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An extended relativization theorem

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Let L be the first order infinitary logic $L_{\omega_1\omega}$ (cf. Feferman [2]) without individual constants or function symbols, V an unary predicate symbol not in L, and $L(V)$ the first order logic obtained from L by adding V as a new predicate symbol. For any formula A in L, we denote by A^V, the formula in $L(V)$ obtained from A by relativizing every quantifier in A by V. Then, we have

RELATIVIZATION THEOREM. *Let A, B be any sentences in L. If* $(\exists v)V(v) \underset{L(V)}{\vdash} A^V \supset B$, *then there is an existential sentence C such that* $\underset{L}{\vdash} A \supset C$ *and* $\underset{L}{\vdash} C \supset B$. (Cf. Motohashi [5].)

This theorem is a syntactical counterpart of Łos-Tarski's theorem on extensions in model theory.

In this paper, we shall extend this theorem to a form which can be considered as a syntactical counterpart of Chang-Łos' theorem on ω-ascending unions (cf. Chang [1]), Keisler's theorem on unions (cf. Keisler [3]) and Nebres' theorem on unions (cf. Nebres [8]). To accomplish this purpose we use a binary predicate symbol $U(*, *)$ instead of the unary predicate symbol $V(*)$ and a two sorted first order logic $L(U)$, which has two kinds of (individual) variables such as $x, y, \cdots$ (free variables of type 1), $u, v, \cdots$ (bound variables of type 1), $\alpha, \beta, \cdots$ (free variables of type 2) and $\xi, \eta, \cdots$ (bound variables of type 2). We assume that every variables in L is of type 1 and every atomic formula in $L(U)$ has one of the forms: $U(\alpha, x)$ or $P(x_1, \cdots, x_n)$, where P is a predicate symbol in L. For any free variable α of type 2 and any formula A in L, A^α is the formula in $L(U)$ obtained from A by relativizing every quantifier in A by $U(\alpha, *)$. Then we have

MAIN THEOREM. *Let k be a non negative integer and A, B sentences in L. If* $(\forall\xi)(\exists u)U(\xi, u), (\forall u_1)\cdots(\forall u_k)(\exists\xi)(U(\xi, u_1)\wedge\cdots\wedge U(\xi, u_k)) \underset{L(U)}{\vdash} (\forall\xi)A^\xi \supset B$, *then there is a sentence C of the form* $C = (\forall u_1)\cdots(\forall u_k)E(u_1, \cdots, u_k)$ *such that* $\underset{L}{\vdash} A \supset C$ *and* $\underset{L}{\vdash} C \supset B$, *where $E(x_1, \cdots, x_k)$ is an existential formula in L.*

This theorem implies the relativization theorem as a special case $k=0$. For simplicity we shall prove this theorem in the case $k=1$, L is finitary and has no equality symbol: There is no difficulty to prove it in general case by the same method.

PROOF OF MAIN THEOREM. First of all, we assume that every formula in L or $L(U)$ is in the negation normal form, i.e. the negation symbol $\neg$ is only applied to atomic formulas, and L, $L(U)$ are formulated in Gentzen style with the following modifications: (1) A sequent in L ($L(U)$) is of the form $\Gamma \to \Theta$, where Γ, Θ are *finite sets* of formulas in L ($L(U)$), although we shall use $\Gamma \to \Theta, A$ instead of $\Gamma \to \Theta \cup \{A\}$, etc. (2) $L(U)$ has no structural inference rules. (3) $L(U)$ has no inference rules with respect to the negation symbol $\neg$. (4) A sequent $\Gamma \to \Theta$ in $L(U)$ is an axiom sequent if for some atomic formula A, we have $\{A, \neg A\} \subseteqq \Gamma$ or $\{A, \neg A\} \subseteqq \Theta$ or $A \in \Gamma \cap \Theta$ or $\neg A \in \Gamma \cap \Theta$. It is obvious that these modifications do not alter the set of provable formulas.

Suppose $(\forall \xi)(\exists u)U(\xi, u), (\forall u)(\exists \xi)U(\xi, u) \vdash_{L(U)} (\forall \xi)A^{\xi} \supset B$. Then there is a proof figure P_0 in $L(U)$ whose end sequent is $(\forall \xi)(\exists u)U(\xi, u)$, $(\forall u)(\exists \xi)U(\xi, u)$, $(\forall \xi)A^{\xi} \to B$, indicated by the figure:

$$\begin{array}{c} \Downarrow P_0 \\ (\forall \xi)(\exists u)U(\xi, u),\ (\forall u)(\exists \xi)U(\xi, u),\ (\forall \xi)A^{\xi} \longrightarrow B\,. \end{array}$$

Notice that every formula in the succedent of any sequent in P_0 is a formula in L. Now we shall transform P_0 into more simple proof figures with the same end-sequent as follows. By the normal derivation theorem in Motohashi [4], we can transform P_0 into P_1 such that every inference rule with respect to quantification for the antecedent of any sequent in P_1 is one of the following forms:

$$(U)_1 \frac{U(\alpha, x),\ \Gamma \longrightarrow \Theta}{(\forall \xi)(\exists u)U(\xi, u),\ \Gamma \longrightarrow \Theta}\,,$$ where x does not appear in the lower sequent.

$$(U)_2 \frac{U(\alpha, x),\ \Gamma \longrightarrow \Theta}{(\forall u)(\exists \xi)U(\xi, u),\ \Gamma \longrightarrow \Theta}\,,$$ where α does not appear in the lower sequent.

$$(\forall \to)_0 \frac{A^{\alpha},\ \Gamma \longrightarrow \Theta}{(\forall \xi)A^{\xi},\ \Gamma \longrightarrow \Theta}\,.$$

$$(\forall \to)_1^U \frac{\neg U(\alpha, x),\ \Gamma \longrightarrow \Theta \qquad F^{\alpha}(x),\ \Gamma \longrightarrow \Theta}{((\forall v)F(v))^{\alpha},\ \Gamma \longrightarrow \Theta}\,.$$

$$(\exists \to)^U \frac{U(\alpha, x),\ F^{\alpha}(x),\ \Gamma \longrightarrow \Theta}{((\exists v)F(v))^{\alpha},\ \Gamma \longrightarrow \Theta}\,,$$ where x does not appear in the lower sequent.

Observe that this proof figure P_1 has the following property (*) (cf. Motohashi [7]):

(*) If $\dfrac{\quad\downarrow\quad}{\neg U(\alpha, x),\ \Gamma \longrightarrow \Theta}$ is a subproof figure of P_1,

then $\underset{L(U)}{\vdash} \Gamma \to \Theta$ or $U(\alpha, x) \in \Gamma$.

Hence we can replace the inference rule $(\forall\to)_1^U$ in P_1 by the following inference rule $(\forall\to)_2^U$ and transform P_1 into P_2.

$$(\forall\to)_2^U \ \frac{U(\alpha, x),\ F^\alpha(x),\ \Gamma \longrightarrow \Theta}{U(\alpha, x),\ ((\forall v)F(v))^\alpha,\ \Gamma \longrightarrow \Theta}.$$

Then there is no occurrence of $\neg U(\alpha, x)$ in the antecedent of any sequent in P_2.

Next we introduce the following notion to express an important property of the proof figure P_2.

For any set Γ of formulas in L and any free variable α of type 2, let $\Gamma^\alpha = \{F^\alpha : F \in \Gamma\}$. If $\vec{x} = \langle x_1, \cdots, x_n\rangle$, then $U(\alpha, \vec{x})$ means $\{U(\alpha, x_1), \cdots, U(\alpha, x_n)\}$. Occasionally we identify $\vec{x}$ as $\{x_1, \cdots, x_n\}$.

A $2n$-tuple $\Delta = (U(\alpha_1, \vec{x}_1), \cdots, U(\alpha_n, \vec{x}_n), \Gamma_1^{\alpha_1}, \cdots, \Gamma_n^{\alpha_n})$ of sets of formulas is said to be a *U-partition* of Γ if Γ and Δ satisfy the following conditions (1)—(3):

(1) $\alpha_1, \alpha_2, \cdots, \alpha_n$ are all distinct and $\Gamma_1^{\alpha_1}, \cdots, \Gamma_n^{\alpha_n}$ are all disjoint.

(2) All the free variables in Γ_i are among $\vec{x}_i$, $i = 1, 2, \cdots, n$.

(3) $U(\alpha_1, \vec{x}_1) \cup \cdots \cup U(\alpha_n, \vec{x}_n) \cup \Gamma_1^{\alpha_1} \cup \cdots \cup \Gamma_n^{\alpha_n} = \Gamma - \{(\forall\xi)(\exists u)U(\xi, u), (\forall u)(\exists\xi)U(\xi, u), (\forall\xi)A^\xi\}$.

Note that we do not require that $\vec{x}_1, \cdots, \vec{x}_n$ are disjoint or $\Gamma_1 \cdots \Gamma_n$ $U(\alpha_1, \vec{x}_1)$ $\cdots$ $U(\alpha_n, \vec{x}_n)$ are all non-empty.

Δ is a U-partition of $\Gamma \to \Theta$ if Δ is a U-partition of Γ. A sequent $\Gamma \to \Theta$ in $L(U)$ is said to be *U-partitionizable* if there is a U-partition of this sequent and all the formulas in Θ are formulas in L.

Then we can easily see that every sequent in P_2 is U-partitionizable. In fact the end-sequent is U-partitionizable, and for any inference rule used in P_2, if the lower sequent of this inference rule is U-partitionizable, then its upper sequent is also U-partitionizable.

Now we shall prove (**) below by the induction on subproof figures of P_2, i.e.

(**) For each sequent $\Gamma \to \Theta$ in P_2 and each U-partition $\Delta = (U(\alpha_1, \vec{x}_1), \cdots, U(\alpha_n, \vec{x}_n), \Gamma_1^{\alpha_1}, \cdots, \Gamma_n^{\alpha_n})$ of this sequent, there are formulas $C_0, C_1, \cdots, C_n$ in L satisfying the following (4)—(7):

(4) C_0 is a sentence of the form $(\forall v)E(v)$, where $E(x)$ is an existential formula in L.

(5) C_i is an existential formula in L whose free variables are among $\vec{x}_i$ for $i = 1, 2, \cdots, n$.

(6) $\underset{L}{\vdash} A \supset C_0$ and $\underset{L}{\vdash} A, \Gamma_i \rightarrow C_i$ for $i=1, 2, \cdots, n$.

(7) $\underset{L}{\vdash} C_0, C_1, \cdots, C_n \rightarrow \Theta$.

If $\Gamma \rightarrow \Theta$ is an axiom sequent such that $P(y_1, \cdots, y_m) \in \Gamma \cap \Theta$. Without loss of generality, we can assume that $P(y_1, \cdots, y_m) \in \Gamma_1^{\alpha_1}$. Let $C_0 = (\forall v)(P(v, \cdots, v) \vee \neg P(v, \cdots, v))$, $C_1 = P(y_1, \cdots, y_m)$, $C_i = (\exists v)(P(v, \cdots, v) \vee \neg P(v, \cdots, v))$ for $i=2, \cdots, n$. The case where $\Gamma \rightarrow \Theta$ is another kind of axiom sequents can be similarly treated.

In the following, we shall only consider the case where $\Gamma \rightarrow \Theta$ is the lower sequent of one of the inference rules; $(\vee \rightarrow)$, $(U)_1$, $(U)_2$, $(\forall \rightarrow)_0$, $(\exists \rightarrow)^U$ because other cases are trivial.

$$\text{CASE 1.} \quad (\vee \rightarrow) \frac{\overset{\Downarrow}{F, \Gamma' \longrightarrow \Theta} \qquad \overset{\Downarrow}{G, \Gamma' \longrightarrow \Theta}}{F \vee G, \Gamma' \longrightarrow \Theta} .$$

Let $\Delta = (U(\alpha_1, \vec{x}_1), \cdots, U(\alpha_n, \vec{x}_n), \Gamma_1^{\alpha_1}, \cdots, \Gamma_n^{\alpha_n})$ be a U-partition of $F \vee G, \Gamma' \rightarrow \Theta$. Without loss of generality, we can assume that $F \vee G \in \Gamma_1^{\alpha_1}$, $F \notin \Gamma'$, $G \notin \Gamma'$ and $F \vee G \in \Gamma'$.

Then

$$\Delta_1 = (U(\alpha_1, \vec{x}_1), \cdots, U(\alpha_n, \vec{x}_n), \Gamma_1^{\alpha_1} \cup \{F\}, \Gamma_2^{\alpha_2}, \cdots, \Gamma_n^{\alpha_n}),$$

$$\Delta_2 = (U(\alpha_1, \vec{x}_1), \cdots, U(\alpha_n, \vec{x}_n), \Gamma_1^{\alpha_1} \cup \{G\}, \Gamma_2^{\alpha_2}, \cdots, \Gamma_n^{\alpha_n})$$

are U-partitions of $F, \Gamma' \rightarrow \Theta$ and $G, \Gamma' \rightarrow \Theta$ respectively. By the hypotheses of induction, there are formulas $C_0', C_1', \cdots, C_n', C_0'', \cdots, C_n''$ such that $C_0', \cdots, C_n'$ satisfy (4)—(7) with respect to Δ_1 and $F, \Gamma' \rightarrow \Theta$ and $C_0'', \cdots, C_n''$ with respect to Δ_2 and $G, \Gamma' \rightarrow \Theta$.

Let $C_0 = (\forall v)(E'(v) \wedge E''(v))$, where $C_0' = (\forall v)E'(v)$

$$C_0'' = (\forall v)E''(v),$$

$$C_1 = C_1' \vee C_1'',$$

$$C_i = C_i' \wedge C_i'' \qquad \text{for } i=2, 3, \cdots, n.$$

Then obviously these $C_0, C_1, \cdots, C_n$ satisfy (4)—(7) with respect to Δ and $F \vee G, \Gamma' \rightarrow \Theta$.

$$\text{CASE 2.} \quad (U)_1 \frac{\overset{\Downarrow}{U(\alpha, x), \Gamma' \longrightarrow \Theta}}{(\forall \xi)(\exists u)U(\xi, u), \Gamma' \longrightarrow \Theta},$$ where x does not appear in the lower sequent.

Let $\Delta = (U(\alpha_1, \vec{x}_1), \cdots, U(\alpha_n, \vec{x}_n), \Gamma_1^{\alpha_1}, \cdots, \Gamma_n^{\alpha_n})$ be a U-partition of $(\forall \xi)(\exists u) U(\xi, u), \Gamma' \rightarrow \Theta$. Without loss of generality, we can assume that $\alpha = \alpha_1$. Let $\Delta_1 = (U(\alpha_1, \vec{x}_1 {}^\frown x), U(\alpha_2, \vec{x}_2), \cdots, U(\alpha_n, \vec{x}_n), \Gamma_1^{\alpha_1}, \cdots, \Gamma_n^{\alpha_n})$, where $\vec{x}_1 {}^\frown x = \langle x_{11}, \cdots, x_{1m}, x \rangle$, $\vec{x}_1 = \langle x_{11}, \cdots, x_{1m} \rangle$. Then Δ_1 is a U-partition of $U(\alpha, x), \Gamma' \rightarrow \Theta$. By the hypothesis of induction, there are formulas $C'_0, C'_1, \cdots, C'_n$ satisfying (4)—(7) with respect to Δ_1 and $U(\alpha, x), \Gamma' \to \Theta$.

Let $C_0 = C'_0$, $C_1 = (\exists u)C'_1(v)$, $C_i = C'_i$ for $i = 2, 3, \cdots, n$, where $C'_1 = C'_1(x)$.

$$\Downarrow$$

CASE 3. $(U)_2 \dfrac{U(\alpha, x), \Gamma' \longrightarrow \Theta}{(\forall u)(\exists \xi)U(\xi, u), \Gamma' \longrightarrow \Theta}$, where α does not appear in the lower sequent.

Let $\Delta = (U(\alpha_1, \dot{x}_1), \cdots, U(\alpha_n, \dot{x}_n), \Gamma_1^{\alpha_1}, \cdots, \Gamma_n^{\alpha_n})$ be a U-partition of $(\forall u)(\exists \xi) U(\xi, u), \Gamma' \to \Theta$. Since α does not appear in this sequent, we can assume $\alpha \neq \alpha_1, \cdots, \alpha_n$. Let $\Delta_1 = (U(\alpha_1, \dot{x}_1), \cdots, U(\alpha_n, \dot{x}_n), \{U(\alpha, x)\}, \Gamma_1^{\alpha_1}, \cdots, \Gamma_n^{\alpha_n}, \Gamma_{n+1}^{\alpha})$, where Γ_{n+1} is the empty set. Then Δ_1 is a U-partition of $U(\alpha, x), \Gamma' \to \Theta$.

By the hypothesis of induction, there are formulas $C'_0, C'_1, \cdots, C'_n, C'_{n+1}$ satisfying (4)—(7) with respect to Δ_1 and $U(\alpha, x), \Gamma' \to \Theta$. Let $C_0 = (\forall v)(E(v) \wedge C'_{n+1}(v))$, where $C_0 = (\forall v)E(v)$ and $C'_{n+1} = C'_{n+1}(x)$, $C_i = C'_i$, for $i = 1, 2, \cdots, n$.

$$\Downarrow$$

CASE 4. $(\forall \to)_0 \dfrac{A^{\alpha}, \Gamma' \longrightarrow \Theta}{(\forall \xi)A^{\xi}, \Gamma' \longrightarrow \Theta}$.

Let $\Delta = (U(\alpha_1, \dot{x}_1), \cdots, U(\alpha_n, \dot{x}_n), \Gamma_1^{\alpha_1}, \cdots, \Gamma_n^{\alpha_n})$ be a U-partition of $(\forall \xi)A^{\xi}$, $\Gamma' \to \Theta$. Without loss of generality, we can assume $\alpha = \alpha_1$. Let $\Delta_1 = (U(\alpha_1, \dot{x}_1), \cdots, U(\alpha_n, \dot{x}_n), (\Gamma_1 \cup \{A\})^{\alpha_1}, \Gamma_2^{\alpha_2}, \cdots, \Gamma_n^{\alpha_n})$. Then Δ_1 is a U-partition of A^{α}, $\Gamma' \to \Theta$. Formulas satisfying (4)—(7) with respect to Δ_1 and $A^{\alpha}, \Gamma' \to \Theta$ clearly satisfy (4)—(7) with respect to Δ and $(\forall \xi)A^{\xi}, \Gamma' \to \Theta$.

$$\Downarrow$$

CASE 5. $(\exists \to)^U \dfrac{U(\alpha, x), F^{\alpha}(x), \Gamma' \longrightarrow \Theta}{((\exists v)F(v))^{\alpha}, \Gamma' \longrightarrow \Theta}$, where x does not appear in the lower sequent.

Let $\Delta = (U(\alpha_1, \dot{x}_1), \cdots, U(\alpha_n, \dot{x}_n), \Gamma_1^{\alpha_1}, \cdots, \Gamma_n^{\alpha_n})$ be a U-partition of $((\exists v)F(v))^{\alpha}$, $\Gamma' \to \Theta$. Without loss of generality, we can assume $\alpha = \alpha_1$. Let $\Delta_1 = (U(\alpha_1, \dot{x}_1{}^\frown x), U(\alpha_2, \dot{x}_2), \cdots, U(\alpha_n, \dot{x}_n), (\Gamma_1 \cup \{F(x)\})^{\alpha_1}, \Gamma_2^{\alpha_2}, \cdots, \Gamma_n^{\alpha_n})$. Then Δ_1 is a U-partition of $U(\alpha, x), F^{\alpha}(x), \Gamma' \to \Theta$. By the hypothesis of induction, there are formulas $C'_0, C'_1, \cdots, C'_n$ satisfying (4)—(7) with respect to Δ_1 and $U(\alpha, x)$, $F^{\alpha}(x), \Gamma' \to \Theta$. Let $C_0 = C'_0$, $C_1 = (\exists v)C'_1(v)$, $C_i = C'_i$ for $i = 2, \cdots, n$, where $C'_1 = C'_1(x)$.

Hence we have proved (**).

Consider the end-sequent $(\forall \xi)(\exists u)U(\xi, u), (\forall u)(\exists \xi)U(\xi, u), (\forall \xi)A^{\xi} \to B$ and its empty U-partition Δ (i.e. $n = 0$ and $\Delta =$ the empty set). Then by (**) there is a sentence $C = (\forall v)E(v)$, where $E(x)$ is an existential formula in L

such that $\vdash_L A \supset C$ and $\vdash_L C \supset B$. This proves our main theorem.

REMARK 1. Our proof of the main theorem works even in the case that L has the equality symbol $\doteq$ unless we take the equality axioms for U as axiom sequents in $L(U)$. Hence $x \doteq y$, $U(\alpha, x)$, $\neg U(\alpha, y) \rightarrow$ is not an axiom sequent in $L(U)$. Note that this restriction does not make any influence on the model theoretic applications of our main theorem stated below.

REMARK 2. Using the same way as above, we can also prove the theorem of the following type:

If $(\forall \xi)(\exists u)U(\xi, u)$, $\bigwedge_{k<\omega} (\forall u_1) \cdots (\forall u_k)(\exists \xi)(U(\xi, u_1) \wedge \cdots \wedge U(\xi, u_k)) \vdash_{L(U)} (\forall \xi)A^{\xi} \supset B$, then there is a sentence C obtained from existential formulas by applying $\wedge$ and $\forall$, i.e. $C \in \{\wedge, \forall\}$ (the set of existential formulas), such that $\vdash_L A \supset C$ and $\vdash_L C \supset B$.

REMARK 3. We can state our main theorem in terms of object logic and morphism logic (cf. [4], [5]) and show that it holds also in the logic treated in [6].

By these remarks we can assert that our results are syntactical counterparts of Chang-Łos' theorem on ω-ascending unions (cf. [1]), Keisler's theorem on unions (cf. [3]) in the finitary case and Nebres' theorem on unions in the infinitary case (cf. [8]).

Finally the author wants to state his conjecture that the technique developed in this paper and the author's another paper [7] will serve for a complete solution of the problem to find a syntactical form of $L_{\omega_1\omega}$-sentences preserved under ω-ascending unions (see Weinstein [9]).

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