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Semi-groups of operators in locally convex spaces

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In this paper we shall deal with the generation of not necessarily equicontinuous semi-groups of operators in locally convex spaces in a simple way which is different from that of T. Kōmura [2].

The theory of semi-groups has made progress after the fundamental works of E. Hille and K. Yosida. H. Komatsu [1], L. Schwartz [3], K. Yosida [5] and others extended the theory of semi-groups of operators in Banach spaces to locally convex spaces. They discussed it under the condition that semi-groups are equicontinuous. Their arguments are based on the following fact which plays an essential role in their theory.

For an equicontinuous semi-group $\{T_t;\ t\geqq 0\}$ *in a locally convex space* E, *the Laplace transform of* T_t *exists, and it is connected with its infinitesimal generator* A *in the following way:*

$$\int_0^\infty e^{-\lambda t}T_t x\,dt=(\lambda-A)^{-1}x\,, \tag{0.1}$$

for any $x\in E$ *and* $\mathrm{Re}\,\lambda>0$.

Without assumption of equicontinuity of a semi-group T_t, neither the Laplace transform of T_t nor the resolvents of its infinitesimal generator A ever exist.

In T. Kōmura [2], she dealt with semi-groups which are not necessarily equicontinuous but locally equicontinuous. To avoid the difficulty that the relation (0.1) does not necessarily hold, she introduced the notion of generalized resolvents. Generalized resolvents play an important role in her theory. To define generalized resolvents and to get their properties she used the theories of vector valued distributions and linear topological spaces attached to them and their related properties. Hence it seems for the author that the notion of generalized resolvents is not simple.

In the following of this paper we discuss the generation of semi-groups without the notion of generalized resolvents. Instead of it, we introduce the notion of asymptotic resolvents to complete our theory. Roughly speaking, asymptotic resolvents are almost resolvents, or they are parametrix of $(\lambda-A)$, where A is the infinitesimal generator of a semi-group T_t, modulo linear operators decaying with the exponential order of λ. Our notion of asymptotic resolvents is different from that in L. Waelbroeck [4].

The notion of asymptotic resolvents in this paper is simpler than that of generalized resolvents in T. Kōmura [2]. Our arguments are easy, and in construction of semi-groups we do not make use of the Laplace inversion transform.

In § 1 we state several properties of semi-groups in locally convex spaces.

In § 2 we are concerned with the generation of semi-groups. The definition of asymptotic resolvents is given and we give elementary properties of them. A complete characterization of locally equicontinuous semi-groups is stated in our way. It is easy to see that our characterization is a generalization of well-known Hille-Yosida's theorem in the theory of semi-groups.

In § 3 we give the exponential formula of semi-groups making use of asymptotic resolvents.

§ 1. Semi-groups of operators.

Let E be a locally convex sequentially complete linear topological space. We denote by $L(E)$ all of continuous linear operators in E. A family $\mathfrak{M}$ in $L(E)$ is said to be equicontinuous, if for any continuous semi-norm p there is a continuous semi-norm q such that $p(Tx) \leqq q(x)$ for any $T \in \mathfrak{M}$ and any $x \in E$. A family $\mathfrak{N} = \{U_\lambda \in L(E); \lambda > \omega\}$ is denoted by $U_\lambda x = O(\varphi(\lambda))x$, where $\varphi(\lambda)$ is a positive continuous function defined for $\lambda > \omega$, if for any continuous semi-norm p there is a continuous semi-norm q such that $p(U_\lambda x) \leqq \varphi(\lambda) q(x)$ for $\lambda > \omega$ and any $x \in E$.

Definition 1.1. A system $\{T_t; t \geqq 0\}$ in $L(E)$ is called a semi-group, if it satisfies the conditions:

$$T_t T_s = T_{t+s} \quad \text{for any } t, s \geqq 0, \tag{1.1}$$

$$T_0 = I \quad \text{(the identity operator)}, \tag{1.2}$$

$$\lim_{t \to s} T_t x = T_s x \quad \text{for any } s \geqq 0 \text{ and any } x \in E. \tag{1.3}$$

In particular, a semi-group is said to be locally equicontinuous, if for any fixed $s > 0$ the subsystem $\{T_t; 0 \leqq t \leqq s\}$ is equicontinuous, and it is said to be equicontinuous, if the system $\{T_t; 0 \leqq t < \infty\}$ is equicontinuous.

The infinitesimal generator A of a semi-group $\{T_t; t \geqq 0\}$ is defined by

$$Ax = \lim_{h \downarrow 0} \frac{1}{h}(T_h - I)x, \tag{1.4}$$

when the limit exists. The domain of A is denoted by $D(A)$.

Now we put for $a > 0$

$$R(\lambda)x=\int_0^a e^{-\lambda t}T_t x\,dt\,. \tag{1.5}$$

PROPOSITION 1.1. *Let E be a locally convex sequentially complete space and let $\{T_t\,;\ t\geqq 0\}$ be a locally equicontinuous semi-group in E. Then its infinitesimal generator A is a closed linear operator and the domain $D(A)$ is dense. $R(\lambda)$ defined by* (1.5) *maps E to $D(A)$ and we have*

$$(\lambda-A)R(\lambda)x=x-e^{-a\lambda}T_a x\,, \tag{1.6}$$

$$R(\lambda)R(\mu)x=R(\mu)R(\lambda)x \quad \textit{for any } x\in E\,, \tag{1.7}$$

$$R(\lambda)Ax=AR(\lambda)x \quad \textit{for any } x\in D(A)\,. \tag{1.8}$$

The proof of Proposition 1.1 is omitted. We refer the reader to H. Komatsu [1], T. Kōmura [2], L. Schwartz [3] and K. Yosida [5] for the proof.

REMARK 1.1. Several properties of semi-groups can be shown without the condition that the space E is sequentially complete, or without the condition that semi-groups are locally equicontinuous.

PROPOSITION 1.2. *Let $\{T_t\,;\ t\geqq 0\}$ be a locally equicontinuous semi-group in a locally convex sequentially complete space E. Then $R(\lambda)x$ defined by* (1.5) *is an E-valued holomorphic function of λ for any $x\in E$. $R(\lambda)$ belongs to $L(E)$ for any λ, and the family of operators*

$$\left\{\frac{\lambda^{n+1}}{n\,!}\frac{d^n}{d\lambda^n}R(\lambda)\,;\ \lambda>0 \textit{ and } n=0,1,2,\cdots\right\} \tag{1.9}$$

is equicontinuous.

PROOF. From (1.5) we have

$$\frac{\lambda^{n+1}}{n\,!}\frac{d^n}{d\lambda^n}R(\lambda)x=(-1)^n\lambda^{n+1}\int_0^a e^{-\lambda t}\frac{t^n}{n\,!}T_t x\,dt\,. \tag{1.10}$$

For any continuous semi-norm p there exists a continuous semi-norm q such that $p(T_t x)\leqq q(x)$ for $0\leqq t\leqq a$ and $x\in E$. Hence, for real positive λ

$$p\left(\lambda^{n+1}\int_0^a e^{-\lambda t}\frac{t^n}{n\,!}T_t x\,dt\right)\leqq q(x)\frac{\lambda^{n+1}}{n\,!}\int_0^a e^{-\lambda t}t^n\,dt\leqq q(x)\,,$$

which shows by (1.10) that the family of operators (1.9) is equicontinuous. Other statements of the proposition are obvious.

REMARK 1.2. Results similar to the statements of Propositions 1.1 and 1.2 hold, if we define $R(\lambda)$ suitably in the way different from (1.5). For example, set

$$R(\lambda)x=\int_0^\infty e^{-\lambda t}\varphi(t)T_t x\,dt\,, \tag{1.11}$$

where $\varphi(t)$ is smooth, $\varphi(t)=1$ $(t\leqq a)$, $\varphi(t)=0$ $(t\geqq 2a)$. Then except (1.6) in

Proposition 1.1 same results hold, and instead of (1.6) the relation

(1.12) $$(\lambda-A)R(\lambda)x=x+S(\lambda)x$$

holds, where $S(\lambda)x=\int_0^\infty e^{-\lambda t}\varphi'(t)T_t x\,dt$.

§ 2. Generation of semi-groups.

The purpose of this section is to construct a locally equicontinuous semi-group T_t with a given generator A in a sequentially complete space E.

DEFINITION 2.1. A family $\{R(\lambda);\lambda>\omega\}$ $(\omega>0)$ in $L(E)$ is called an asymptotic resolvent of a closed linear operator A with the domain $D(A)$, if it satisfies conditions:

(2.1) $R(\lambda)x$ is an infinitely differentiable function of λ in $\Sigma=\{\lambda;\ \lambda>\omega\}$ for any $x\in E$ and $R(\lambda)$ maps E to $D(A)$.

(2.2) $AR(\lambda)=R(\lambda)A$ on $D(A)$ and $R(\lambda)R(\mu)=R(\mu)R(\lambda)$ for $\lambda,\ \mu\in\Sigma$.

(2.3) $(\lambda-A)R(\lambda)=I+S(\lambda)$, where $S(\lambda)\in L(E)$ and it satisfies the following condition: For any $x\in E$ $S(\lambda)x$ is infinitely differentiable in λ, and for any continuous semi-norm p there is a continuous semi-norm q such that for a constant a

$$p((d^k/d\lambda^k)S(\lambda)x)\leqq a^k e^{-a\lambda}q(x),\qquad \text{for any } x\in E,\ \lambda>\omega,\ \text{and } k=0,1,2,\cdots.$$

We now attain to the following main theorem which is a generalization of the Hille-Yosida's theorem that gives a criterion for generation of semi-groups. In it the notion of asymptotic resolvents in Definition 2.1 is useful.

THEOREM 2.1. *Let E be a sequentially complete locally convex space. Then a linear operator A is the infinitesimal generator of a locally equicontinuous semi-group, if and only if it satisfies the following conditions:*

(2.4) *A is a closed linear operator with a dense domain $D(A)$.*

(2.5) *There is an asymptotic resolvent $R(\lambda)$ of A such that the family of operators*

$$\left\{\frac{\lambda^{n+1}}{n!}\frac{d^n}{d\lambda^n}R(\lambda);\ \lambda>\omega \text{ and } n=0,1,2,\cdots\right\}$$

is equicontinuous.

The necessity part follows from Propositions 1.1 and 1.2. In the remainder of this section we shall prove the sufficient condition of Theorem 2.1. In the following we shall make use of the notations $R^{(n)}(\lambda)$ and $S^{(n)}(\lambda)$ which denote $(d^n/d\lambda^n)R(\lambda)$ and $(d^n/d\lambda^n)S(\lambda)$ respectively.

Let A satisfy the conditions (2.4) and (2.5). Then we have

LEMMA 2.1. (i) *For any* $x \in E$, *we have* $\lim_{\lambda\to\infty} \lambda R(\lambda)x = x$.

(ii) *For any* $x \in D(A)$, *set* $A_\lambda x = \{-\lambda+\lambda^2 R(\lambda)\} x$. *Then we have* $\lim_{\lambda\to\infty} A_\lambda x = Ax$.

(iii) *The family of operators*

$$\left\{\frac{\lambda^{k+2}\exp(a\lambda)}{(a\lambda)^{k+1}+(k+1)!}(R^{(k+1)}(\lambda)+(k+1)R(\lambda)R^{(k)}(\lambda))\,;\ \lambda>\omega,\ k=0, 1, 2, \cdots\right\} \tag{2.6}$$

is equicontinuous.

PROOF. (i) For $x \in D(A)$, from (2.2) and (2.3) we have $\lambda R(\lambda)x = x + R(\lambda)Ax + O(\exp(-a\lambda))x$. Hence we have $\lim_{\lambda\to\infty} \lambda R(\lambda)x = x$ for $x \in D(A)$. Since the family of operators $\{\lambda R(\lambda)\,;\ \lambda>\omega\}$ is equicontinuous and $D(A)$ is dense, we get $\lim_{\lambda\to\infty} \lambda R(\lambda)x = x$ for any $x \in E$ with the aid of the Banach-Steinhaus theorem.

(ii) From (2.2) and (2.3) we have for $x \in D(A)$ $A_\lambda x = (-\lambda+\lambda^2 R(\lambda))x = \lambda R(\lambda)Ax + O(\lambda\exp(-a\lambda))x$. Thus we have $\lim_{\lambda\to\infty} A_\lambda x = Ax$ by (i) of this lemma.

(iii) Differentiate (2.3) $k+1$ times in λ. We have

$$(\lambda-A)R^{(k+1)}(\lambda)x + (k+1)R^{(k)}(\lambda)x = S^{(k+1)}(\lambda)x\,, \tag{2.7}$$

$$R(\lambda)(\lambda-A)R^{(k+1)}(\lambda)x + (k+1)R(\lambda)R^{(k)}(\lambda)x = R(\lambda)S^{(k+1)}(\lambda)x\,. \tag{2.8}$$

From (2.3) we obtain

$$R^{(k+1)}(\lambda)x + (k+1)R(\lambda)R^{(k)}(\lambda)x = R(\lambda)S^{(k+1)}(\lambda)x - S(\lambda)R^{(k+1)}(\lambda)x\,. \tag{2.9}$$

Therefore, after simple calculations we can show the equicontinuity of the family of operators (2.6) from (2.5) and the condition on $S(\lambda)$ in (2.3). q. e. d.

Now set for $x \in E$, $\lambda>\omega$ and $0 \leqq t \leqq a/4$

$$T_t(\lambda)x = \exp(-\lambda t)\left\{I + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\frac{(\lambda^2 t)^{k+1}}{(k+1)!}R^{(k)}(\lambda)\right\}x\,. \tag{2.10}$$

Under the condition (2.5) the power series of (2.10) converges and the family of operators

$$\{T_t(\lambda)\,;\ \lambda>\omega \text{ and } 0 \leqq t \leqq a/4\} \tag{2.11}$$

is equicontinuous.

LEMMA 2.2. $T_t(\lambda)x$ *defined by* (2.10) *converges uniformly on* $[0, a/4]$ *when* λ *tends to infinity.*

PROOF. $T_t(\lambda)x$ is differentiable in t:

$$\begin{aligned}\frac{d}{dt}T_t(\lambda)x &= -\lambda T_t(\lambda)x + \lambda^2\exp(-\lambda t)\sum_{k=0}^{\infty}\frac{(-1)^k(\lambda^2 t)^k}{k!\;k!}R^{(k)}(\lambda)x \\ &= -\lambda T_t(\lambda)x + \lambda^2\exp(-\lambda t)\Big\{R(\lambda) \\ &\qquad + \sum_{k=0}^{\infty}\frac{(-\lambda^2 t)^{k+1}}{(k+1)!\,(k+1)!}R^{(k+1)}(\lambda)\Big\}x\,.\end{aligned} \tag{2.12}$$

On writing $S_k(\lambda)x=\{R^{(k+1)}(\lambda)+(k+1)R(\lambda)R^{(k)}(\lambda)\}x$, we have

$$\frac{d}{dt}T_t(\lambda)=-\lambda T_t(\lambda)x+\lambda^2\exp(-\lambda t)\Big\{R(\lambda)+\sum_{k=0}^{\infty}\frac{(-1)^k(\lambda^2t)^{k+1}}{k!\,(k+1)!}R(\lambda)R^{(k)}(\lambda)\Big\}x+\lambda^2\exp(-\lambda t)\sum_{k=0}^{\infty}\frac{(-\lambda^2t)^{k+1}}{(k+1)!\,(k+1)!}S_k(\lambda)x\,. \tag{2.13}$$

We shall estimate the last term of (2.13). Let p be an arbitrary continuous semi-norm. Then there is a continuous semi-norm q such that

$$\begin{aligned}&p\Big(\lambda^2\exp(-\lambda t)\sum_{k=0}^{\infty}\frac{(-\lambda^2t)^{k+1}}{(k+1)!\,(k+1)!}S_k(\lambda)x\Big)\\ &\leqq\lambda^2\exp(-\lambda t)\sum_{k=0}^{\infty}\frac{(\lambda^2t)^{k+1}}{(k+1)!\,(k+1)!}p(S_k(\lambda)x)\\ &\leqq\lambda\exp(-\lambda(t+a))\Big\{\sum_{k=0}^{\infty}\frac{(\lambda^2at)^{k+1}}{(k+1)!\,(k+1)!}+\sum_{k=0}^{\infty}\frac{(\lambda t)^{k+1}}{(k+1)!}\Big\}q(x)\\ &\qquad\text{(by Lemma 2.1 (iii))}\\ &\leqq\lambda\{\exp(-\lambda(\sqrt{t}-\sqrt{a})^2)+\exp(-\lambda a)\}q(x)\\ &\leqq2\lambda\exp(-\lambda(\sqrt{t}-\sqrt{a})^2)q(x)\,.\end{aligned} \tag{2.14}$$

Thus we conclude that the last term of (2.13) decays rapidly.

Set $A_\lambda x=\{-\lambda+\lambda^2R(\lambda)\}x$. We have, from (2.13) and (2.14),

$$(d/dt)T_t(\lambda)x=A_\lambda T_t(\lambda)x+O(\lambda\exp(-\lambda(a/4)))x \tag{2.15}$$

for $0\leqq t\leqq a/4$.

Now let us note $T_0(\lambda)x=x$. For $0\leqq t\leqq a/4$ and $x\in D(A)$ we have

$$\begin{aligned}T_t(\lambda)x-T_t(\mu)x&=\int_0^t(d/ds)T_s(\lambda)T_{t-s}(\mu)x\,ds\\ &=\int_0^tT_s(\lambda)(A_\lambda-A_\mu)T_{t-s}(\mu)x\,ds\\ &\quad+O(\lambda\exp(-\lambda(a/4)))x+O(\mu\exp(-\mu(a/4)))x\,.\end{aligned} \tag{2.16}$$

Since $A_\lambda T_t(\mu)=T_t(\mu)A_\lambda$ obviously in view of the construction of $T_t(\mu)$ and (2.2), we obtain

$$T_t(\lambda)x-T_t(\mu)x=\int_0^tT_s(\lambda)T_{t-s}(\mu)(A_\lambda-A_\mu)x\,ds+S(\lambda,\mu)x\,, \tag{2.17}$$

where $S(\lambda,\mu)x\to0$ when λ and $\mu\to\infty$. Hence for an arbitrary continuous semi-norm p we have a continuous semi-norm q such that

$$p(T_t(\lambda)x-T_t(\mu)x)\leqq q(A_\lambda x-A_\mu x)+q(S(\lambda,\mu)x)\,. \tag{2.18}$$

The above inequality follows from the equicontinuity of the family of operators (2.11). For $x \in D(A)$ $A_\lambda x \to Ax$ because of Lemma 2.1 (ii). Therefore $T_t(\lambda)x$ converges when $\lambda \to \infty$. Convergence of $T_t(\lambda)x$ for an arbitrary $x \in E$ is shown easily by the Banach-Steinhaus theorem, for the family of operators (2.11) is equicontinuous. q. e. d.

Define

$$T_t x = \lim_{\lambda \to \infty} T_t(\lambda)x, \qquad \text{for } x \in E \text{ and } 0 \leqq t \leqq a/4. \tag{2.19}$$

It follows easily from the proof of Lemma 2.2 that $T_t x$ is continuous and $T_0 x = x$. We shall show in the following that T_t defined by (2.19) can be extended to the interval $[0, \infty)$, and it is the desired locally equicontinuous semi-group with the infinitesimal generator A.

LEMMA 2.3. *Let T_t be the operator defined by* (2.19).

(i) *For $x \in D(A)$, $T_t x$ is continuously differentiable in t and*

$$(d/dt)T_t x = AT_t x = T_t Ax \tag{2.20}$$

holds in $t \in [0, a/4]$.

(ii) *For $t \geqq 0$, $s \geqq 0$ and $t+s \leqq a/4$, we have the semi-group property of T_t:*

$$T_t T_s = T_{t+s}. \tag{2.21}$$

PROOF. In order to prove (i) we note the relation

$$AT_t(\lambda)x = T_t(\lambda)Ax \qquad \text{for } x \in D(A). \tag{2.22}$$

This relation is easily proved with the aid of the commutative relation (2.2) and closedness of A. Letting $\lambda \to \infty$ in (2.22), since A is closed, we obtain

$$AT_t x = T_t Ax. \tag{2.23}$$

On the other hand $T_t(\lambda)A_\lambda x = A_\lambda T_t(\lambda)x$ converges to $T_t Ax$ for $x \in D(A)$ when λ tends to infinity. Therefore, from (2.15) $(d/dt)T_t(\lambda)x$ converges to $(d/dt)T_t x$ and (2.20) is true.

Next we show that $\{T_t ; 0 \leqq t \leqq a/4\}$ has the semi-group property. Under the condition (ii) for t and s, for any $x \in D(A)$ $T_{t-u}T_{s+u}x$ is continuously differentiable in u, for $\{T_t ; 0 \leqq t \leqq a/4\}$ is equicontinuous and $T_t x$ is continuously differentiable for $x \in D(A)$. So we have

$$T_{t+s}x - T_t T_s x = \int_0^t \frac{d}{du}(T_{t-u}T_{s+u}x)du = \int_0^t T_{t-u}T_{s+u}(A-A)xdu = 0. \tag{2.24}$$

As $D(A)$ is dense, we have $T_{t+s}x = T_t T_s x$ for all $x \in E$. q. e. d.

Making use of the semi-group property shown above, we can extend T_t for $t \geqq a/4$. This extension is again denoted by T_t. Thus we get a locally equicontinuous semi-group $\{T_t ; t \geqq 0\}$.

Next we show that the infinitesimal generator of $\{T_t;\ t\geqq 0\}$ constructed above is precisely A. Let A_1 be the infinitesimal generator of T_t. Then, by Lemma 2.3, we have $A\subset A_1$, that is, if $x\in D(A)$, then we have $x\in D(A_1)$ and $Ax=A_1x$. We shall show $A\supset A_1$. Since $R(\lambda)$ maps E to $D(A)$, we see

$$A_1R(\lambda)x=AR(\lambda)x \qquad \text{for any } x\in E. \tag{2.25}$$

Since

$$((T_h-I)/h)R(\lambda)x=R(\lambda)((T_h-I)/h)x,$$

letting $h\to+0$, we have for $x\in D(A_1)$

$$A_1R(\lambda)x=R(\lambda)A_1x. \tag{2.26}$$

Therefore, for $x\in D(A_1)$ we obtain

$$\lambda R(\lambda)A_1x=A_1\lambda R(\lambda)x=A\lambda R(\lambda)x. \tag{2.27}$$

Letting $\lambda\to\infty$ in (2.27), the left side of (2.27) converges to A_1x and the right converges to Ax because of closedness of A and the convergence of $\lambda R(\lambda)x$ to x. Thus we get $A=A_1$.

Finally we shall show that the locally equicontinuous semi-group T_t is uniquely determined by A. Let $\{S_t;\ t\geqq 0\}$ be a locally equicontinuous semi-group and A be its generator. Then for $x\in D(A)$

$$T_tx-S_tx=\int_0^t\frac{d}{ds}S_{t-s}T_sx\,ds=\int_0^t S_{t-s}T_s(A-A)x\,ds=0.$$

Hence $S_t=T_t$.

Thus the proof of Theorem 2.1 is complete.

REMARK 2.1. For an equicontinuous semi-group T_t with the infinitesimal generator A, the resolvent of A $R(\lambda)$ exists and it is represented by the formula

$$R(\lambda)x=\int_0^\infty e^{-\lambda t}T_tx\,dt \qquad \text{for } \operatorname{Re}\lambda>0. \tag{2.28}$$

Obviously the resolvent $R(\lambda)$ is an asymptotic resolvent of A and $S(\lambda)$ in Definition 2.1 is identically zero.

REMARK 2.2. In Definition 2.1 we can relax the condition on the estimate of derivatives of $S(\lambda)$ in the following way:

For any continuous semi-norm p there is a continuous semi-norm q such that for any $x\in E$, $\lambda>\omega$ and $k=0, 1, 2, \cdots$

$$p((d^k/d\lambda^k)S(\lambda)x)\leqq b^ke^{-a\lambda}q(x), \tag{2.29}$$

where $b\geqq a>0$.

For example, $S(\lambda)$ which corresponds to $R(\lambda)$ defined by (1.11) in Remark 1.2 satisfies this condition. It is easy to check that semi-groups can be constructured under this condition (2.29) analogously.

REMARK 2.3. From the proof of Theorem 2.1 we find that a semi-group can be constructed, if an asymptotic resolvent exists on a positive unbounded set.

DEFINITION 2.2. A family $\{R(\lambda);\ \lambda\in\Sigma\}$ in $L(E)$ is called a sequentially asymptotic resolvent of a closed linear operator A with the domain $D(A)$, if it satisfies conditions:

(2.30) For any $x\in E$ $R(\lambda)x$ is an infinitely differentiable function of λ in an open set Σ in R^1, and $R(\lambda)$ maps E to $D(A)$.

(2.31) $$AR(\lambda)=R(\lambda)A \text{ on } D(A) \quad\text{and}\quad R(\lambda)R(\mu)=R(\mu)R(\lambda)$$

for $\lambda,\ \mu\in\Sigma$.

(2.32) $$(\lambda-A)R(\lambda)=I+S(\lambda) \qquad \text{for } \lambda\in\Sigma,$$

where $S(\lambda)\in L(E)$ and is infinitely differentiable in λ.

(2.33) There is a sequence $\Lambda=\{\lambda_i\}_{i=1}^{\infty}\subset\Sigma$ such that $\lim_{i\to\infty}\lambda_i=\infty$, and for any continuous semi-norm p there exists a semi-norm q such that for $\lambda_i\in\Lambda$, any $x\in E$ and $k=0, 1, 2, \cdots$

$$p((d^k/d\lambda^k)S(\lambda)x)|_{\lambda=\lambda_i}\leqq b^k e^{-a\lambda_i}q(x), \quad \text{where } b\geqq a>0.$$

Making use of Definition 2.2 we have

THEOREM 2.2. *Let E be a sequentially complete locally convex space. Then a linear operator A is the infinitesimal generator of a locally equicontinuous semi-group, if and only if it satisfies the conditions:*

(2.34) *A is a closed linear operator with a dense domain $D(A)$.*

(2.35) *There is a sequentially asymptotic resolvent $\{R(\lambda);\ \lambda\in\Sigma\}$ of A such that the family of operators*

$$\{[(\lambda^{n+1}/n!)(d^n/d\lambda^n)R(\lambda)]_{\lambda=\lambda_i};\ \lambda_i\in\Lambda \text{ and } n=0, 1, 2, \cdots\}$$

is equicontinuous, where Λ is that in Definition 2.2.

§ 3. Exponential formula of semi-groups.

In this section we shall show the exponential formula of semi-groups. This formula is represented by means of asymptotic resolvents. Before we state the exponential formula, we give a lemma.

LEMMA 3.1. *Suppose that a linear operator A satisfies the conditions (2.4) and (2.5). Then for $n=0, 1, 2, \cdots$ we have*

(3.1) $$\lim_{\lambda\to\infty}\frac{(-1)^n\lambda^{n+1}}{n!}R^{(n)}(\lambda)x=x \qquad \text{for any } x\in E.$$

PROOF. We prove (3.1) by induction of n. In the case of $n=0$, by Lemma 2.1 (i), we have (3.1). Assume that (3.1) is true for $n \leqq k$. Lemma 2.1 (iii) implies that

$$\frac{(-1)^{k+1}\lambda^{k+2}}{(k+1)!} R^{(k+1)}(\lambda)x = (-1)^k \frac{\lambda^{k+1}}{k!} R^{(k)}(\lambda)\lambda R(\lambda)x + O(\lambda^{k+1}\exp(-a\lambda))x. \tag{3.2}$$

Noting the condition (2.5) of the equicontinuity, we can conclude that (3.1) is true from the assumption. q. e. d.

Let A satisfy the conditions (2.4) and (2.5). Now we set for $x \in E$

$$S_n(t)x = \frac{(-1)^{n-1}}{(n-1)!}(n/t)^n R^{(n-1)}(n/t)x \qquad \text{for } 0<t<a, \tag{3.3}$$

$$S_n(0)x = x.$$

$S_n(t)x$ is a continuous function of t in $0 \leqq t < a$, because of Lemma 3.1. Since the linear operator A satisfies the conditions (2.4) and (2.5), the family of operators $\{S_n(t)\,;\ 0 \leqq t < a,\ n > a\omega\}$ is equicontinuous.

THEOREM 3.1. *Suppose that a linear operator A in a locally convex sequentially complete space E satisfies the conditions* (2.4) *and* (2.5). *Then we have the following formula, so called the exponential formula, of the semi-group $\{T_t\,;\ t \geqq 0\}$ which is uniquely determined by A,*

$$T_t x = \lim_{n\to\infty} S_n(t)x \qquad \text{for any } x \in E \text{ and } 0 \leqq t < a. \tag{3.4}$$

This convergence is uniform with respect to t on any compact interval in $[0, a)$.

To prove this theorem, we show a lemma which gives the estimate of the derivative of $S_n(t)x$.

LEMMA 3.2. *For $x \in E$, we have*

$$(d/dt)S_n(t)x = \{-(n/t)+(n/t)^2 R(n/t)\}S_n(t)x + V_n(t)x, \tag{3.5}$$

where $V_n(t)x$ converges to zero uniformly in t on any compact interval in $[0, a)$ when $n \to \infty$.

PROOF. The derivative of $S_n(t)x$ is given by

$$\frac{d}{dt}S_n(t)x = -\frac{n}{t}S_n(t)x + \frac{(-1)^n}{(n-1)!}\left(\frac{n}{t}\right)^n \frac{n}{t^2} R^{(n)}\left(\frac{n}{t}\right)x. \tag{3.6}$$

Now we apply Lemma 2.1 (iii) to (3.6), then we obtain

$$\frac{d}{dt}S_n(t)x = -\frac{n}{t}S_n(t)x + \frac{(-1)^{n-1}}{(n-1)!}\left(\frac{n}{t}\right)^{n+2} R\left(\frac{n}{t}\right)R^{(n-1)}\left(\frac{n}{t}\right)x + \frac{1}{n!}\left(\frac{n}{t}\right)^{n+2}G_n(t)x = \left\{-\frac{n}{t}+\left(\frac{n}{t}\right)^2 R\left(\frac{n}{t}\right)\right\}S_n(t)x + \frac{1}{n!}\left(\frac{n}{t}\right)^{n+2}G_n(t)x, \tag{3.7}$$

where $G_n(t)$ has the property:

For any continuous semi-norm p there is a continuous semi-norm q such that

$$p(G_n(t)x) \leqq \exp\left(-a\frac{n}{t}\right)\left(\frac{t}{n}a^n+\left(\frac{t}{n}\right)^{n+1}n!\right)q(x). \tag{3.8}$$

Set

$$V_n(t)x=\frac{1}{n!}\left(\frac{n}{t}\right)^{n+2}G_n(t)x. \tag{3.9}$$

We shall prove that $V_n(t)$ defined by (3.9) has the desired property stated in the lemma. Let p be any continuous semi-norm. Then there is a continuous semi-norm q such that the following estimate holds:

$$\begin{aligned} p(V_n(t)x) &\leqq \frac{1}{n!}\left(\frac{n}{t}\right)^{n+2}\left\{\left(\frac{t}{n}\right)^{n+1}n!+\frac{a^n t}{n}\right\}\exp\left(-a\frac{n}{t}\right)q(x) \\ &\leqq K\left\{\frac{n}{t}+\frac{1}{n!}\left(\frac{na}{t}\right)^{n+1}\right\}\exp\left(-a\frac{n}{t}\right)q(x). \end{aligned} \tag{3.10}$$

Noting the Stirling's formula $n!\sim n^n e^{-n}\sqrt{2\pi n}$, we have

$$\begin{aligned} p(V_n(t)x) &\leqq K'\left(\frac{n}{t}+e^n\sqrt{n}\left(\frac{a}{t}\right)^{n+1}\right)\exp\left(-a\frac{n}{t}\right)q(x) \\ &=K'\left(\frac{n}{t}\exp\left(-a\frac{n}{t}\right)+\sqrt{n}\left(\frac{a}{t}\right)^{n+1}\exp\left(n\left(1-\frac{a}{t}\right)\right)\right)q(x). \end{aligned} \tag{3.11}$$

Let us note that the function $f(x)=x^{n+1}e^{-nx}$ decreases on $[1+1/n, \infty)$.

If $0\leqq t\leqq t_0<a$, here t_0 is a fixed positive number, then we have

$$\left(\frac{a}{t}\right)^{n+1}\exp\left(-n\frac{a}{t}\right)\leqq\left(\frac{a}{t_0}\right)^{n+1}\exp\left(-n\frac{a}{t_0}\right) \tag{3.12}$$

for n satisfying $a/t_0\geqq 1+1/n$, because $a/t\geqq a/t_0\geqq 1+1/n$. Set $\alpha=a/t_0>1$, from (3.11) we have for large n

$$p(V_n(t)x)\leqq K''(\sqrt{n}\alpha^{n+1}e^{(1-\alpha)n}+n\alpha e^{-\alpha n})q(x). \tag{3.13}$$

Since $\alpha e^{1-\alpha}<1$, (3.13) implies that $V_n(t)x$ converges to zero uniformly in t on any compact interval in $[0, a)$, when $n\to\infty$. q. e. d.

PROOF OF THEOREM 3.1. For $x\in D(A)$, we have

$$\begin{aligned} S_n(t)x-T_tx &=\int_0^t\frac{d}{ds}S_n(s)T_{t-s}x\,ds \\ &=\int_0^t S_n(s)T_{t-s}\left\{-\left(\frac{n}{s}\right)+\left(\frac{n}{s}\right)^2R\left(\frac{n}{s}\right)-A\right\}x\,ds+U_n(t)x, \end{aligned} \tag{3.14}$$

where $U_n(t)x$ tends to zero when $n\to\infty$, by Lemma 3.2. If $0\leqq s\leqq t<a$, then $n/s\geqq n/t>n/a$. Hence $\lim_{n\to\infty}(-(n/s)+(n/s)^2R(n/s))x=Ax$ uniformly in $0\leqq s\leqq t$ by Lemma 2.1 (ii). Letting $n\to\infty$ in (3.14), $S_n(t)x$ converges to T_tx. For any

$x \in E$ convergence of $S_n(t)x$ to $T_t x$ is proved by means of equicontinuity of the family of operators $\{S_n(t);\ 0 \leqq t < a,\ n > a\omega\}$.

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