11) we have proved that a projective Π-module P is a quasi-permutation Π-module if and only if, for any $m|n$, P^{Φ_m} is $Z[\zeta_m]$-free. This is clearly equivalent to the assertion that $\tilde{C}(Z\Pi)=C^q(Z\Pi)$.

In this paper we will first give, as a generalization of this result,

[I] *For any finite abelian group* Π, $\tilde{C}(Z\Pi)=C^q(Z\Pi)$.

It seems natural to ask whether $C^q(Z\Pi)$ coincides with $\tilde{C}(Z\Pi)$ for any finite group Π or not. In fact, we will prove

[II] *Let* Π *be one of the following groups:*

(i) *p-groups where p is an odd prime;*

(ii) *dihedral groups,* D_{p^l}, *where p is a prime and l is a positive integer;*

(iii) *the quaternion group* H_2, *the alternating group* A_4 *and the symmetric group* S_4.

Then $\tilde{C}(Z\Pi)=C^q(Z\Pi)$.

To prove [II] we use the Jacobinski-Roiter's results in [**3**] and [**6**]. Furthermore using them, we can show the following refinement of [I].

[III] *Let* Π *be a finite abelian group and let* $\mathfrak{a}$ *be a projective (left) ideal of* $Z\Pi$. *Then* $\mathfrak{a}$ *is a quasi-permutation* Π*-module if and only if* $\mathfrak{a}\oplus\sum_{\Pi'\in\mathfrak{S}}\oplus Z\Pi/\Pi'\cong Z\Pi\oplus\sum_{\Pi'\in\mathfrak{S}}\oplus Z\Pi/\Pi'$, *where* $\mathfrak{S}$ *denotes the set of all subgroups,* Π' $(\neq 1)$, *of* Π *such that* Π/Π' *is cyclic.*

It is much more difficult to examine the properties of non projective quasi-permutation modules. Here we will consider only the case where Π is a cyclic group.

[IV] *Let* Π *be a cyclic p-group of order* p^l.

(i) *A* Π*-module M is a quasi-permutation module if and only if, for any* $0\leqq m\leqq l$, $M^{\Phi_{p^m}}$ *is a free* $Z[\zeta_{p^m}]$*-module.*

(ii) *If M is a quasi-permutation* Π*-module, then the dual module* $M^*=\mathrm{Hom}_Z(M, Z)$ *of M is also a quasi-permutation* Π*-module.*

(iii) *Let* $0\to M'\to M\to M''\to 0$ *be an exact sequence of* Π*-modules. If two of* M', M *and* M'' *are quasi-permutation* Π*-modules, then the rest of them is a quasi-permutation* Π*-module.*

Let Π be a finite group and K/k be a Galois extension with group $\cong\Pi$. Let M be a Π-module with a Z-free basis $\{u_1, u_2, \cdots, u_n\}$. Denote by $K(X_1, X_2, \cdots, X_n)$ the rational function field with n-variables $X_1, X_2, \cdots, X_n$ over K and define the action of Π on $K(X_1, X_2, \cdots, X_n)$, as an extension of the action of Π on K, by putting

$$\sigma(X_i)=\prod_{j=1}^{n}X_j^{m_{ij}} \qquad \text{when} \qquad \sigma\cdot u_i=\sum_{j=1}^{n}m_{ij}u_j\,,\quad m_{ij}\in Z$$

for any $\sigma\in\Pi$ and $1\leqq i\leqq n$. We denote $K(X_1, X_2, \cdots, X_n)$ with this action of Π by $K(M)$.

[V] *Let Π be a cyclic p-group and K/k be a Galois extension with group $\cong\Pi$. In case of $p\neq 2$, assume that k is infinite. If M is a quasi-permutation Π-module, then $K(M)^{\Pi}/k$ is rational.*

§ **1.** We give, as a slight generalization of [**2**], (1.1),

PROPOSITION 1.1. *Let K/k be a Galois extension with group Π and $K(X_1, X_2, \cdots, X_n)$ be the rational function field with n variables $X_1, X_2, \cdots, X_n$ over K. Further suppose that Π acts on $K(X_1, X_2, \cdots, X_n)$ as follows:*

$$\sigma(X_i)=\sum_{i=1}^{n}\alpha_{ij}(\sigma)X_j+\beta_i(\sigma)\,, \qquad \alpha_{ij}(\sigma)\,,\ \ \beta_i(\sigma)\in K\,.$$

Then $K(X_1, X_2, \cdots, X_n)^{\Pi}$ is rational over k.

PROOF. We denote by $Aff(n, K)$ the affine transformation group of the n-dimensional affine space over K. Then we have an exact sequence of Π-groups:

$$1\longrightarrow K^{(n)}\longrightarrow Aff(n, K)\longrightarrow GL(n, K)\longrightarrow 1\,.$$

From this we get an exact sequence:

$$H^1(\Pi, K)^{(n)}\longrightarrow H^1(\Pi, Aff(n, K))\longrightarrow H^1(\Pi, GL(n, K))\,.$$

By the Hilbert's theorem 90 $H^1(\Pi, K)^{(n)}=H^1(\Pi, GL(n, K))=1$, and so $H^1(\Pi, Aff(n, K))=1$. The proposition is clearly a restatement of the fact that $H^1(\Pi, Aff(n, K))=1$ (cf. [**2**]).

Let E, F be extensions of a field k. We define a relation $E\underset{(r)_k}{-}F$ if there exist variables $X_1, X_2, \cdots, X_s$ and $Y_1, Y_2, \cdots, Y_t$ such that $E(X_1, X_2, \cdots, X_s)$ is k-isomorphic to $F(Y_1, Y_2, \cdots, Y_t)$. An extension E/k is said to be quasi-rational if $E\underset{(r)_k}{-}k$.

Let Π be a finite group and denote by $\boldsymbol{C}_{Z\Pi}$ the class of all Π-modules. Let $M, N\in\boldsymbol{C}_{Z\Pi}$. We define an equivalence relation $M\underset{(r)}{-}N$ if, for any Galois extension K/k with group $\cong\Pi$, $K(M)^{\Pi}\underset{(r)_k}{-}K(N)^{\Pi}$. If $M_1\underset{(r)}{-}N_1$ and $M_2\underset{(r)}{-}N_2$, then $M_1\oplus M_2\underset{(r)}{-}N_1\oplus N_2$. Let $T(\Pi)$ be the set of all equivalence classes in $\boldsymbol{C}_{Z\Pi}$. We define an addition in $T(\Pi)$ by $[M]+[N]=[M\oplus N]$. Then this makes $T(\Pi)$ an abelian semigroup.

Further let $M, N\in\boldsymbol{C}_{Z\Pi}$. We write $M\underset{(r)}{=}N$ if, for any Galois extension K/k with group $\cong\Pi$, $K(M)^{\Pi}$ is k-isomorphic to $K(N)^{\Pi}$.

The following fundamental theorem is essentially due to R. G. Swan (cf. [**13**], [**14**]. Also see [**2**], (1.6).)

THEOREM 1.2. *Let Π be a finite group and let M be a Π-module. Then the following conditions are equivalent:*

(1) *M is a quasi-permutation Π-module.*

(2) *For a fixed Galois extension K/k with group $\cong \Pi$, $K(M)^{\Pi}/k$ is quasi-rational.*

(3) *For any Galois extension K/k with group $\cong \Pi$, $K(M)^{\Pi}/k$ is quasi-rational, i.e.,* $M \underset{(r)}{\sim} 0$.

COROLLARY 1.3. *Let Π be a finite group. Let*

$$0 \longrightarrow M \longrightarrow N \longrightarrow S \longrightarrow 0$$

be an exact sequence where M, N are Π-modules and S is a permutation Π-module. Then $N \underset{(r)}{\approx} M \oplus S$ and $N \underset{(r)}{\sim} M$. Especially N is a quasi-permutation Π-module if and only if M is a quasi-permutation Π-module.

PROPOSITION 1.4. *Let Π be a finite group. Let M be a Π-module and L be a quasi-permutation Π-module. If $M \oplus L$ is a quasi-permutation Π-module, then M is also a quasi-permutation Π-module.*

PROOF. By (1.2) we have $L \underset{(r)}{\sim} 0$ and $M \oplus L \underset{(r)}{\sim} 0$. Therefore $M \underset{(r)}{\sim} M \oplus L \underset{(r)}{\sim} 0$. Again by (1.2) we can conclude that M is a quasi-permutation Π-module.

LEMMA 1.5. *Let Π be a finite group. Let M be a Π-module and let $\Pi'_1, \cdots, \Pi'_s$ be subgroups of Π. Then the following conditions are equivalent:*

(1) *$H^1(\Pi'_i, M) = 0$ for any $1 \leqq i \leqq s$.*

(2) *Every exact sequence $0 \rightarrow M \rightarrow N \rightarrow \sum_{i=1}^{s} \oplus (Z\Pi/\Pi'_i)^{(t_i)} \rightarrow 0$, where $t_1, \cdots, t_s$ are non-negative integers, splits.*

PROOF. For any Π-module L and any subgroup Π' of Π there exists a natural isomorphism $H^1(\Pi', L) \cong \mathrm{Ext}^1_{Z\Pi}(Z\Pi/\Pi', L)$. From this the lemma follows immediately.

PROPOSITION 1.6. *Let Π be a finite group. For any Π-module M the following conditions are equivalent:*

(1) *M is a quasi-permutation Π-module and $H^1(\Pi', M) = 0$ for any subgroup Π' of Π.*

(2) *There exist permutation Π-modules S, S' such that $M \oplus S' \cong S$. Especially a projective Π-module P is a quasi-permutation Π-module if and only if there exist permutation Π-modules S, S' such that $P \oplus S' \cong S$.*

PROOF. Assume (1). Then there is an exact sequence $0 \rightarrow M \rightarrow S \rightarrow S' \rightarrow 0$ where S and S' are permutation Π-modules. By (1.5) this sequence splits, and hence $M \oplus S' \cong S$. This proves (1) $\Rightarrow$ (2). Conversely assume (2). Then it is clear by the definition that M is a quasi-permutation Π-module. Further we see that $H^1(\Pi', S) = H^1(\Pi', S') = 0$ for any subgroup Π' of Π. Therefore $H^1(\Pi', M) = 0$ for any subgroup Π' of Π. Thus (2) $\Rightarrow$ (1). The second part of the proposition follows from the first part.

It should be noted that the second part of (1.6) can be directly proved by dualizing the exact sequence $0 \rightarrow P \rightarrow S \rightarrow S' \rightarrow 0$ where S, S' are permutation Π-modules.

PROPOSITION 1.7. *For any finite group Π, $C^q(Z\Pi)$ is a subgroup of $C(Z\Pi)$.*

PROOF. Let $\mathfrak{a}$ be a quasi-permutation, projective ideal of $Z\Pi$. Then there exists a projective ideal $\mathfrak{b}$ of $Z\Pi$ such that $\mathfrak{a}\oplus\mathfrak{b}\cong Z\Pi\oplus Z\Pi$. By (1.6) (or by (1.4)) $\mathfrak{b}$ is a quasi-permutation Π-module and we have $-([\mathfrak{a}]-[Z\Pi])=[\mathfrak{b}]-[Z\Pi]$.

§ **2.** Let Π be a cyclic group of order n and σ be a generator of Π. Let $Z[T]$ be the polynomial ring with a variable T over Z and Φ, Ψ be monic polynomials of $Z[T]$ such that $\Phi\cdot\Psi=T^n-1$. If M is a Π-module, there are three ways to construct a $Z\Pi/(\Phi(\sigma))$-module from M, i.e., putting $\Psi M=\Psi(\sigma)M$, $M_\Phi=M/\Phi(\sigma)M$ and $M^\Phi=\{u\in M \mid \Phi(\sigma)u=0\}\cong \mathrm{Hom}_{Z\Pi}(Z\Pi/(\Phi(\sigma)), M)$, ΨM, M_Φ and M^Φ can be regarded as $Z\Pi/(\Phi(\sigma))$-modules. Then $\Psi M\subseteq M^\Phi$. We define an epimorphism $\theta_M: M_\Phi\to\Psi M$ by $\theta_M(\bar{u})=\Psi(\sigma)u$. Especially, if we take the m-th cyclotomic polynomial $\Phi_m(T)$ $(m|n)$ as $\Phi(T)$, we have $Z\Pi/(\Phi_m(\sigma))\cong Z[\zeta_m]$ where ζ_m denotes the primitive m-th root of unity. We can easily prove the following two lemmas (cf. [**13**]).

LEMMA 2.1. *Let Π be a cyclic group of order n and σ be a generator of Π. For any positive integers m, l dividing n, we have*

$$(Z\Pi/[\sigma^l])^{\Phi_m}\cong\begin{cases} Z[\zeta_m] & \text{when } m\mid l \\ 0 & \text{when } m\nmid l. \end{cases}$$

If S is a permutation Π-module, then S^{Φ_m} is a free $Z[\zeta_m]$-module.

LEMMA 2.2. *Let Π be a cyclic group of order n and $\Phi(T)$, $\Psi(T)$ be monic polynomials such that $\Phi(T)\Psi(T)=T^n-1$. Let P be a projective Π-module. Then $\Psi P=P^\Phi$ and $\theta_P: P_\Phi\to\Psi P$ is an isomorphism.*

The following proposition has been proved essentially in [**2**], (1.11).

PROPOSITION 2.3. *Let Π be a finite abelian group. Then the maximal order Ω_Π of $Q\Pi$ which contains $Z\Pi$ is a quasi-permutation Π-module.*

PROOF. We can express Ω_Π as a direct sum of $Z[\zeta]$'s where each ζ is the root of unity. Therefore it suffices to prove that each $Z[\zeta]$ is a quasi-permutation Π-module. Let us denote by Π' the kernel of the natural projection of Π on $[\zeta]$. Then Π/Π' is cyclic and $Z[\zeta]$ can be regarded as a Π/Π'-module. Hence we may suppose that Π is a cyclic group of order n with a generator σ and $Z[\zeta]\cong Z\Pi/[\Phi_n(\sigma)]$. Then, using the Möbius' inversion formula, we obtain the following exact sequences of Π-modules:

$$0\longrightarrow M_1\longrightarrow Z\Pi\longrightarrow Z\Pi/\Pi_0\longrightarrow 0$$

$$0\longrightarrow M_1\longrightarrow M_2\longrightarrow Z\Pi/\Pi_1\longrightarrow 0$$

$$0\longrightarrow M_3\longrightarrow M_2\longrightarrow Z\Pi/\Pi_2\longrightarrow 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$0 \longrightarrow M_{t-1} \longrightarrow M_t \longrightarrow Z\Pi/\Pi_{t-1} \longrightarrow 0$$

$$0 \longrightarrow Z\Pi/(\Phi_n(\sigma)) \longrightarrow M_t \longrightarrow Z\Pi/\Pi_t \longrightarrow 0,$$

where $\Pi_0, \cdots, \Pi_t$ are subgroups of Π. Applying (1.3) to these exact sequences repeatedly, we see that $Z\Pi/(\Phi_n(\sigma))$ is a quasi-permutation Π-module.

LEMMA 2.4. *Let Π be a finite group and $\mathfrak{a}$ be a projective (left) ideal of $Z\Pi$. Let Λ be an order in $Q\Pi$ which contains $Z\Pi$. Then $\mathfrak{a}\oplus\Lambda \cong Z\Pi\oplus\Lambda\mathfrak{a}$ as Π-modules.*

PROOF. Let n be the order of Π. Then we know $n\Lambda \subseteq Z\Pi$ and there exists an ideal $\mathfrak{a}_0$ of $Z\Pi$ such that $\mathfrak{a}_0 \cong \mathfrak{a}$ and $(\mathrm{Ann}_Z \Lambda/\mathfrak{a}_0, n)=1$ ([11]). Hence we may assume that $(\mathrm{Ann}_Z \Lambda/\mathfrak{a}, n)=1$. Now we can make the commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{a} & \xrightarrow{\beta_1} & \Lambda\mathfrak{a} & \xrightarrow{\gamma_1} & \Lambda\mathfrak{a}/\mathfrak{a} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \delta} & & \\ 0 & \longrightarrow & Z\Pi & \xrightarrow{\beta_2} & \Lambda & \xrightarrow{\gamma_2} & \Lambda/Z\Pi & \longrightarrow & 0 \end{array}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are natural injections. Let p be a rational prime. If $p\nmid n$, then $Z_p\Pi$ is a maximal Z_p-order in $Q\Pi$, therefore both $(\beta_1)_p$ and $(\beta_2)_p$ are isomorphisms. Hence δ_p $(=0)$ is an isomorphism. On the other hand, if $p|n$, then $p\nmid \mathrm{Ann}_Z Z\Pi/\mathfrak{a}$ so that both $(\alpha_1)_p$ and $(\alpha_2)_p$ are isomorphisms, hence δ_p is also an isomorphism. Accordingly, for any p, δ_p is an isomorphism. Thus δ must be an isomorphism. We identify $\Lambda\mathfrak{a}/\mathfrak{a}$ with $\Lambda/Z\Pi$ through δ and denote it by A. Forming the pullback of γ_1 and γ_2, we get the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & Z\Pi & = & Z\Pi & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{a} & \longrightarrow & L & \longrightarrow & \Lambda & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \gamma_2} & & \\ 0 & \longrightarrow & \mathfrak{a} & \longrightarrow & \Lambda\mathfrak{a} & \xrightarrow{\gamma_1} & A & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Since $\mathfrak{a}$, $Z\Pi$ are Π-projective and L, $\Lambda\mathfrak{a}$, Λ are torsion-free, the second row and the second column are Π-split. Therefore $Z\Pi\oplus\Lambda\mathfrak{a} \cong L \cong \mathfrak{a}\oplus\Lambda$, which completes the proof.

Now we prove

THEOREM 2.5. *Let Π be a finite abelian group. Then*

$$\tilde{C}(Z\Pi)=C^{q}(Z\Pi).$$

PROOF. Let $\mathfrak{a}$ be a projective ideal of $Z\Pi$ and let Ω_{Π} be the maximal order in $Q\Pi$ which contains $Z\Pi$. By the definitions of $\tilde{C}(Z\Pi)$ and $C^{q}(Z\Pi)$, it suffices to prove that $\mathfrak{a}$ is a quasi-permutation Π-module if and only if $\Omega_{\Pi}\mathfrak{a}\cong\Omega_{\Pi}$ as Ω_{Π}-modules.

Suppose that $\mathfrak{a}$ is a quasi-permutation Π-module. Then there exist permutation Π-modules S, S' such that $\mathfrak{a}\oplus S'\cong S$ by (1.6). Since Π is abelian, Ω_{Π} is expressible as the direct sum of $Z[\zeta_m]$'s where each ζ_m is a primitive m-th root of unity. To show $\Omega_{\Pi}\mathfrak{a}\cong\Omega_{\Pi}$, it suffices to show $Z[\zeta_m]\mathfrak{a}\cong Z[\zeta_m]$ for every component $Z[\zeta_m]$ of Ω_{Π}. We denote by Π' the kernel of the natural projection of Π on $[\zeta_m]$. Then Π/Π' is cyclic and $Z[\zeta_m]$ can be regarded as a Π/Π'-module. $Z\Pi/\Pi'\underset{Z\Pi}{\otimes}\mathfrak{a}$ is a quasi-permutation Π/Π'-module and $Z[\zeta_m]\mathfrak{a}\cong Z[\zeta_m]\cdot(Z\Pi/\Pi'\underset{Z\Pi}{\otimes}\mathfrak{a})$. Hence we may assume that Π is cyclic. In this case we have $\mathfrak{a}^{\Phi_m}\oplus S'^{\Phi_m}\cong S^{\Phi_m}$, and therefore $Z[\zeta_m]\cong\mathfrak{a}^{\Phi_m}\cong\mathfrak{a}_{\Phi_m}\cong Z[\zeta_m]\mathfrak{a}$ by (2.1) and (2.2). This proves the only if part.

Conversely suppose that $\Omega_{\Pi}\mathfrak{a}\cong\Omega_{\Pi}$ as Ω_{Π}-modules. Then, by virtue of (2.4), $\mathfrak{a}\oplus\Omega_{\Pi}\cong Z\Pi\oplus\Omega_{\Pi}$. Further, by (2.3), Ω_{Π} is a quasi-permutation Π-module. Therefore applying (1.4), we can conclude that $\mathfrak{a}$ is a quasi-permutation Π-module. Thus the proof of the theorem is completed.

COROLLARY 2.6. *Let Π be a finite abelian group and let $\Omega_{\Pi}=\sum_{i=1}^{s}\oplus\Omega_i$ be the decomposition of Ω_{Π} into Dedekind domains. Then a Z-free Ω_{Π}-module M is a quasi-permutation Π-module if and only if $M\cong\sum_{i=1}^{s}\oplus\Omega_i^{(t_i)}$ for $t_i\geqq 0$.*

PROOF. The if part of the corollary follows immediately from (2.3). Suppose that M is a quasi-permutation Π-module. Since each Ω_i is a Dedekind domain and a quasi-permutation Π-module ((2.3)), we may suppose that $M\cong\sum_{i=1}^{s}\oplus\mathfrak{a}_i$ where each $\mathfrak{a}_i$ is a non-zero ideal of Ω_i. The natural homomorphism $C(Z\Pi)\rightarrow C(\Omega_{\Pi})$ is an epimorphism. Therefore there exists a projective ideal $\mathfrak{a}$ of $Z\Pi$ such that $\Omega_{\Pi}\mathfrak{a}\cong M$. According to (2.4), we have $\mathfrak{a}\oplus\Omega_{\Pi}\cong Z\Pi\oplus M$, and it follows from (2.3) that $\mathfrak{a}\underset{(r)}{\text{—}}M\underset{(r)}{\text{—}}0$. Hence $\mathfrak{a}$ is a quasi-permutation Π-module. By virtue of (2.5) we can conclude that $M\cong\Omega_{\Pi}\mathfrak{a}\cong\Omega_{\Pi}$, which completes the proof of the only if part.

REMARK 2.7. Let Π be a finite abelian group and let $\Omega_{\Pi}=\sum_{i=1}^{s}\oplus\Omega_i$ be the decomposition of Ω_{Π} into Dedekind domains. Let P be a projective Π-module. Then we have $P\underset{(r)}{\text{—}}\sum_{i=1}^{s}\oplus\Omega_iP$. Especially, if Π is cyclic, we have

$P \underset{(r)}{=\!=} \sum_{i=1}^{s} \oplus \Omega_i P.$

PROOF. The first part follows directly from (2.3) and (2.4), and the second part can be proved along the same line as in the proof of [2], (1.11).

§3. We sketch the Jacobinski-Roiter's results on orders which will be used in §§3 and 4.

Let K be an algebraic number field and R be the ring of all algebraic integers in K. Let Σ be a semi-simple K-algebra and Λ be an R-order in Σ. A Λ-module is called a Λ-lattice if it is a finitely generated projective R-module and we denote by $\boldsymbol{C}_\Lambda$ the class of all Λ-lattices. Given $M, N \in \boldsymbol{C}_\Lambda$, we write $M \sim N$ if, for any prime ideal $\mathfrak{p}$ of R, $M_\mathfrak{p} \cong N_\mathfrak{p}$ as $\Lambda_\mathfrak{p}$-modules.

We say that a Λ-lattice M satisfies (ε) if $\mathrm{End}_\Sigma(KM)$ does not have any totally definite quaternion algebra as its simple component.

Let Ω be a maximal R-order in Σ which contains Λ. Given $M, N \in \boldsymbol{C}_\Lambda$ we write $M \approx N$ if $M \sim N$ and $\Omega M \cong \Omega N$ as Ω-modules. We put $\gamma_M = \{N \in \boldsymbol{C}_\Lambda | N \approx M\}$ and denote by $|\gamma_M|$ the number of all isomorphism types in γ_M. If M satisfies (ε), then γ_M does not depend on the choice of Ω.

(A) ([6]). *If N is a local direct summand of M, then there is a decomposition $M \cong N' \oplus L$ with $N' \sim N$.*

(B) ([6]). *If M is Λ-faithful and $N \sim N'$, then there is M' such that $M \oplus N \cong M' \oplus N'$.*

(C) ([6]). *Suppose that N is a local direct summand of M and that every simple Σ-module S which occurs in KM occurs strictly more times in KM than in KN. Then N is a direct summand of M.*

(D) ([3]). *Let M satisfy* (ε) *and X be a local direct summand of $M^{(l)}$ for some l. Then $X \oplus M \cong X \oplus N$ implies $M \cong N$.*

(E) ([3]). (i) *If M is an Ω-lattice which satisfies* (ε), *we have $|\gamma_M| = 1$ as a Λ-lattice.* (ii) *Let T be a Λ-faithful, Λ-lattice satisfying* (ε) *such that $|\gamma_T| = 1$. Then $M \approx N$ if and only if $M \oplus T \cong N \oplus T$.*

(E′) ([3]). *Let M be a Λ-lattice. Suppose that* $\mathrm{End}_\Sigma(KM)$ *is a commutative field and that* $\mathrm{End}_\Lambda(M)$ *is the integral closure of R in* $\mathrm{End}_\Sigma(KM)$. *Then $|\gamma_M| = 1$ as a Λ-lattice.*

It is noted that (E′) is a special case of (E), (i).

In (2.5) we have shown that $\tilde{C}(Z\Pi) = C^q(Z\Pi)$ for any finite abelian group Π. Here it is natural to ask whether $C^q(Z\Pi)$ coincides with $\tilde{C}(Z\Pi)$ for any finite group Π or not. In this section we will prove that $\tilde{C}(Z\Pi) = C^q(Z\Pi)$ for some types of finite groups.

Let Π be a finite group. Let $Q\Pi = \Sigma_1 \oplus \Sigma_2 \oplus \cdots \oplus \Sigma_t$ be the decomposition of $Q\Pi$ into simple algebras. Denote by K_i the center of Σ_i and let R_i be the ring of all algebraic integers in K_i.

A finite group Π is said to be of split type (over Q) if each Σ_i is a full matrix algebra over K_i.

PROPOSITION 3.1. *Let Π be a finite group of split type. Suppose that, for each i, there is a quasi-permutation Π-module T_i such that* $\mathrm{End}_{\Sigma_i}(QT_i)\cong K_i$ *and* $\mathrm{End}_{Z\Pi}(T_i)\cong R_i$. *Then* $\tilde{C}(Z\Pi)\subseteq C^q(Z\Pi)$.

PROOF. Put $T=\sum_{i=1}^{t}\oplus T_i$. Then T is a faithful quasi-permutation Π-module. By (E′) we have $|\gamma_{T_i}|=1$ for each i. Then, using (A), we easily see that $|\gamma_T|=1$. Let $\mathfrak{a}$ be a projective ideal of $Z\Pi$ such that $\mathfrak{a}\approx Z\Pi$. By virtue of (E), (ii) we have $\mathfrak{a}\oplus T\cong Z\Pi\oplus T$. According to (1.4) this implies that $\mathfrak{a}$ is a quasi-permutation Π-module, which completes the proof.

Let S be a Dedekind domain with quotient field L. Let Π be a finite group of automorphisms of L and put $K=L^\Pi$ and $R=S^\Pi$. We denote by $\Delta(\Pi, S)$ ($\Delta(\Pi, L)$) the twisted group algebra of Π over S (L). Then $\Delta(\Pi, L)$ is isomorphic to a full matrix algebra over K and $\Delta(\Pi, S)$ can be regarded as an R-order in $\Delta(\Pi, L)$.

Especially, if S/R is tamely ramified, then $\Delta(\Pi, S)$ is hereditary, as is well known, and any finitely generated projective $\Delta(\Pi, S)$-module is expressible as a direct sum of ambiguous ideals of S (cf. [8]).

LEMMA 3.2. *Suppose that S/R is tamely ramified. Then* $|\gamma_{\Delta(\Pi,S)}|=1$.

PROOF. Let $\mathfrak{a}$ be an ambiguous ideal of S. Then we have $\mathrm{End}_{\Delta(\Pi,S)}\mathfrak{a}\cong S^\Pi=R$, hence, by (E′), $|\gamma_{\mathfrak{a}}|=1$. Now we can write $\Delta(\Pi, S)\cong\sum_{i=1}^{l}\oplus\mathfrak{a}_i$ as $\Delta(\Pi, S)$-modules where each $\mathfrak{a}_i$ is an ambiguous ideal of S. Hence, using (C), we easily see that $|\gamma_{\Delta(\Pi,S)}|=1$.

LEMMA 3.3. *Let Π be a finite group. Suppose that there is an order Λ in $Q\Pi$ containing $Z\Pi$ which is a quasi-permutation Π-module with $|\gamma_\Lambda|=1$. Then* $\tilde{C}(Z\Pi)\subseteq C^q(Z\Pi)$.

PROOF. This follows directly from (2.4) (or (E)) and (1.4).

We denote by $\Pi_{n,m,r}$ the metacyclic group with generators σ and τ satisfying the relations:

$$\tau^{-1}\sigma\tau=\sigma^r\,,\qquad \sigma^n=\tau^m=I$$

where $(r, n)=1$ and $r^m\equiv 1 \bmod n$. It is remarked that the group $\Pi_{n,2,-1}$ means the dihedral group D_n of order $2n$.

PROPOSITION 3.4. *Let Π be one of the following groups:*

(1) *nilpotent groups of odd order;*

(2) *metacyclic groups $\{\Pi_{n,q,r}\}$ where q is a prime such that $q\nmid n$;*

(3) *dihedral groups $\{D_n\}$;*

(4) *the alternating group A_4 and the symmetric group S_4.*

Then $\tilde{C}(Z\Pi)\subseteq C^q(Z\Pi)$.

PROOF. (1) Let Π be a finite nilpotent group of odd order n. Let

$Q\Pi=\Sigma_1\oplus\Sigma_2\oplus\cdots\oplus\Sigma_t$ be the decomposition of $Q\Pi$ into simple algebras and denote by K_i the center of Σ_i. By the well-known Witt-Roquette's theorem ([7]), for each i, the simple algebra Σ_i is a full matrix algebra over K_i and the field K_i is a cyclotomic field $Q(\zeta_{n_i})$ for some $n_i|n$. Further let V_i be a simple Σ_i-module and let χ_i be the character of Π afforded by V_i. Then there exist a subgroup Π_i of Π and a one dimensional $K_i\Pi_i$-module K_i with character ρ_i such that $V_i\cong Q\Pi\underset{Q\Pi_i}{\otimes}K_i$ and $Q(\chi_i)=Q(\rho_i)=K_i$. Now we put $T_i=Z\Pi\underset{Z\Pi_i}{\otimes}Z[\zeta_{n_i}]$. Then we see that $\mathrm{End}_{Z\Pi}(T_i)\cong Z[\zeta_{n_i}]$, and from (2.3) it follows that each T_i is a quasi-permutation Π-module. So we have $\tilde{C}(Z\Pi)\subseteq C^q(Z\Pi)$ from (3.1).

(2) Let $\Pi=\Pi_{n,q,r}$ where q is a prime such that $q\nmid n$. Put $\Pi_0=[\tau]$, $\Pi_1=[\sigma]$ and $m=(r-1, n)$. We can write $Q\Pi=\sum_{l|n}\oplus Q\Pi/(\Phi_l(\sigma))$. If l divides m, then $\Pi/[\sigma^l]$ is a cyclic group and therefore we have $Q\Pi/(\Phi_l(\sigma))\cong Q(\zeta_l)\oplus Q(\zeta_{ql})$. Then the images of $Z\Pi$ by the projections on $Q(\zeta_l)$ and $Q(\zeta_{ql})$ are $Z[\zeta_l]$ and $Z[\zeta_{ql}]$, respectively. Since both $Z[\zeta_l]$ and $Z[\zeta_{ql}]$ are regarded as $\Pi/[\sigma^l]$-modules, according to (2.3) these are quasi-permutation Π-modules and, by (E′), $|\gamma_{Z[\zeta_l]}|=|\gamma_{Z[\zeta_{ql}]}|=1$. On the other hand, if l does not divide m, then $Q\Pi/(\Phi_l(\sigma))$ is isomorphic to the twisted group algebra $\Delta(\Pi_0, Q(\zeta_l))$ because q is a prime and the order $Z\Pi/(\Phi_l(\sigma))$ in $Q\Pi/(\Phi_l(\sigma))$ is also isomorphic to the twisted group algebra $\Delta(\Pi_0, Z[\zeta_l])$. From the assumption that q is a prime such that $q\nmid n$ it is easily seen that $Z[\zeta_l]/Z[\zeta_l]^{\Pi_0}$ is tamely ramified, and hence, by (3.2), $|\gamma_{\Delta(\Pi_0,Z[\zeta_l])}|=1$. It is clear that $\Delta(\Pi_0, Z[\zeta_l])\cong Z\Pi\underset{Z\Pi_1}{\otimes}Z[\zeta_l]$ as Π-modules and so, by (2.3), $\Delta(\Pi_0, Z[\zeta_l])$ is a quasi-permutation Π-module. We put $\Lambda=\sum_{l|m}\oplus(Z[\zeta_l]\oplus Z[\zeta_{ql}])\oplus\sum_{l\nmid m}\oplus\Delta(\Pi_0, Z[\zeta_l])$. Then Λ is a hereditary order in $Q\Pi$ containing $Z\Pi$ and a quasi-permutation Π-module such that $|\gamma_\Lambda|=1$. Hence, from (3.3), we get $\tilde{C}(Z\Pi)\subseteq C^q(Z\Pi)$.

(3) Let $\Pi=D_n=\Pi_{n,2,-1}$. When $2\nmid n$ this is a special case of (2). Hence we have only to prove the assertion when $2|n$. We can write $Q\Pi=\sum_{l|n}\oplus Q\Pi/(\Phi_l(\sigma))$. Here $Q\Pi/(\Phi_1(\sigma))\cong Q\oplus Q\Pi/(\sigma-1, \tau+1)$ and $Q\Pi/(\Phi_2(\sigma))\cong Q\Pi/(\sigma+1, \tau-1)\oplus Q\Pi/(\sigma+1, \tau+1)$. For each $l|n$, $l>2$, $Q\Pi/(\Phi_l(\sigma))$ is isomorphic to the twisted group algebra $\Delta(\Pi_0, Q(\zeta_l))$ and the order $Z\Pi/(\Phi_l(\sigma))$ in $Q\Pi/(\Phi_l(\sigma))$ is also isomorphic to the twisted group algebra $\Delta(\Pi_0, Z[\zeta_l])$. We put $T^{(1)}=Z$, $T^{(2)}=Z\Pi/(\sigma-1, \tau+1)$, $T^{(3)}=Z\Pi/(\sigma+1, \tau-1)$ and $T^{(4)}=Z\Pi/(\sigma+1, \tau+1)$. Then it is clear that $T^{(i)}$ is a quasi-permutation Π-module with $|\gamma_{T^{(i)}}|=1$. Further, putting $T_l=\Delta(\Pi_0, Z[\zeta_l])(\tau-1)=Z\Pi/(\Phi_l(\sigma), \tau+1)$ for any $l|n$, $l>2$, we can show using the same method as in (2.3) that T_l is a quasi-permutation Π-module. We easily see that $\mathrm{End}_{Z\Pi}(T_l)=Z[\zeta_l+\zeta_l^{-1}]=Z[\zeta_l]^{\Pi_0}$, and therefore, by (E′), we have $|\gamma_{T_l}|=1$. Thus we conclude by (3.1) that $\tilde{C}(Z\Pi)\subseteq C^q(Z\Pi)$.

(4) Both A_4 and S_4 are of split type, as is well known. The assertion can be proved using (3.1).

To show the inverse inclusion $C^q(Z\Pi)\subseteq\tilde{C}(Z\Pi)$ we must refer to a Conlon's result.

LEMMA 3.5 ([1]). *Let Π be a finite group of order n. Suppose that $\sum_{i=1}^{s}\oplus Z\Pi/\Pi'_i\oplus L\cong\sum_{j=1}^{t}\oplus Z\Pi/\Pi''_j\oplus L$ where Π'_i and Π''_j are subgroups of Π each of which is a cyclic extension of a p-subgroup of Π for some prime $p|n$ and L is a Π-module. Then $s=t$ and the Π''_j can be reordered so that $Z\Pi/\Pi'_i\cong Z\Pi/\Pi''_j$ for any $1\leqq i\leqq s$.*

PROPOSITION 3.6. *Let Π be a finite group which is a cyclic extension of a p-group. Then $C^q(Z\Pi)\subseteq\tilde{C}(Z\Pi)$. Further let $\mathfrak{S}$ be a complete set of non-conjugate subgroups of Π, and put $T=\sum_{\Pi'\in\mathfrak{S}}\oplus Z\Pi/\Pi'$ in the case where no simple component of $Q\Pi$ is a totally definite quaternion algebra and $T=Z\Pi\oplus\sum_{\Pi'\in\mathfrak{S}}\oplus Z\Pi/\Pi'$ in the other case. Then, for any quasi-permutation projective ideal $\mathfrak{a}$ of $Z\Pi$, $\mathfrak{a}\oplus T\cong Z\Pi\oplus T$, and $|\gamma_T|=[\tilde{C}(Z\Pi):C^q(Z\Pi)]$.*

PROOF. Let $\mathfrak{a}$ be a quasi-permutation projective ideal of $Z\Pi$. Then we have $\mathfrak{a}\oplus\sum_{i=1}^{s}\oplus Z\Pi/\Pi'_i\cong\sum_{j=1}^{t}\oplus Z\Pi/\Pi''_j$ for some subgroups Π'_i, Π''_j of Π. However $\mathfrak{a}^{(k)}\cong Z\Pi^{(k)}$ for some $k>0$ because $C(Z\Pi)$ is a finite group ([11]). Hence $Z\Pi^{(k)}\oplus\sum_{i=1}^{s}\oplus(Z\Pi/\Pi'_i)^{(k)}\cong\sum_{j=1}^{t}\oplus(Z\Pi/\Pi''_j)^{(k)}$. By (3.5) we have $s=t$ and $\Pi''_0=I$, $Z\Pi/\Pi''_i\cong Z\Pi/\Pi'_i$ for any $1\leqq i\leqq s$ by reordering the Π''_j. Therefore $\mathfrak{a}\oplus\sum_{i=1}^{s}\oplus Z\Pi/\Pi'_i\cong Z\Pi\oplus\sum_{i=1}^{s}\oplus Z\Pi/\Pi'_i$. From this and (D) we get $\Omega_\Pi\mathfrak{a}\oplus\Omega_\Pi\cong\Omega_\Pi\oplus\Omega_\Pi$ where Ω_Π denotes a maximal order in $Q\Pi$ containing $Z\Pi$. This shows that $C^q(Z\Pi)\subseteq\tilde{C}(Z\Pi)$. By (D) we have also $\mathfrak{a}\oplus T\cong Z\Pi\oplus T$. Since T is $Z\Pi$-faithful, we easily see using (B) and (D) that $|\gamma_T|=[\tilde{C}(Z\Pi):C^q(Z\Pi)]$.

REMARK 3.7. Let Π be a finite group of split type. Let $F=K_1\oplus K_2\oplus\cdots\oplus K_t$ be the center of $Q\Pi$. Suppose that, for each i, the class number of K_i is 1. Then $C^q(Z\Pi)\subseteq\tilde{C}(Z\Pi)=C(Z\Pi)$.

PROOF. Since Π is of split type, we have $C(\Omega_\Pi)=C(F)$ ([5]). By the assumption, $C(F)=\sum_{i=1}^{t}\oplus C(K_i)=0$, hence $C(\Omega_\Pi)=0$. Thus $\tilde{C}(Z\Pi)=C(Z\Pi)$.

We denote by H_n ($n\geqq 2$) the generalized quaternion group of order $4n$, i.e., the group with generators σ and τ satisfying the relations:

$$\sigma^{2n}=I,\qquad \sigma^n=\tau^2,\qquad \tau^{-1}\sigma\tau=\sigma^{-1}.$$

Let $N_n=[\sigma^n]=[\tau^2]$ and $H'_n=[\sigma]$. Then $H_n/N_n\cong D_n$ and $QH_n\cong QD_n\oplus QH_n/(\sigma^n+I)$. Let $n=2^m n_0$, $2\nmid n_0$. Then $QH_n/(\sigma^n+I)\cong\sum_{2^m|l|n}\oplus QH_n/(\Phi_{2l}(\sigma))$. For any $2^m|l|n$, $l>1$, $\Sigma_l=QH_n/(\Phi_{2l}(\sigma))$ is a quaternion algebra over $Q(\zeta_{2l}+\zeta_{2l}^{-1})$.

In case n is odd there is a simple component $QH_n/(\Phi_2(\sigma)) \cong Q(i)$ in $QH_n/(\sigma^n+I)$. For each $2^m|l|n$, $ZH_n/(\Phi_{2l}(\sigma)) \cong ZH_n \underset{ZH'_n}{\otimes} Z[\zeta_{2l}]$ and so $\Lambda_l = ZH_n/(\Phi_{2l}(\sigma))$ is a quasi-permutation H_n-module. Let Ω_{D_n} be a maximal order in QD_n which contains ZD_n. We put $\Lambda_{H_n} = \Omega_{D_n} \oplus \sum_{2^m|l|n} \oplus \Lambda_l$. Then Λ_{H_n} is an order in QH_n which contains ZH_n. Let Ω_{H_n} be a maximal order in QH_n which contains Λ_{H_n}. There are natural epimorphisms $\alpha_n: C(ZH_n) \to C(\Lambda_{H_n})$ and $\beta_n: C(\Lambda_{H_n}) \to C(\Omega_{H_n})$ induced by $\Lambda_{H_n} \underset{ZH_n}{\otimes} \cdot$ and $\Omega_{H_n} \underset{\Lambda_{H_n}}{\otimes} \cdot$, respectively.

PROPOSITION 3.8. *For any prime power p^l there is an exact sequence:*

$$0 \longrightarrow C^q(ZH_{p^l}) \longrightarrow C(ZH_{p^l}) \xrightarrow{\alpha_{p^l}} C(\Lambda_{H_{p^l}}) \longrightarrow 0 .$$

We have $\tilde{C}(ZH_{p^l}) = C^q(ZH_{p^l})$ if and only if $\beta_{p^l}: C(\Lambda_{H_{p^l}}) \to C(\Omega_{H_{p^l}})$ is an isomorphism. Especially $C(\Lambda_{H_2}) = 0$ and $C^q(ZH_2) = \tilde{C}(ZH_2) = C(ZH_2)$.

PROOF. Let $\mathfrak{a}$ be a quasi-permutation projective ideal of ZH_{p^l}. Then $\mathfrak{a}/(\sigma^{p^l}-1)\mathfrak{a}$ is also a quasi-permutation projective ideal of ZD_{p^l}. By (3.4) and (3.6) we have $\tilde{C}(ZD_{p^l}) = C^q(ZD_{p^l})$. Hence we see $\Omega_{D_{p^l}}\mathfrak{a} \cong \Omega_{D_{p^l}}$. On the other hand, since each Σ_{p^m} is a division algebra, for any subgroup H of H_{p^l}, $\Lambda_{p^m} \cdot ZH_{p^l}/H \cong \Lambda_{p^m}$ or 0. From this it follows that $\Lambda_{p^m}\mathfrak{a} \oplus \Lambda_{p^m} \cong \Lambda_{p^m} \oplus \Lambda_{p^m}$. Thus we have $\Lambda_{H_{p^l}}\mathfrak{a} \oplus \Lambda_{H_{p^l}} \cong \Lambda_{H_{p^l}} \oplus \Lambda_{H_{p^l}}$. Conversely let $\mathfrak{a}$ be a projective ideal of ZH_{p^l} such that $\Lambda_{H_{p^l}}\mathfrak{a} \oplus \Lambda_{H_{p^l}} \cong \Lambda_{H_{p^l}} \oplus \Lambda_{H_{p^l}}$. Then $\Omega_{H_{p^l}}\mathfrak{a} \cong \Omega_{H_{p^l}}$ and so $\mathfrak{a}/(\sigma^{p^l}-1)\mathfrak{a}$ is a quasi-permutation H_{p^l}-module. We have also that, for any $m \leqq l$, $\Lambda_{p^m}\mathfrak{a} \oplus \Lambda_{p^m} \cong \Lambda_{p^m} \oplus \Lambda_{p^m}$. Since Λ_{p^m} is a quasi-permutation H_{p^l}-module, according to (1.4), $\Lambda_{p^m}\mathfrak{a}$ is also a quasi-permutation H_{p^l}-module. Using the same method as in (2.3) we can show that $\mathfrak{a}/(\sigma^{p^l}+1)\mathfrak{a}$ is a quasi-permutation H_{p^l}-module. Furthermore, considering an exact sequence:

$$0 \longrightarrow \mathfrak{a}/(\sigma^{p^l}+1)\mathfrak{a} \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{a}/(\sigma^{p^l}-1)\mathfrak{a} \longrightarrow 0 ,$$

it can easily be seen that $\mathfrak{a}$ is a quasi-permutation H_{p^l}-module. This completes the proof of Kernel $\alpha_{p^l} = C^q(ZH_{p^l})$. The second part of the proposition is obvious. Hence we will prove only $C(\Lambda_{H_2}) = 0$. We have $\Lambda_{H_2} = \Omega_{D_2} \oplus \Lambda_2$, and therefore it suffices to show that any projective ideal of Λ_2 is principal. The quaternion algebra Σ_2 is generated by i, j, k over Q such that $i^2 = j^2 = -1$, $ij = -ji$ and $k = ij$ and Λ_2 can be identified with the order $Z+Zi+Zj+Zk$ in Σ_2. Let Γ be the order in Σ_2 generated by $i, j, k, \frac{1}{2}(1+i+j+k)$. Then, as is well known, Γ is a maximal order in Σ_2 which contains Λ_2 and the class number $c(\Gamma)$ of Γ is 1. Let $\mathfrak{a}$ be a projective ideal of Λ_2. Then there is an ideal $\mathfrak{b}$ of Λ_2 such that $\mathfrak{a} \cong \mathfrak{b}$ and $2 \nmid [\Lambda_2 : \mathfrak{b}]$. Therefore we may suppose $2 \nmid [\Lambda_2 : \mathfrak{a}]$. Since $[\Gamma : \Lambda_2] = 2$, we have $\mathfrak{a} = \Gamma\mathfrak{a} \cap \Lambda_2$. Because of $c(\Gamma) = 1$ there exists $\omega \in \Gamma$ such that $\Gamma\mathfrak{a} = \Gamma\omega$. Then the norm of ω is odd. Therefore we can find a unit ε of Γ such that $\varepsilon\omega \equiv 1 \bmod 2$ by the well-known result on quaternions. Accordingly we can show that $\mathfrak{a}=\Lambda_2\varepsilon\omega$, $\varepsilon\omega\in\Lambda_2$, which completes the proof of the proposition.

We remark here that Reiner-Ullom ([4]) has proved that $C(ZH_2)\cong Z/2Z$.

From (3.4), (3.6), (3.7) and (3.8) we get

THEOREM 3.9. *Let Π be one of the following groups:*

(1) *p-groups with $p\neq 2$;*

(2) *metacyclic groups $\{\Pi_{p^l,q,r}\}$ where p, q are distinct primes;*

(3) *dihedral groups $\{D_{p^l}\}$ where p is a prime;*

(4) *the quaternion group H_2, the alternating group A_4 and the symmetric group S_4.*

Then we have $\tilde{C}(Z\Pi)=C^q(Z\Pi)$.

§4. In this section we will give a basic result on projective quasi-permutation modules over a finite abelian group which is a refinement of (2.5).

LEMMA 4.1. *Let $\Pi=[\sigma]$ be a cyclic group of order n and $\mathfrak{a}$ be a projective ideal of $Z\Pi$ such that $\mathfrak{a}^{\Phi_n}\cong Z[\zeta_n]$. Then there exist a projective ideal $\mathfrak{c}$ of $Z\Pi$ with $\mathfrak{c}^{\Phi_l}\cong Z[\zeta_l]$ for any $l|n$, a permutation Π-module S and a projective ideal $\mathfrak{a}_m$ of $Z\Pi/[\sigma^m]$ for any $m|n$ such that n/m is a prime, such that*

$$\mathfrak{a}\oplus\sum_m\oplus Z\Pi/[\sigma^m]\oplus S\cong\mathfrak{c}\oplus\sum_m\oplus\mathfrak{a}_m\oplus S.$$

PROOF. Let $M_1=\mathfrak{a}\oplus\sum_m\oplus[\mathfrak{a}/(\sigma^m-1)\mathfrak{a}]^{-1}$. Then $M_1^{\Phi_n}=\mathfrak{a}^{\Phi_n}$ and $M_1^{\Phi_m}=\mathfrak{a}^{\Phi_m}\oplus(\mathfrak{a}^{-1})^{\Phi_m}\cong Z[\zeta_m]^{(2)}$ for any $m|n$ such that n/m is a prime. Also, for any $l|n$ with $n/l=p_1p_2$ for primes p_1, p_2, $M_1^{\Phi_l}\cong Z[\zeta_l]^{(2)}\oplus[(\mathfrak{a}^{-1})^{\Phi_l}]^{(\alpha_l)}$ for some $\alpha_l\geqq 0$. We now put $M_2=M_1\oplus\sum_{n/l=p_1p_2}\oplus[\mathfrak{a}/(\sigma^l-1)\mathfrak{a}]^{(\alpha_l)}$. Then, by (2.1), we have

$$M_2^{\Phi_n}\cong Z[\zeta_n],$$

$$M_2^{\Phi_m}\cong Z[\zeta_m]^{(2)}\qquad\text{when}\quad n/m=p,$$

$$M_2^{\Phi_l}\cong Z[\zeta_l]^{(2+2\alpha_l)}\qquad\text{when}\quad n/l=p_1p_2,$$

and

$$M_2^{\Phi_k}\cong Z[\zeta_k]^{(\beta_k)}\oplus[(\mathfrak{a}^{\pm1})^{\Phi_k}]^{(\gamma_k)}\qquad\text{for some}\quad \beta_k,\gamma_k\geqq 0$$

when $n/k=p_1p_2p_3$. We further put

$$M_3=M_2\oplus\sum_{n/k=p_1p_2p_3}\oplus[(\mathfrak{a}/(\sigma^k-1)\mathfrak{a})^{\mp1}]^{(\gamma_k)}$$

and repeat the same procedure to M_3 as to M_2. Continuing this procedure, we finally find $s_l\geqq 0$ and $\varepsilon(l)=\pm1$ for any $l|n$ such that, putting $M=\mathfrak{a}\oplus\sum_{\substack{l|n\\ l<n}}\oplus[(\mathfrak{a}/(\sigma^l-1)\mathfrak{a})^{\varepsilon(l)}]^{(s_l)}$, M^{Φ_k} is $Z[\zeta_k]$-free for any $k|n$. Since $\mathfrak{a}$ is $Z\Pi$-faithful and $\mathfrak{a}/(\sigma^l-1)\mathfrak{a}\sim Z\Pi/[\sigma^l]$, by (C), there is a projective ideal of $Z\Pi$

such that $M\cong\mathfrak{c}\oplus\sum_{\substack{l\mid n\\ l<n}}\oplus[Z\Pi/[\sigma^l]]^{(s_l)}$. Then we see easily that $\mathfrak{c}^{\Phi_k}\cong Z[\zeta_k]$ for any $k\mid n$. Therefore $\mathfrak{c}$ is as required. Furthermore we have

$$M\oplus\sum_{\substack{l\mid n\\ l<n}}\oplus[(\mathfrak{a}/(\sigma^l-1)\mathfrak{a})^{-\varepsilon(l)}]^{(s_l)}\cong\mathfrak{a}\oplus\sum_{\substack{l\mid n\\ l<n}}\oplus[Z\Pi/[\sigma^l]]^{(2s_l)}$$

$$\cong\mathfrak{c}\oplus\sum_{\substack{l\mid n\\ l<n}}\oplus[Z\Pi/[\sigma^l]\oplus(\mathfrak{a}/(\sigma^l-1)\mathfrak{a})^{-\varepsilon(l)}]^{(s_l)}.$$

Again, by (C), there is a Π-module L such that

$$L\sim\sum_{n/m=p}\oplus Z\Pi/[\sigma^m]$$

and

$$\sum_{\substack{l\mid n\\ l<n}}\oplus[Z\Pi/[\sigma^l]\oplus(\mathfrak{a}/(\sigma^l-1)\mathfrak{a})^{-\varepsilon(l)}]^{(s_l)}$$

$$\cong L\oplus\sum_{n/m=p}\oplus[Z\Pi/[\sigma^m]]^{(2s_m-1)}\oplus\sum_{\substack{n/l=p_1p_2\cdots p_t\\ t\geqq 2}}\oplus[Z\Pi/[\sigma^l]]^{(2s_l)}.$$

Using (A), we can write $L\cong\sum_{n/m=p}\oplus\mathfrak{a}_m$ for some $\mathfrak{a}_m\sim Z\Pi/[\sigma^m]$. Let

$$S=\sum_{n/m=p}\oplus[Z\Pi/[\sigma^m]]^{(2s_m-1)}\oplus\sum_{\substack{n/l=p_1p_2\cdots p_t\\ t\geqq 2}}\oplus[Z\Pi/[\sigma^l]]^{(2s_l)}.$$

Then we obtain

$$\mathfrak{a}\oplus\sum_{n/m=p}\oplus[Z\Pi/[\sigma^m]]\oplus S\cong\mathfrak{c}\oplus\sum_{n/m=p}\oplus\mathfrak{a}_m\oplus S,$$

and this completes the proof of the lemma.

THEOREM 4.2. *Let Π be a finite abelian group and $\mathfrak{a}$ be a projective ideal of $Z\Pi$. Then $\mathfrak{a}$ is a quasi-permutation Π-module if and only if $\mathfrak{a}\oplus\sum_{\Pi'\in\mathfrak{S}}\oplus Z\Pi/\Pi'\cong Z\Pi\oplus\sum_{\Pi'\in\mathfrak{S}}\oplus Z\Pi/\Pi'$, where $\mathfrak{S}$ denotes the set of all subgroups, Π' ($\neq 1$), of Π such that Π/Π' is cyclic.*

PROOF. The if part is evident. Hence we have only to prove the only if part. Suppose that $\mathfrak{a}$ is a quasi-permutation Π-module. Then $\mathfrak{a}\oplus S'\cong S$ for some permutation Π-modules S, S' by (1.6), and $\Omega_\Pi\mathfrak{a}\cong\Omega_\Pi$ by (2.5), where Ω_Π denotes the maximal order in $Q\Pi$ containing $Z\Pi$.

First suppose that Π is a cyclic group of order n with a generator σ. In this case we can write

$$S=\sum_{k\mid n}\oplus[Z\Pi/[\sigma^k]]^{(N_k)},\qquad S'=\sum_{k\mid n}\oplus[Z\Pi/[\sigma^k]]^{(N'_k)}$$

for some N_k, N'_k ($\geqq 0$). Since $\mathfrak{S}=\{[\sigma^k]\}_{k\mid n,k<n}$ in this case, it suffices by (D) to prove $N_n=N'_n+1$ and $N_k=N'_k$ for any $k<n$. By our assumption we have $Z[\zeta_n]\oplus S'^{\Phi_n}\cong S^{\Phi_n}$. But, by (2.1), $S'^{\Phi_n}\cong[Z[\zeta_n]]^{(N'_n)}$ and $S^{\Phi_n}\cong[Z[\zeta_n]]^{(N_n)}$. Hence $N_n=N'_n+1$. Let $m\mid n$, $m<n$ and suppose that $N_n=N'_n+1$ and $N_k=$

N'_k for each $m<k<n$. Then we have $Z[\zeta_m]\oplus S'^{\Phi_m}\cong S^{\Phi_m}$ and, again applying (2.1),

$$S'^{\Phi_m}\cong[Z[\zeta_m]]^{(N_n-1)}\oplus\sum_{\substack{m\mid k\mid n\\ m<k<n}}\oplus[Z[\zeta_m]]^{(N_k)}\oplus[Z[\zeta_m]]^{(N'_m)}$$

$$S^{\Phi_m}\cong[Z[\zeta_m]]^{(N_n)}\oplus\sum_{\substack{m\mid k\mid n\\ m<k<n}}\oplus[Z[\zeta_m]]^{(N_k)}\oplus[Z[\zeta_m]]^{(N_m)}.$$

This shows $N_m=N'_m$.

Now suppose that Π is a non-cyclic abelian group. We denote by $\mathfrak{S}_0$ the subset of $\mathfrak{S}$ consisting of all minimal members of $\mathfrak{S}$. We define $\mathfrak{S}_{k+1}$ to be the subset of $\mathfrak{S}$ consisting of $\mathfrak{S}_k$ and all minimal elements of $\mathfrak{S}-\mathfrak{S}_k$. Then we obtain an ascending chain of the subsets of $\mathfrak{S}$: $\mathfrak{S}_0\subsetneqq\mathfrak{S}_1\subsetneqq\cdots\subsetneqq\mathfrak{S}_t=\mathfrak{S}$. Let $\tilde{\mathfrak{S}}_k$ be the set of all maximal elements in $\mathfrak{S}_k$. To prove our assertion it suffices to prove that, for any $0\leqq k\leqq t$, there exist some $\mathfrak{a}_{\Pi'}$ with

$$\left.\begin{aligned}&\mathfrak{a}_{\Pi'}\sim Z\Pi/\Pi'\ (\Pi'\in\tilde{\mathfrak{S}}_k)\ \text{such that}\\ &\mathfrak{a}\oplus\sum_{\Pi'\in\mathfrak{S}_k}\oplus Z\Pi/\Pi'\oplus\sum_{\Pi''\in\mathfrak{S}-\mathfrak{S}_k}\oplus Z\Pi/\Pi''\\ &\qquad\cong Z\Pi\oplus\sum_{\Pi'\in\mathfrak{S}_k-\tilde{\mathfrak{S}}_k}\oplus Z\Pi/\Pi'\oplus\sum_{\Pi'\in\tilde{\mathfrak{S}}_k}\oplus\mathfrak{a}_{\Pi'}\oplus\sum_{\Pi''\in\mathfrak{S}-\mathfrak{S}_k}\oplus Z\Pi/\Pi''\end{aligned}\right\}\cdots\cdots(*)_k.$$

In fact $(*)_t$ implies our assertion, because $\mathfrak{S}_t=\mathfrak{S}$ and $\tilde{\mathfrak{S}}_t=\{\Pi\}$. We will prove $(*)_k$ by induction on k. Since $\sum_{\Pi'\in\mathfrak{S}}\oplus Z\Pi/\Pi'$ is $Z\Pi$-faithful, by (B), we have

$$\mathfrak{a}\oplus\sum_{\Pi'\in\mathfrak{S}}\oplus Z\Pi/\Pi'\cong Z\Pi\oplus N$$

for some N with $N\approx\sum_{\Pi'\in\mathfrak{S}}\oplus Z\Pi/\Pi'$. Then, by (A) and (C), we can find $\mathfrak{a}_{\Pi'}$ with $\mathfrak{a}_{\Pi'}\sim Z\Pi/\Pi'$ for all $\Pi'\in\mathfrak{S}_0$ such that

$$N\cong\sum_{\Pi'\in\mathfrak{S}_0}\oplus\mathfrak{a}_{\Pi'}\oplus\sum_{\Pi'\in\mathfrak{S}-\mathfrak{S}_0}\oplus Z\Pi/\Pi'.$$

Hence we obtain that

$$\mathfrak{a}\oplus\sum_{\Pi'\in\mathfrak{S}_0}\oplus Z\Pi/\Pi'\oplus\sum_{\Pi''\in\mathfrak{S}-\mathfrak{S}_0}\oplus Z\Pi/\Pi''\cong Z\Pi\oplus\sum_{\Pi'\in\mathfrak{S}_0}\oplus\mathfrak{a}_{\Pi'}\oplus\sum_{\Pi''\in\mathfrak{S}-\mathfrak{S}_0}\oplus Z\Pi/\Pi''.$$

Thus $(*)_0$ is proved. Next suppose $(*)_k$, $k\geqq 0$. For each $\Pi'\in\tilde{\mathfrak{S}}_k$, $\mathfrak{a}_{\Pi'}$ is considered as a Π/Π'-module. We denote by n' the order of Π/Π'. Then $\mathfrak{a}_{\Pi'}^{\Phi_{n'}}\cong Z[\zeta_{n'}]$. By (4.1) there exist a projective ideal $\mathfrak{c}_{\Pi'}$ of $Z\Pi/\Pi'$ with $\Omega_{\Pi/\Pi'}\mathfrak{c}_{\Pi'}\cong\Omega_{\Pi/\Pi'}$, a permutation Π/Π'-module $S'_{\Pi'}$ and a projective ideal $\mathfrak{a}_{\tilde{\Pi}'}^{(\Pi')}$ of $Z\Pi/\tilde{\Pi}'$ for any subgroup $\tilde{\Pi}'$ of Π such that $\tilde{\Pi}'/\Pi'$ is a cyclic group of prime order such that

$$\mathfrak{a}_{\Pi'}\oplus\sum_{\tilde{\Pi}'\in\mathfrak{S}_{\Pi'}}\oplus Z\Pi/\tilde{\Pi}'\oplus S'_{\Pi'}\cong\mathfrak{c}_{\Pi'}\oplus\sum_{\tilde{\Pi}'\in\mathfrak{S}_{\Pi'}}\oplus\mathfrak{a}_{\tilde{\Pi}'}^{(\Pi')}\oplus S'_{\Pi'}$$

where $\mathfrak{S}_{\Pi'}$ denotes the set of all subgroups $\tilde{\Pi}'$ of Π such that $\Pi'\subset\tilde{\Pi}'$ and $[\tilde{\Pi}':\Pi']$ is a prime. However we have already proved the assertion in case Π is cyclic. Therefore, for any $\Pi'\in\tilde{\mathfrak{S}}_k$, there is a permutation Π/Π'-module $S''_{\Pi'}$ such that $\mathfrak{c}_{\Pi'}\oplus S''_{\Pi'}\cong Z\Pi/\Pi'\oplus S''_{\Pi'}$. Accordingly we have

$$\mathfrak{a}_{\Pi'}\oplus\sum_{\tilde{\Pi}'\in\mathfrak{S}_{\Pi'}}\oplus Z\Pi/\tilde{\Pi}'\oplus S_{\Pi'}\cong Z\Pi/\Pi'\oplus\sum_{\tilde{\Pi}'\in\mathfrak{S}_{\Pi'}}\oplus\mathfrak{a}_{\tilde{\Pi}'}^{(\Pi')}\oplus S_{\Pi'}$$

for some permutation Π/Π'-module $S_{\Pi'}$. From this and $(*)_k$ it follows immediately that

$$\mathfrak{a}\oplus\sum_{\Pi'\in\mathfrak{S}_k}\oplus Z\Pi/\Pi'\oplus\sum_{\Pi''\in\mathfrak{S}-\mathfrak{S}_k}\oplus Z\Pi/\Pi''\oplus\sum_{\Pi'\in\tilde{\mathfrak{S}}_k}\sum_{\tilde{\Pi}'\in\mathfrak{S}_{\Pi'}}\oplus Z\Pi/\tilde{\Pi}'\oplus\sum_{\Pi'\in\tilde{\mathfrak{S}}_k}\oplus S_{\Pi'} \qquad\cdots\cdots(**)$$

$$\cong Z\Pi\oplus\sum_{\Pi'\in\mathfrak{S}_k}\oplus Z\Pi/\Pi'\oplus\sum_{\Pi''\in\mathfrak{S}-\mathfrak{S}_k}\oplus Z\Pi/\Pi''\oplus\sum_{\Pi'\in\tilde{\mathfrak{S}}_k}\sum_{\tilde{\Pi}'\in\mathfrak{S}_{\Pi'}}\oplus\mathfrak{a}^{(\Pi')}\oplus\sum_{\Pi'\in\tilde{\mathfrak{S}}_k}\oplus S_{\Pi'}\,.$$

It is easily seen that any $\tilde{\Pi}'$ of $\bigcup_{\Pi'\in\tilde{\mathfrak{S}}_k}\mathfrak{S}_{\Pi'}$ contains some Π'' of $\mathfrak{S}_{k+1}$. According to (B), for fixed $\Pi''\in\mathfrak{S}_{k+1}$, we have

$$Z\Pi/\Pi''\oplus\sum_{\substack{\Pi''\subseteq\tilde{\Pi}'\\ \tilde{\Pi}'\in\mathfrak{S}_{\Pi'}}}\oplus\mathfrak{a}_{\tilde{\Pi}'}^{(\Pi')}\cong\mathfrak{a}_{\Pi''}\oplus\sum_{\substack{\Pi''\subseteq\tilde{\Pi}'\\ \tilde{\Pi}'\in\mathfrak{S}_{\Pi'}}}\oplus[Z\Pi/\tilde{\Pi}']^{(l_{\tilde{\Pi}'})}$$

for a projective ideal $\mathfrak{a}_{\Pi''}$ of $Z\Pi/\Pi''$ and $l_{\tilde{\Pi}'}\geqq 0$. Hence from (**) we obtain

$$\mathfrak{a}\oplus\sum_{\Pi''\in\mathfrak{S}_{k+1}}\oplus Z\Pi/\Pi''\oplus\sum_{\Pi'''\in\mathfrak{S}-\mathfrak{S}_{k+1}}\oplus Z\Pi/\Pi'''\oplus T$$

$$\cong Z\Pi\oplus\sum_{\Pi''\in\mathfrak{S}_{k+1}-\tilde{\mathfrak{S}}_{k+1}}\oplus Z\Pi/\Pi''\oplus\sum_{\Pi''\in\tilde{\mathfrak{S}}_{k+1}}\oplus\mathfrak{a}_{\Pi''}\oplus\sum_{\Pi'''\in\mathfrak{S}-\mathfrak{S}_{k+1}}\oplus Z\Pi/\Pi'''\oplus T$$

for some permutation Π-module T such that $T\cong\sum_{\Pi'\in\mathfrak{S}}\oplus[Z\Pi/\Pi']^{(j_{\Pi'})}$, $j_{\Pi'}\geqq 0$. Since $\mathfrak{a}\oplus\sum_{\Pi'\in\mathfrak{S}}\oplus Z\Pi/\Pi'\sim Z\Pi\oplus\sum_{\Pi'\in\mathfrak{S}}\oplus Z\Pi/\Pi'$, applying (D), we can omit T from both sides, i.e.,

$$\mathfrak{a}\oplus\sum_{\Pi''\in\mathfrak{S}_{k+1}}\oplus Z\Pi/\Pi''\oplus\sum_{\Pi'''\in\mathfrak{S}-\mathfrak{S}_{k+1}}\oplus Z\Pi/\Pi'''$$

$$\cong Z\Pi\oplus\sum_{\Pi''\in\mathfrak{S}_{k+1}-\tilde{\mathfrak{S}}_{k+1}}\oplus Z\Pi/\Pi''\oplus\sum_{\Pi''\in\tilde{\mathfrak{S}}_{k+1}}\oplus\mathfrak{a}_{\Pi''}\oplus\sum_{\Pi'''\in\mathfrak{S}-\mathfrak{S}_{k+1}}\oplus Z\Pi/\Pi'''\,.$$

Thus we obtain $(*)_{k+1}$, which completes the proof.

COROLLARY 4.3. *Let Π be a finite abelian group. Then*

$$|\gamma_{Z\Pi\oplus\sum_{\Pi'\in\mathfrak{S}}\oplus Z\Pi/\Pi'}|=1\,,$$

where $\mathfrak{S}$ denotes the set of all subgroups, Π', of Π such that Π/Π' is cyclic.

§5. In §§5 and 6 we will study non-projective quasi-permutation modules over finite cyclic groups.

Let Π be a cyclic group of order n. Let $m|n$ and let $\Phi_m(T)$ be the m-th cyclotomic polynomial. We put $\Psi_m(T)=(T^n-1)/\Phi_m(T)$.

We begin with

LEMMA 5.1. *Let Π be a cyclic group of order n. Let M be a Π-module. Then, for any $m|n$, $(M^{\Phi m})^* \cong \Psi_m M^*$.*

PROOF. Consider the exact sequence: $0 \to M^{\Phi m} \to M \to M/M^{\Phi m} \to 0$. By dualizing this sequence we get an exact sequence

$$0 \longrightarrow (M/M^{\Phi m})^* \longrightarrow M^* \longrightarrow (M^{\Phi m})^* \longrightarrow 0 .$$

Then we have $\Psi_m(M/M^{\Phi m})^*=0$ and therefore there is an epimorphism: $(M^{\Phi m})^* \to \Psi_m M^*$. However $\mathrm{rank}_Z(M^{\Phi m})^* = \mathrm{rank}_Z \Psi_m M^*$. Hence $\Psi_m M^* \cong (M^{\Phi m})^*$.

PROPOSITION 5.2. *Let Π be a cyclic group of order n. Then the following conditions are equivalent:*

(1) *For any $m|n$, all prime divisors of n in $Z[\zeta_m]$ are principal.*

(2) *For any exact sequence $0 \to M' \to M \to M'' \to 0$ of Π-modules and any $m|n$, $M^{\Phi m} \cong M'^{\Phi m} \oplus M''^{\Phi m}$.*

(3) *For any Π-module M and any $m|n$, $M^{\Phi m} \cong \Psi_m M$.*

(4) *For any Π-module M and any $m|n$, $(M^*)^{\Phi m} \cong (M^{\Phi m})^*$.*

If n is a prime power or if the class number of $Q(\zeta_n)$ is 1, then the above conditions are satisfied.

PROOF. The implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) were proved by Swan ([**12**], p. 108), and the implication (3) $\Leftrightarrow$ (4) follows directly from (5.1). Let $m|n$ $(n \neq 1)$ and let $\mathfrak{p}$ be a prime divisor of n in $Z[\zeta_m]$. Then, by a direct computation, we see that $\Psi_m(\zeta_m) \in \mathfrak{p}$ and therefore there is a divisor $d \neq m$ of n such that $\Phi_d(\zeta_m) \in \mathfrak{p}$. Now put $A=Z[\zeta_m]/\mathfrak{p}$. Since A can be considered as a $Z[\zeta_d]$-module, we can construct an exact sequence: $0 \to \mathfrak{q} \to Z[\zeta_d] \to A \to 0$ where $\mathfrak{q}$ is an ideal of $Z[\zeta_d]$. Forming the pullback as $Z\Pi$-modules, we get the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathfrak{p} & = & \mathfrak{p} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{q} & \longrightarrow & M & \longrightarrow & Z[\zeta_m] & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{q} & \longrightarrow & Z[\zeta_d] & \longrightarrow & A & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 . & &
\end{array}$$

Then $M^{\Phi d} \cong \mathfrak{q}$ and $M^{\Phi m} \cong \mathfrak{p}$. Suppose that the condition (2) is satisfied. Because of the exactness of the second row, we have $M^{\Phi m} \cong Z[\zeta_m]$, hence $\mathfrak{p} \cong Z[\zeta_m]$, which implies (1). On the other hand, suppose that (4) is satisfied.

By dualizing the second row, we have an exact sequence $0 \to Z[\zeta_m] \to M^* \to \mathfrak{q}^{-1} \to 0$. Hence $(M^*)^{\Phi_m} \cong Z[\zeta_m]$. Since $(M^*)^{\Phi_m} \cong (M^{\Phi_m})^*$, $\mathfrak{p} \cong Z[\zeta_m]$, which implies also (1). The second part of the proposition is obvious. Thus the proof of the proposition is completed.

Recently T. Sumioka ([**10**]) proved the equivalence of (1) and (2) in (5.2) in a little more general form.

We should remark that the smallest integer n which does not satisfy the conditions in (5.2) is $39 = 3 \cdot 13$. In fact, $\mathfrak{p} = (\zeta_{39} - 3, 13)$ is a non-principal prime ideal of $Z[\zeta_{39}]$ which divides 39.

LEMMA 5.3 ([**13**]). *Let Π be a cyclic group of order n and let $0 \to M \to N \to S \to 0$ be an exact sequence where M, N are Π-modules and S is a permutation Π-module. Then $N^{\Phi_n} \cong M^{\Phi_n} \oplus S^{\Phi_n} \cong M^{\Phi_n} \oplus Z[\zeta_n]^{(t)}$ for some $t \geqq 0$. Especially, if M is a quasi-permutation Π-module, then M^{Φ_n} is a free $Z[\zeta_n]$-module.*

PROPOSITION 5.4. *Let Π be a cyclic group of order n and let M be a Π-module. Then we have $M \underset{(r)}{\text{---}} M^{\Phi_n} \oplus M/M^{\Phi_n}$. Especially M is a quasi-permutation Π-module if and only if M^{Φ_n} is a free $Z[\zeta_n]$-module and M/M^{Φ_n} is a quasi-permutation Π-module.*

PROOF. The second part of the proposition follows directly from the first part and (5.3). Hence we have only to prove the first part. First suppose that M^{Φ_n} is $Z[\zeta_n]$-free. Let t be the rank of M^{Φ_n}. Then we have an exact sequence

$$0 \longrightarrow M^{\Phi_n} \longrightarrow Z\Pi^{(t)} \longrightarrow (Z\Pi/(\Psi_n(\sigma)))^{(t)} \longrightarrow 0$$

where σ denotes a generator of Π. Put $F = Z\Pi^{(t)}$ and $L = (Z\Pi/(\Psi_n(\sigma)))^{(t)}$. Using the same method as in the proof of (2.3) we can show that L is a quasi-permutation Π-module. Since $\hat{H}^0(\Pi', Z[\zeta_n]) = 0$ for any subgroup Π' of Π, we have $\hat{H}^1(\Pi', L) = 0$ for any subgroup Π' of Π. Hence, by (1.6), there exist permutation Π-modules S, S' such that $L \oplus S' \cong S$. Forming the pushout of $M^{\Phi_n} \to F$ and $M^{\Phi_n} \to M$, we get the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M^{\Phi_n} & \longrightarrow & M & \longrightarrow & M/M^{\Phi_n} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & F & \longrightarrow & F \oplus M/M^{\Phi_n} & \longrightarrow & M/M^{\Phi_n} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & L & = & L & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

From the second column, we get an exact sequence

$$0 \longrightarrow M \longrightarrow S' \oplus F \oplus M/M^{\Phi_n} \longrightarrow S \longrightarrow 0.$$

According to (1.3), we have $M \underset{(r)}{\text{---}} S' \oplus F \oplus M/M^{\Phi_n} \underset{(r)}{\text{---}} M/M^{\Phi_n}$, and therefore $M \underset{(r)}{\text{---}} M/M^{\Phi_n} \underset{(r)}{\text{---}} M^{\Phi_n} \oplus M/M^{\Phi_n}$. In the general case we put $M_0 = M \oplus (M^{\Phi_n})^*$. Then $M_0/M_0^{\Phi_n} \cong M/M^{\Phi_n}$ and $M_0^{\Phi_n} \cong M^{\Phi_n} \oplus (M^{\Phi_n})^*$. Since $M^{\Phi_n} \oplus (M^{\Phi_n})^*$ is $Z[\zeta_n]$-free, $M_0^{\Phi_n}$ is $Z[\zeta_n]$-free. Therefore we have $M \oplus (M^{\Phi_m})^* \cong M_0 \underset{(r)}{\text{---}} M_0/M_0^{\Phi_n} \cong M/M^{\Phi_n}$, and hence $M \oplus (M^{\Phi_n})^* \oplus M^{\Phi_n} \underset{(r)}{\text{---}} M/M^{\Phi_n} \oplus M^{\Phi_n}$. Because $M^{\Phi_n} \oplus (M^{\Phi_n})^*$ is $Z[\zeta_n]$-free, this shows that $M \underset{(r)}{\text{---}} M/M^{\Phi_n} \oplus M^{\Phi_n}$. Thus the proof of the proposition is completed.

THEOREM 5.5. *Let Π be a cyclic p-group of order p^l.*

(0) *Let M be a Π-module. Then* $M \underset{(r)}{\text{---}} \sum_{m=1}^{l} \oplus M^{\Phi_{p^m}}$.

(1) *A Π-module M is a quasi-permutation Π-module if and only if, for any $1 \leqq m \leqq l$, $M^{\Phi_{p^m}}$ is $Z[\zeta_{p^m}]$-free.*

(2) *If M is a quasi-permutation Π-module, then the dual module M^* is also a quasi-permutation Π-module.*

(3) *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of Π-modules. If two of M', M and M'' are quasi-permutation Π-modules, then the rest of them is also a quasi-permutation Π-module.*

PROOF. We will prove (0) by induction on l. For $l=0$ this is obvious. Suppose that $l \geqq 1$. Now, by (5.4), we have $M \underset{(r)}{\text{---}} M^{\Phi_{p^l}} \oplus M/M^{\Phi_{p^l}}$. Let Π_1 be the subgroup of Π of order p. Then $M/M^{\Phi_{p^l}}$ can be regarded as a Π/Π_1-module. Therefore, by induction, $M/M^{\Phi_{p^l}} \underset{(r)}{\text{---}} \sum_{m=1}^{l-1} \oplus (M/M^{\Phi_{p^l}})^{\Phi_{p^m}}$. However, by (5.2), $M^{\Phi_{p^m}} \cong (M/M^{\Phi_{p^l}})^{\Phi_{p^m}}$ for any $0 \leqq m \leqq l-1$, and so $M/M^{\Phi_{p^l}} \underset{(r)}{\text{---}} \sum_{m=1}^{l-1} \oplus M^{\Phi_{p^m}}$. Thus we get $M \underset{(r)}{\text{---}} \sum_{m=1}^{l} \oplus M^{\Phi_{p^m}}$.

The assertion (1) follows directly from (0) and (2.6) (or (5.4)), and both (2) and (3) are immediate consequences of (1) and (5.2).

COROLLARY 5.6. *Let Π be a cyclic p-group and let Ω_Π be the maximal order in $Q\Pi$ which contains $Z\Pi$. Then the abelian semigroup $T(\Pi)$ is a group isomorphic to $C(\Omega_\Pi)$.*

More generally, for a cyclic group Π of order n, we consider the following statements:

(1) A Π-module M is a quasi-permutation Π-module if and only if, for any $m|n$, M^{Φ_m} is $Z[\zeta_m]$-free.

(2) If M is a quasi-permutation Π-module, then M^* is also a quasi-permutation Π-module.

(3) Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of Π-modules. If two of M', M and M'' are quasi-permutation Π-modules, then the rest of them is a quasi-permutation Π-module.

We conjecture that, for any cyclic group of order n which satisfies the conditions in (5.2), all of (1), (2) and (3) are true. However we did not succeed in proving this in the general case.

REMARK 5.7. Let Π be a cyclic group of order n which satisfies the conditions in (5.2). If (1) is true for Π, then both (2) and (3) are true for Π.

REMARK 5.8. Let Π be a cyclic group of order n which does not satisfy the conditions in (5.2). Then both (2) and (3) are false for Π.

REMARK 5.9. Let p, q be primes such that $q|p-1$ and suppose that there is a non-principal prime divisor $\mathfrak{q}$ of p in $Z[\zeta_q]$. Let Π be a cyclic group of order pq. Then (1) is false for Π (cf. [12]). The smallest pair of primes satisfying the assumption is $\{47, 23\}$.

From these remarks it seems fairly difficult to generalize (5.5), (0) and (1) to any finite cyclic group.

PROPOSITION 5.10. *Let $\Pi=[\sigma]$ be a cyclic group of order n and let $p^l|n$ where p is a prime. Let $0\rightarrow N\rightarrow M\rightarrow Z[\zeta_{p^l}]^{(t)}\rightarrow 0$ be an exact sequence of Π-modules where N has no non-zero element invariant under $\sigma^{p^{l-1}}$ (when $l\geqq 1$) and t is a non-negative integer. Let K/k be a Galois extension with group Π. Then $K(M)$ can be identified with $K(N\oplus Z[\zeta_{p^l}]^{(t)})$.*

PROOF. It suffices to prove this in the case of $t=1$ and $l\geqq 1$. Let $n=p^l d$, $q=p^l-p^{l-1}$ and $r=p^{l-1}$. We can identify $K(M)$ with $K(N)(X_1, X_2, \cdots, X_q)$ with the action of Π such that $\sigma(X_i)=X_{i+1}$, $1\leqq i\leqq q-1$ and $\sigma(X_q)=\sigma^q(X_1)$ $=\dfrac{a}{X_1X_{r+1}X_{2r+1}\cdots X_{(p-2)r+1}}$ for some $a\in N$. Then $\sigma^r(\sigma^q(X_1))=\sigma^{p^l}(X_1)=$ $\dfrac{\sigma^r a}{a}\cdot X_1$. Since $\sigma^n(X_1)=\sigma^{dp^l}(X_1)=X_1$, we have

$$\frac{\sigma^r a}{a}\cdot\sigma^{p^l}\Big(\frac{\sigma^r a}{a}\Big)\cdot\sigma^{2p^l}\Big(\frac{\sigma^r a}{a}\Big)\cdots\cdots\sigma^{(d-1)p^l}\Big(\frac{\sigma^r a}{a}\Big)=1\,.$$

Hence $a\cdot\sigma^{p^l}a\cdot\sigma^{2p^l}a\cdots\sigma^{(d-1)p^l}a$ is an element of N invariant under σ^r. By the assumption we have

$$a\cdot\sigma^{p^l}a\cdot\sigma^{2p^l}a\cdots\sigma^{(d-1)p^l}a=1\,.$$

If we put $K'=K(N)^{[\sigma^{p^l}]}$, then $N_{K(N)/K'}(a)=1$. By the Hilbert's theorem 90, there is $b\in K(N)$ such that $a=b/\sigma^{p^l}b$. Further put $c=\sigma^r b/b$ and $Z_i=\sigma^{i-1}c\cdot X_i$, $1\leqq i\leqq q$. Then $\sigma^{i-1}c\in K(N)$ and so $K(M)=K(N)(Z_1, Z_2, \cdots, Z_q)$. We easily see that $c\cdot\sigma^r c\cdot\sigma^{2r}c\cdots\sigma^{(p-1)r}c=a^{-1}$ and so we get

$$\sigma(Z_q)=\frac{1}{Z_1Z_{r+1}Z_{2r+1}\cdots Z_{(p-2)r+1}}\,.$$

Therefore the group generated by $Z_1, Z_2, \cdots, Z_q$ is isomorphic to $Z[\zeta_{p^l}]$. Thus $K(M)$ can be identified with $K(N\oplus Z[\zeta_{p^l}])$.

COROLLARY 5.11. *Let $\Pi=[\sigma]$ be a cyclic group of order n and let $p^l|n$ where p is a prime. Let $0\rightarrow N\rightarrow M\rightarrow L\rightarrow 0$ be an exact sequence of Π-modules* *where N has no non-zero element invariant under $\sigma^{p^{l-1}}$ (when $l \geqq 1$) and L is a $Z[\zeta_{p^l}]$-module. Then $M \underset{(r)}{-\!\!-} N \oplus L$.*

PROOF. We have an exact sequence

$$0 \longrightarrow N \longrightarrow M \oplus L^* \longrightarrow L \oplus L^* \longrightarrow 0 .$$

Since $L \oplus L^*$ is $Z[\zeta_{p^l}]$-free, this sequence satisfies the assumption in (5.10) and hence $M \oplus L^* \underset{(r)}{-\!\!-} N \oplus L \oplus L^*$. Therefore $M \oplus L^* \oplus L \underset{(r)}{-\!\!-} N \oplus L \oplus L^* \oplus L$. Since $L \oplus L^* \underset{(r)}{-\!\!-} 0$, this shows that $M \underset{(r)}{-\!\!-} N \oplus L$.

PROPOSITION 5.12. *Let Π be a cyclic group of order $p^l q$ where p, q are distinct primes and l is a positive integer. Let M be a Π-module. Let $M_1 = M^{\Psi_1}$, $M_2 = M_1^{\Psi_p}, \cdots, M_l = M_{l-1}^{\Psi_{p^{l-1}}}$ and $M'_1 = \Phi_{p^l q} M^{\Phi_{p^l q} \Phi_{p^{l-1} q} \cdots \Phi_q}$, $M'_2 = \Phi_{p^{l-1} q} M'_1, \cdots, M'_l = \Phi_{pq} M'_{l-1}$. Then*

$$M \underset{(r)}{-\!\!-} M^{\Phi_{p^l q}} \oplus \sum_{i=1}^{l} \oplus M_i'^{\Phi_{p^{l-i} q}} \oplus \sum_{j=1}^{l} \oplus \Psi_{p^j} M_j .$$

Especially M is a quasi-permutation Π-module if and only if $M^{\Phi_{p^l q}}$ is $Z[\zeta_{p^l q}]$-free, each $M_i'^{\Phi_{p^{l-i} q}}$ is $Z[\zeta_{p^{l-i} q}]$-free and each $\Psi_{p^j} M_j$ is $Z[\zeta_{p^j}]$-free.

PROOF. We consider the following exact sequences:

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow \Psi_1 M \longrightarrow 0$$

$$0 \longrightarrow M_2 \longrightarrow M_1 \longrightarrow \Psi_p M_1 \longrightarrow 0$$

....

....

$$0 \longrightarrow M_l \longrightarrow M_{l-1} \longrightarrow \Psi_{p^{l-1}} M_{l-1} \longrightarrow 0$$

$$0 \longrightarrow M_l^{\Psi_{p^l}} \longrightarrow M_l \longrightarrow \Psi_{p^l} M_l \longrightarrow 0 .$$

Here $\Psi_1 M$ is expressible as a direct sum of the copies of the trivial Π-module Z and, for each $1 \leqq j \leqq l$, $\Psi_{p^j} M_j$ can be regarded as a $Z[\zeta_{p^l}]$-module. Let σ be a generator of Π. Then M_j has no non-zero element invariant under $\sigma^{p^{j-1}}$. Therefore, according to (5.11), $M \underset{(r)}{-\!\!-} M_1$ and, for each $1 \leqq j \leqq l$, $M_j \underset{(r)}{-\!\!-} M_{j+1} \oplus \Psi_{p^j} M_j$. Thus we get

$$M \underset{(r)}{-\!\!-} M_l^{\Psi_{p^l}} \oplus \sum_{j=1}^{l} \oplus \Psi_{p^j} M_j .$$

We easily see that $M_l^{\Psi_{p^l}} = M^{\Phi_{p^l q} \Phi_{p^{l-1} q} \cdots \Phi_q}$. Put $M' = M_l^{\Psi_{p^l}}$. Consider the following exact sequences:

$$0 \longrightarrow M'^{\Phi_{p^l q}} \longrightarrow M' \longrightarrow M'_1 \longrightarrow 0$$

$$0 \longrightarrow M_1'^{\Phi_{p^{l-1} q}} \longrightarrow M'_1 \longrightarrow M'_2 \longrightarrow 0$$

....

....

$$0 \longrightarrow M_{l-1}'^{\Phi_{pq}} \longrightarrow M_{l-1}' \longrightarrow M_l' \longrightarrow 0$$
$$0 \longrightarrow M_l'^{\Phi_q} \longrightarrow M_l' \longrightarrow 0\,.$$

Here $M'^{\Phi_{p^lq}} = M^{\Phi_{p^lq}}$. Then, by (5.4), we have $M' \underset{(r)}{—} M^{\Phi_{p^lq}} \oplus M_1'$. Let Π_i' be the subgroup of Π of order p^i, for each $1 \leqq i \leqq l$. Then M_i' can be regarded as a Π/Π_i'-module. By virtue of (5.4), we have $M_i' \underset{(r)}{—} M_i'^{\Phi_{p^{l-i}q}} \oplus M_{i+1}'$. Hence we get

$$M' \underset{(r)}{—} M^{\Phi_{p^lq}} \oplus \sum_{i=1}^{l} \oplus M_i'^{\Phi_{p^{l-i}q}}\,.$$

Consequently we get

$$M \underset{(r)}{—} M' \oplus \sum_{j=1}^{l} \oplus \Psi_{p^j} M_j \underset{(r)}{—} M^{\Phi_{p^lq}} \oplus \sum_{i=1}^{l} \oplus M_i'^{\Phi_{p^{l-i}q}} \oplus \sum_{j=1}^{l} \oplus \Psi_{p^j} M_j\,.$$

This completes the proof of the first part. The second part of the proposition follows immediately from the first part and (2.6).

LEMMA 5.13. *Let Π be a cyclic group of order n and let m_1, m_2 be divisors of n such that $m_1 \nmid m_2$ and $m_2 \nmid m_1$. Then* $\mathrm{Ext}^1_{Z\Pi}(M_1, M_2) = 0$ *for any $Z[\zeta_{m_1}]$-module M_1 and any $Z[\zeta_{m_2}]$-module M_2.*

PROOF. From the fact that $(\Phi_{m_1}(T), \Phi_{m_2}(T)) = Z[T]$ this follows immediately.

PROPOSITION 5.14. *Let Π be a cyclic group of order $p_1p_2p_3$ where p_1, p_2, p_3 are distinct primes. Let M be a Π-module. Let $M' = \Phi_{p_1p_2p_3} M^{\Phi_{p_1p_2p_3}\Phi_{p_1p_2}\Phi_{p_1p_3}\Phi_{p_2p_3}}$ and $M'' = M^{\Psi_1}$. Then*

$$M \underset{(r)}{—} M^{\Phi_{p_1p_2p_3}} \oplus M'^{\Phi_{p_1p_2}} \oplus M'^{\Phi_{p_1p_3}} \oplus M'^{\Phi_{p_2p_3}} \oplus \Psi_{p_1} M'' \oplus \Psi_{p_2} M'' \oplus \Psi_{p_3} M''\,.$$

Especially M is a quasi-permutation Π-module if and only if $M^{\Phi_{p_1p_2p_3}}$ is $Z[\zeta_{p_1p_2p_3}]$-free, each $M'^{\Phi_{p_ip_j}}$ is $Z[\zeta_{p_ip_j}]$-free and each $\Psi_{p_i} M''$ is $Z[\zeta_{p_i}]$-free.

PROOF. We have only to prove the first part. Consider the exact sequence $0 \to M'' \to M \to \Psi_1 M \to 0$. Then $\Psi_1 M \cong Z^{(t)}$ for some $t \geqq 0$ and so $M \underset{(r)}{—} M''$. Now put $\Phi = \Phi_{p_1p_2p_3} \cdot \Phi_{p_1p_2} \cdot \Phi_{p_1p_3} \cdot \Phi_{p_2p_3}$. Then $M^{\Phi} = M''^{\Phi}$. Hence we have an exact sequence $0 \to M^{\Phi} \to M'' \to \Phi M'' \to 0$. Then $\Phi M''$ can be regarded as a $Z\Pi/(\Phi_{p_1}(\sigma)\Phi_{p_2}(\sigma)\Phi_{p_3}(\sigma))$-module where σ denotes a generator of Π. Therefore by (5.13) we can write $M'' = M_1 \oplus M_2 \oplus M_3$ where each M_i is a $Z[\zeta_{p_i}]$-module. Each M_i is clearly isomorphic to $\Psi_{p_i} M''$. Applying (5.11) repeatedly, we get

$$M'' \underset{(r)}{—} M^{\Phi} \oplus \Psi_{p_1} M'' \oplus \Psi_{p_2} M'' \oplus \Psi_{p_3} M''\,.$$

Next consider the exact sequence $0 \to M^{\Phi_{p_1p_2p_3}} \to M^{\Phi} \to M' \to 0$. According to (5.4), $M^{\Phi} \underset{(r)}{—} M^{\Phi_{p_1p_2p_3}} \oplus M'$. Since M' can be regarded as a $Z\Pi/(\Phi_{p_1p_2}(\sigma)\Phi_{p_1p_3}(\sigma)\Phi_{p_2p_3}(\sigma))$-module, again by (5.13) we can write $M' = M_{12} \oplus M_{13} \oplus M_{23}$ where each M_{ij} is a $Z[\zeta_{p_ip_j}]$-module. It is easily seen that $M_{ij} \cong M'^{\Phi_{p_ip_j}}$. Hence we have

$$M^{\Phi} \underset{(r)}{—} M^{\Phi_{p_1p_2p_3}} \oplus M'^{\Phi_{p_1p_2}} \oplus M'^{\Phi_{p_1p_3}} \oplus M'^{\Phi_{p_2p_3}}\,.$$

Thus we get

$$M \underset{(r)}{\text{---}} M^{\Phi_{p_1p_2p_3}} \oplus M'^{\Phi_{p_1p_2}} \oplus M'^{\Phi_{p_1p_3}} \oplus M'^{\Phi_{p_2p_3}} \oplus \Psi_{p_1}M'' \oplus \Psi_{p_2}M'' \oplus \Psi_{p_3}M'' .$$

COROLLARY 5.15. *Let Π be a cyclic group as in* (5.12) *or* (5.14) *and let Ω_Π be the maximal order in $Q\Pi$ containing $Z\Pi$. Then the abelian semigroup $T(\Pi)$ is a group isomorphic to $C(\Omega_\Pi)$.*

§ **6.** The following lemma is due to P. Samuel ([**9**]).

LEMMA 6.1. *Let k be an infinite field and let K_1, K_2 be extensions of k finitely generated over k. Suppose that there exist elements $x_1, x_2, \cdots, x_n$ which are algebraically independent over K_1 and K_2 such that $K_1(x_1, x_2, \cdots, x_n) = K_2(x_1, x_2, \cdots, x_n)$. Then K_1 is k-isomorphic to K_2.*

LEMMA 6.2. *Let Π be a cyclic p-group of order p^l and let K/k be a Galois extension with group Π. In case of $p \neq 2$, suppose that k is an infinite field. Then $K(Z[\zeta_{p^l}])^\Pi$ is rational over k.*

PROOF. For $l=1$ this has been proved in [2], (1.13). Hence we may suppose that $l \geqq 2$. Let $q = p^{l-1}$ and let σ be a generator of Π.

(i) Case of $p=2$. Take $b \in K$ such that $\sigma^q b \neq b$ and put $a = b/\sigma^q b$. Then $a \cdot \sigma^q a = 1$ and $\sigma^q a \neq a$. Now $K(Z[\zeta_{2^l}])$ is expressed as the rational function field $K(X_1, X_2, \cdots, X_q)$ with the action of Π such that $\sigma(X_i) = X_{i+1}$, $1 \leqq i \leqq q-1$ and $\sigma(X_q) = 1/X_1$. If we put $Y_1 = \dfrac{X_1 + a}{X_1 + \sigma^q a}$ and $Y_{i+1} = \sigma(Y_i)$, $1 \leqq i \leqq q-1$, then $K(Z[\zeta_{2^l}]) = K(Y_1, Y_2, \cdots, Y_q)$ and $\sigma(Y_q) = \dfrac{\sigma^q a}{a} Y_1$, and therefore Π acts semi-linearly on $\sum_{i=1}^{q} KY_i$. Thus, by (1.1), $K(Z[\zeta_{2^l}])^\Pi$ is rational over k.

(ii) Case of $p \neq 2$. Suppose that k is an infinite field. By the definition $K(Z\Pi)$ is the rational function field $K(X_1, X_2, \cdots, X_{p^l})$ with the action of Π such that $\sigma(X_i) = X_{i+1}$, $1 \leqq i \leqq p^l - 1$ and $\sigma(X_{p^l}) = X_1$. Then $K(Z[\zeta_{p^l}])$ can be identified with $K(X_{q+1}^{-1}X_1, X_{q+2}^{-1}X_2, \cdots, X_{p^l}^{-1}X_{q(p-1)})$ because $Z[\zeta_{p^l}] = (\sigma^q - 1)$, and we have $K(Z\Pi) = K(Z[\zeta_{p^l}])(X_1, X_2, \cdots, X_q)$. Let

$$Y = \frac{X_1 + X_{q+1} + X_{2q+1} + \cdots + X_{(p-1)q+1}}{X_1} .$$

Then $Y \in K(Z[\zeta_{p^l}])$. Further let $Z_1 = X_1 + X_{q+1} + X_{2q+1} + \cdots + X_{(p-1)q+1}$ and $Z_{i+1} = \sigma^i Z_1$, $1 \leqq i \leqq q-1$. Then $\sigma^q Z_1 = Z_1$ and $Z_1, Z_2, \cdots, Z_q$ are algebraically independent over $K(Z[\zeta_{p^l}])$ since $Z_i = \sigma^{i-1}(Y) \cdot X_i$, $1 \leqq i \leqq q$. Now put $V = \sum_{j=1}^{p^l} KX_j$ and $W = \sum_{i=1}^{q} KZ_i$. Then $W \subseteqq V$ and Π acts semi-linearly on V and W. By (1.1) there exist $U_1, U_2, \cdots, U_q \in K(W)$ which are invariant under Π such that $K(W) = K(U_1, U_2, \cdots, U_q)$. Then we have

$$K(Z\Pi)^\Pi = [K(Z[\zeta_{p^l}])(W)]^\Pi = K(Z[\zeta_{p^l}])^\Pi(U_1, U_2, \cdots, U_q) .$$

On the other hand we have an exact sequence:

$$0 \longrightarrow W \longrightarrow V \longrightarrow V/W \longrightarrow 0$$

of K-vector spaces. Applying (1.1) to this we can find $U'_1, U'_2, \cdots, U'_{q(p-1)} \in K(V)$ such that $K(Z\Pi)^{\Pi} = K(V)^{\Pi} = K(W)^{\Pi}(U'_1, U'_2, \cdots, U'_{q(p-1)})$. Therefore we get

$$K(Z[\zeta_{pl}])^{\Pi}(U_1, U_2, \cdots, U_q) = K(Z\Pi)^{\Pi}$$
$$= k(U'_1, U'_2, \cdots, U'_{q(p-1)})(U_1, U_2, \cdots, U_q).$$

Here $U_1, U_2, \cdots, U_q$ are algebraically independent over $K(Z[\zeta_{pl}])^{\Pi}$ and $k(U'_1, U'_2, \cdots, U'_{q(p-1)})$. Then, by virtue of (6.1), $K(Z[\zeta_{pl}])^{\Pi}$ is k-isomorphic to the rational function field $k(U'_1, U'_2, \cdots, U'_{q(p-1)})$. This completes the proof of the lemma.

THEOREM 6.3. *Let Π be a cyclic p-group and let K/k be a Galois extension with group Π. In case of $p \neq 2$ suppose that k is an infinite field. If M is a quasi-permutation Π-module, then $K(M)^{\Pi}/k$ is rational.*

PROOF. Let Π be a cyclic group of order p^l. As in (5.12) we put $M_0 = M$, $M_1 = M_0^{\Psi_1}$, $M_2 = M_1^{\Psi_p}, \cdots, M_l = M_{l-1}^{\Psi_{p^{l-1}}}$. Then we have the following exact sequences:

$$0 \longrightarrow M_1 \longrightarrow M_0 \longrightarrow \Psi_1 M_0 \longrightarrow 0$$
$$0 \longrightarrow M_2 \longrightarrow M_1 \longrightarrow \Psi_p M_1 \longrightarrow 0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$0 \longrightarrow M_l \longrightarrow M_{l-1} \longrightarrow \Psi_{p^{l-1}} M_{l-1} \longrightarrow 0$$
$$0 \longrightarrow M_l \longrightarrow \Psi_{p^l} M_l \longrightarrow 0.$$

By (5.2) we have $\Psi_{p^i} M_i \cong M^{\Phi_{p^i}}$ for each $0 \leqq i \leqq l$ and further, by (5.5), $M^{\Phi_{p^i}}$ is $Z[\zeta_{p^i}]$-free for each $0 \leqq i \leqq l$. Therefore, applying (5.10) to the above exact sequences repeatedly, we see that $K(M)$ is k-isomorphic to $K(M^{\Phi_{p^l}} \oplus M^{\Phi_{p^{l-1}}} \oplus \cdots \oplus M^{\Phi_1})$. Thus we can conclude by (6.2) that $K(M)^{\Pi}$ is rational over k.

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