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Some closed subalgebras of measure algebras and a generalization of P. J. Cohen's theorem II

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§ 1. Introduction.

This paper is a continuation of the previous paper [4]. Throughout this paper $G(\tau)$ and $H(\sigma)$ denote LCA groups with underlying groups G and H, and with topologies τ and σ, respectively. In the previous paper [4], we introduced the closed subalgebra $L^*(G(\tau))$ of $M(G(\tau))$, and determined all the homomorphisms of $L^*(G(\tau))$ into $M(H(\sigma))$ as a generalization of Cohen's theorem. In this paper we prove that every homomorphism of $L^*(G(\tau))$ into $M(H(\sigma))$ has a natural norm-preserving extension to a homomorphism of $M(G(\tau))$ into $M(H(\sigma))$ as a generalization of Cohen's theorem.

In § 2 we give some preliminaries, and in § 3 we give the proof of our result for the special case that $H(\sigma)$ is compact. § 4 contains some results on the topology of the maximal ideal space of $M(G(\tau))$, which is used in § 5 to prove our result for the general case.

§ 2. Preliminaries.

We denote by $\mathfrak{T}(G(\tau))$ the set of all the locally compact group topologies on G which are at least as strong as the original topology τ. Let τ_1 and τ_2 be elements of $\mathfrak{T}(G(\tau))$ with $\tau_1 \subset \tau_2$. We denote by $\eta_{\tau_2}^{\tau_1}$ the natural continuous isomorphism of $G(\tau_2)$ onto $G(\tau_1)$. Γ_{τ_i} denotes the dual group of $G(\tau_i)$ and $\varphi_{\tau_2}^{\tau_1}$ denotes the natural continuous isomorphism of Γ_{τ_1} onto a dense subgroup of Γ_{τ_2} such that (cf. Lemma 2.3 of [4])

$$(x, \varphi_{\tau_2}^{\tau_1}(r)) = (\eta_{\tau_2}^{\tau_1}(x), r) \qquad (x \in G(\tau_2),\ r \in \Gamma_{\tau_1}) .$$

For each $\tau' \in \mathfrak{T}(G(\tau))$, there exists a natural norm-preserving isomorphism $\pi_{\tau'}$ of $M(G(\tau'))$ into $M(G(\tau))$ such that (cf. Proposition 2.1 of [4])

$$\pi_{\tau'}(\mu)(E) = \mu(\eta_{\tau'}^{\tau\,-1}(E)) \qquad (E\text{: Borel set of } G(\tau)\,;\ \mu \in M(G(\tau'))) .$$

We identify $L^1(G(\tau'))$ and $M(G(\tau'))$ with the closed subalgebras of $M(G(\tau))$ through $\pi_{\tau'}$, respectively. $M(G(\tau'))^{\perp} = \{\mu \in M(G(\tau))\colon \mu \perp \nu\,;\ \nu \in M(G(\tau'))\}$ is an ideal of $M(G(\tau))$ and the projection $P_{\tau'}$ of $M(G(\tau))$ onto $M(G(\tau'))$ is a homomorphism.

$\mathfrak{M}$ denotes the maximal ideal space of $M(G(\tau))$ and Γ^* denotes the maximal ideal space of $L^*(G(\tau))$ constructed in § 3 of [4]. If $\mu \in M(G(\tau))$, $\hat{\mu}$ denotes the Fourier-Stieltjes transform of μ, and $\hat{\mu}$ denotes the Gelfand transform of μ. If we express by $\hat{\hat{\mu}}$ the function of Γ^* defined by (3.10) of [4], each $r \in \Gamma^*$ has an extension to a complex homomorphism

$$M(G(\tau)) \ni \mu \longmapsto \hat{\hat{\mu}}(r) \in C,$$

and in this way we consider Γ^* as a subset of $\mathfrak{M}$. In this point of view we have

$$\hat{\mu}(r) = \hat{\hat{\mu}}(r) \qquad (r \in \Gamma^* ;\ \mu \in M(G(\tau))).$$

By the Remark in p. 291 of [4], $\Gamma_{\tau'}$ can be considered as a subset of Γ^*, and so as a subset of $\mathfrak{M}$, and it is easy to see that the relative topology of $\Gamma_{\tau'}$ in $\mathfrak{M}$ is equal to the topology of $\Gamma_{\tau'}$ $(\tau' \in \mathfrak{T}(G(\tau)))$.

If $\mu \in M(G(\tau'))$ $(\tau' \in \mathfrak{T}(G(\tau)))$, we express the Fourier-Stieltjes transform of μ with respect to $\Gamma_{\tau'}$ and the Fourier-Stieltjes transform of μ with respect to Γ_τ by the same symbol $\hat{\mu}$, and we make a difference between them by indicating the domain of $\hat{\mu}$. We constantly refer to the previous paper [4], and the notations used in this paper are chosen so that they are consistent with those in [4] as far as possible.

§ 3. Consideration of the case when $H(\sigma)$ is compact.

THEOREM 3.1. *Suppose that $\tau_1, \cdots, \tau_n$ are a finite number of elements in $\mathfrak{T}(G(\tau))$, then there exists a unique $\tau_0 \in \mathfrak{T}(G(\tau))$ such that $\bigcap_{i=1}^{n} M(G(\tau_i)) = M(G(\tau_0))$.*

PROOF. We may suppose $n=2$. Let τ_d be the discrete topology on G, then Γ_{τ_d} is the Bohr compactification of Γ_τ. Choose $\tau_1', \tau_2' \in \mathfrak{T}(\Gamma_{\tau_d})$ so that

$$\eta_i'^{-1} \circ \varphi_{\tau_d}^{\tau_i} \colon \Gamma_{\tau_i} \longrightarrow \Gamma_{\tau_d}(\tau_i') \qquad (i=1, 2)$$

is an open continuous isomorphism, where η_i' is the natural continuous isomorphism of $\Gamma_{\tau_d}(\tau_i')$ onto Γ_{τ_d}. By Theorem 2.8 of [4], there exists $\tau_0' \in \mathfrak{T}(\Gamma_{\tau_d})$ such that

$$L^1(\Gamma_{\tau_d}(\tau_1')) * L^1(\Gamma_{\tau_d}(\tau_2')) \subset L^1(\Gamma_{\tau_d}(\tau_0')), \tag{3.1}$$

$$\tau_0' \subset \tau_1',\ \tau_2'.$$

Furthermore for this τ_0', we see from the proof of Theorem 2.8 of [4] that

$$\Gamma_{\tau_d}(\tau_1') \times \Gamma_{\tau_d}(\tau_2') \ni (x, y) \longmapsto \eta'^{-1}(\eta_1'(x)) + \eta'^{-1}(\eta_2'(y)) \in \Gamma_{\tau_d}(\tau_0')$$

is an open continuous map, where η' denotes the natural continuous isomorphism of $\Gamma_{\tau d}(\tau'_0)$ onto $\Gamma_{\tau d}$. Thus if we put

$$\Pi = \eta'^{-1}(\varphi^{\tau_1}_{\tau d}(\Gamma_{\tau_1}) + \varphi^{\tau_2}_{\tau d}(\Gamma_{\tau_2})), \tag{3.2}$$

then Π is an open subgroup of $\Gamma_{\tau d}(\tau'_0)$.

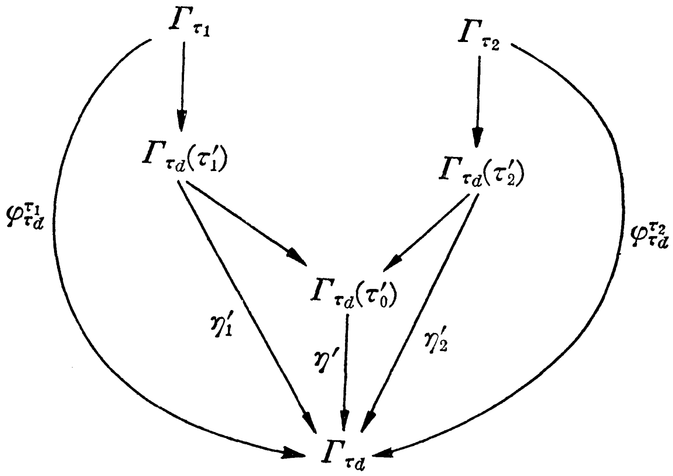

Fig. 1.

Let G' be the dual group of Π. For each $x \in G'$ $\eta_i(x) = x \circ \eta'^{-1} \circ \varphi^{\tau_i}_{\tau d}$ is an element of $G(\tau_i)$, and it is easy to see that η_i is a continuous homomorphism of G' onto $G(\tau_i)$ $(i=1, 2)$.

Let $x \in G'$ such that $\eta_1(x)=0$. Since

$$\varphi^{\tau_1}_{\tau d}(\Gamma_{\tau_1}) \cap \varphi^{\tau_2}_{\tau d}(\Gamma_{\tau_2}) \supset \varphi^{\tau}_{\tau d}(\Gamma_{\tau}),$$

and since $\varphi^{\tau}_{\tau d}(\Gamma_{\tau})$ is dense in $\Gamma_{\tau d}$, we have $\eta^{\tau_2\,-1}_{\tau d}(\eta_2(x))=0$ and consequently $\eta_2(x)=0$. If we remember that Π is generated by $\eta'^{-1}(\varphi^{\tau_1}_{\tau d}(\Gamma_{\tau_1}) \cup \varphi^{\tau_2}_{\tau d}(\Gamma_{\tau_2}))$, $\eta_2(x)=0$ reduces to $x=0$. This shows that η_1 is an isomorphism and in the same way η_2 is an isomorphism. It follows from this that we can choose $\tau_0 \in \mathfrak{T}(G(\tau))$ such that $\tau_0 \supset \tau_1, \tau_2$, $G' \cong G(\tau_0)$, and thus we can identify G' with $G(\tau_0)$ in the natural way.

Let ν be an arbitrary element of $M(G(\tau_1)) \cap M(G(\tau_2))$. By Proposition 2.1 of [4], there exists a σ-compact set K_i in $G(\tau_i)$ $(i=1, 2)$ such that ν is concentrated on $\eta^{\tau}_{\tau_i}(K_i)$, and hence ν is concentrated on $\eta^{\tau}_{\tau_1}(K_1) \cap \eta^{\tau}_{\tau_2}(K_2)$. Let K_{ij} $(j=1, 2, \cdots)$ be a sequence of compact subsets of $G(\tau_i)$ such that $\bigcup_{j=1}^{\infty} K_{ij} = K_i$ $(i=1, 2)$. For each positive integer j_i $(i=1, 2)$, we can find $\nu_{ij_i} \in L^1(\Gamma_{\tau d}(\tau'_i))$ such that

$$\hat{\nu}_{ij_i}(x) = 1 \qquad (x \in K_{ij_i}), \tag{3.3}$$

$$\nu_{ij_i} \text{ is concentrated on } \eta_i'^{-1}(\varphi^{\tau_i}_{\tau d}(\Gamma_{\tau_i})).$$

From (3.1) we have $\nu_{1j_1}*\nu_{2j_2}\in L^1(\Gamma_{\tau d}(\tau_0'))$ and from (3.2) $\nu_{1j_1}*\nu_{2j_2}$ is concentrated on Π. Using the fact that Π is open in $\Gamma_{\tau d}(\tau_0')$, we get $\nu_{1j_1}*\nu_{2j_2}\in L^1(\Pi)$. Consequently we get

$$\widehat{\nu_{1j_1}*\nu_{2j_2}}(x)=\hat{\nu}_{1j_1}(x)\hat{\nu}_{2j_2}(x)=1 \qquad (x\in \eta_{\tau_0}^{\tau_1^{-1}}(K_{1j_1})\cap \eta_{\tau_0}^{\tau_2^{-1}}(K_{2j_2})), \tag{3.4}$$

$$\widehat{\nu_{1j_1}*\nu_{2j_2}}\in C_0(G(\tau_0)).$$

(3.4) means that there exists a compact subset $C(j_1, j_2)$ of $G(\tau_0)$ such that

$$\eta_{\tau_0}^{\tau}(C(j_1, j_2))\supset \eta_{\tau_1}^{\tau}(K_{1j_1})\cap \eta_{\tau_2}^{\tau}(K_{2j_2}). \tag{3.5}$$

Summing (3.5) for j_1 and j_2, we get

$$\bigcup_{j_1,j_2=1}^{\infty}\eta_{\tau_0}^{\tau}(C(j_1, j_2))\supset \bigcup_{j_1,j_2=1}^{\infty}(\eta_{\tau_1}^{\tau}(K_{1j_1})\cap \eta_{\tau_2}^{\tau}(K_{2j_2})) \tag{3.6}$$

$$=\eta_{\tau_1}^{\tau}(\bigcup_{j_1=1}^{\infty}K_{1j_1})\cap \eta_{\tau_2}^{\tau}(\bigcup_{j_2=1}^{\infty}K_{2j_2})=\eta_{\tau_1}^{\tau}(K_1)\cap \eta_{\tau_2}^{\tau}(K_2).$$

From (3.6) and the fact that ν is concentrated on $\eta_{\tau_1}^{\tau}(K_1)\cap\eta_{\tau_2}^{\tau}(K_2)$, we obtain using Proposition 2.1 of [4] that ν belongs to $M(G(\tau_0))$.

Thus $M(G(\tau_1))\cap M(G(\tau_2))$ is contained in $M(G(\tau_0))$ and it is clear that $M(G(\tau_0))$ is contained in $M(G(\tau_1))\cap M(G(\tau_2))$. Since the uniqueness of τ_0 is obvious from Theorem 2.5 of [4], this completes the proof of Theorem 3.1.

We introduce a partial ordering in $\mathfrak{T}(G(\tau))$ such that if $\tau_1, \tau_2\in\mathfrak{T}(G(\tau))$, then $\tau_2\leqq\tau_1$ if and only if $\tau_1\subset\tau_2$.

COROLLARY 3.2[1]. *$\mathfrak{T}(G(\tau))$ is a lattice under the partial ordering $\leqq$.*

PROOF. Let $\tau_1, \tau_2\in\mathfrak{T}(G(\tau))$. By Theorem 2.8 of [4], there exists $\tau_3\in\mathfrak{T}(G(\tau))$, l. u. b. of τ_1 and τ_2 such that

$$L^1(G(\tau_1))*L^1(G(\tau_2))\subset L^1(G(\tau_3)).$$

By Theorem 3.1 there exists $\tau_0\in\mathfrak{T}(G(\tau))$, g. l. b. of τ_1 and τ_2 such that

$$M(G(\tau_1))\cap M(G(\tau_2))=M(G(\tau_0)),$$

and this completes the proof.

COROLLARY 3.3. *Let τ_0, τ_1 and τ_2 be elements of $\mathfrak{T}(G(\tau))$ such that $M(G(\tau_1))\cap M(G(\tau_2))=M(G(\tau_0))$. We regard each Γ_{τ_i} $(i=0, 1, 2)$ as a subgroup of the semigroup Γ^* (cf. Proposition* 3.2 *and* p. 291 *Remark of* [4]), *then we have*

$$\Gamma_{\tau_1}+\Gamma_{\tau_2}=\Gamma_{\tau_0}.$$

PROOF. In the proof of the Theorem 3.1, we can identify Γ_{τ_0} with Π and that we have $\varphi_{\tau_0}^{\tau_1}(\Gamma_{\tau_1})+\varphi_{\tau_0}^{\tau_2}(\Gamma_{\tau_2})=\eta'^{-1}(\varphi_{\tau d}^{\tau_1}(\Gamma_{\tau_1})+\varphi_{\tau d}^{\tau_2}(\Gamma_{\tau_2}))=\Pi$. Thus we have

1) But $\mathfrak{T}(G(\tau))$ is not generally a σ-complete lattice (cf. §5 example).

$$\begin{aligned}\Gamma_{S_{\tau_1}}+\Gamma_{S_{\tau_2}} &= \{(\varphi_{\tau'}^{\tau_1}(r))_{\tau'\in S_{\tau_1}}+(\varphi_{\tau'}^{\tau_2}(r'))_{\tau'\in S_{\tau_2}}\,;\ r\in\Gamma_{\tau_1},\ r'\in\Gamma_{\tau_2}\}\\ &= \{(\varphi_{\tau'}^{\tau_1}(r)+\varphi_{\tau'}^{\tau_2}(r'))_{\tau'\in S_{\tau_0}}\,;\ r\in\Gamma_{\tau_1},\ r'\in\Gamma_{\tau_2}\}\\ &= \{(\varphi_{\tau'}^{\tau_0}(r))_{\tau'\in S_{\tau_0}}\,;\ r\in\Pi=\Gamma_{\tau_0}\}=\Gamma_{S_{\tau_0}}.\end{aligned}$$

If we identify Γ_{τ_i} with $\Gamma_{S_{\tau_i}}$ we get the conclusion of Corollary 3.3.

PROPOSITION 3.4. *Suppose that $H(\sigma)$ is compact, then every homomorphism h of $L^*(G(\tau))$ into $M(H(\sigma))$ has a natural norm-preserving extension to a homomorphism of $M(G(\tau))$ into $M(H(\sigma))$.*

PROOF. Let Λ_σ denote the dual group of $H(\sigma)$. By Theorem 4.1 of [4], there exists a subset Y of Λ_σ and a map α of Y into Γ^* such that for each $\tau'\in\mathfrak{T}(G(\tau))$ $Y_{\tau'}$ is an element of the coset ring of Λ_σ and $\alpha_{\tau'}$ is a piecewise affine map of $Y_{\tau'}$ into $\Gamma_{\tau'}$, where $Y_{\tau'}$ and $\alpha_{\tau'}$ is defined by (4.3) of [4].

Let μ be an element of $M(G(\tau))$ and put

$$\beta_\mu(r)=\begin{cases}\hat{\mu}(\alpha(r))=\hat{\hat{\mu}}(\alpha(r))\,; & r\in Y\\ 0 & ;\ r\in\Lambda_\sigma-Y.\end{cases} \tag{3.7}$$

We show that $\beta_\mu\in B(\Lambda_\sigma)$ and $\|\beta_\mu\|\leqq\|h\|\|\mu\|$, and this will complete the proof of Proposition 3.4 (cf. p. 32 and p. 83 of [5]).

Let $P(x)=\sum_{i=1}^{n}a_i(x,r_i)$ be a non-zero trigonometric polynomial on $H(\sigma)$, and let $\varepsilon>0$. Suppose that $\alpha(r_i)\in\Gamma_{S_i}$ $(i=1,\cdots,m)$ and $r_i\notin Y$ $(i=m+1,\cdots,n)$ (cf. Definition 3.2 of [4]). By Proposition 3.4 of [4], we have a decomposition of μ such that

$$\mu=\mu_1^{(i)}+\mu_2^{(i)},\quad \mu_1^{(i)}\in\overline{\sum_{\tau'\in S_i}M(G(\tau'))},\quad \mu_2^{(i)}\in\overline{\sum_{\tau'\in S_i}M(G(\tau'))}^{\perp}\quad(i=1,\cdots,m). \tag{3.8}$$

Remembering that S_i is a directed set, we may write

$$\mu_1^{(i)}=\lim_{\tau'\in S_i}P_{\tau'}(\mu)\qquad(i=1,\cdots,m), \tag{3.9}$$

and there exists $\tau_i\in S_i$ such that

$$\|\mu_1^{(i)}-P_{\tau_i}(\mu)\|\leqq\varepsilon/2m(1+\varepsilon)\max[|a_k|\,;\ k=1,\cdots,m]\quad(i=1,\cdots,m). \tag{3.10}$$

We have from (3.7) and the definition of $\hat{\hat{\mu}}$ (cf. p. 292 of [4]) that

$$\begin{aligned}|\beta_\mu(r_i)-\widehat{P_{\tau_i}(\mu)}(\alpha(r_i))| &= |\hat{\mu}_1^{(i)}(\alpha(r_i))-\widehat{P_{\tau_i}(\mu)}(\alpha(r_i))|\\ &\leqq\|\mu_1^{(i)}-P_{\tau_i}(\mu)\|\leqq\varepsilon/2m(1+\varepsilon)\max[|a_k|\,;\ k=1,\cdots,m]\quad(i=1,\cdots,m).\end{aligned} \tag{3.11}$$

By induction on m, we can find $\mu_A\in M(G(\tau))$ $(A\subset\{\tau_1,\cdots,\tau_m\})$ such that

$$\begin{aligned}&\mu_A\in M(G(\tau_i))^{\perp} && (\tau_i\in\{\tau_1,\cdots,\tau_m\}-A),\\ &\mu_A\in M(G(\tau_i)) && (\tau_i\in A),\end{aligned} \tag{3.12}$$

$$\sum_{A\subset\{\tau_1,\cdots,\tau_m\}}\mu_A=\mu\,.$$

By Theorem 3.1 there exists $\tau_A\in\mathfrak{T}(G(\tau))$ such that

$$M(G(\tau_A))=\bigcap_{\tau_i\in A}M(G(\tau_i))\qquad (A\subset\{\tau_1,\cdots,\tau_m\})\,, \tag{3.13}$$

where we put $\tau_A=\tau$ if $A=\emptyset$. Choosing $\lambda_A\in L^1(G(\tau_A))$ $(A\subset\{\tau_1,\cdots,\tau_m\})$ such that

$$\|\lambda_A\|\leqq 1+\varepsilon\,;\ \hat{\lambda}_A(\varphi^{S_i}_{\tau_A}(\alpha(r_i)))=1\qquad (\tau_i\in A)\,, \tag{3.14}$$

and putting $\lambda=\sum_{A\subset\{\tau_1,\cdots,\tau_m\}}\lambda_A*\mu_A$, we have

$$\|\lambda\|\leqq(1+\varepsilon)\|\mu\|\,,\qquad \lambda\in L^*(G(\tau))\,. \tag{3.15}$$

It is easy to see from (3.12), (3.13) and the definition of λ that

$$\begin{aligned}&\mu_A\perp\mu_{A'}\ (\{\tau_1,\cdots,\tau_m\}\supset A,\ A'\,;\ A\neq A')\,,\\ &P_{\tau_i}(\mu)=\sum_{A\ni\tau_i}\mu_A\,, \qquad\qquad\qquad\qquad (i=1,\cdots,m)\,.\\ &P_{\tau_i}(\lambda)=\sum_{A\subset\{\tau_1,\cdots,\tau_m\}}P_{\tau_i}(\lambda_A*\mu_A)=\sum_{A\ni\tau_i}\lambda_A*\mu_A\end{aligned} \tag{3.16}$$

Again by Proposition 3.4 of [4], we decompose λ and μ_A $(A\subset\{\tau_1,\cdots,\tau_m\})$ so that

$$\begin{aligned}&\lambda=\lambda_1^{(i)}+\lambda_2^{(i)},\quad \mu_A=\mu_{A,1}^{(i)}+\mu_{A,2}^{(i)}\,, \qquad\qquad (i=1,\cdots,m)\,.\\ &\mu_{A,1}^{(i)},\ \lambda_1^{(i)}\in\overline{\sum_{\tau'\in S_i}M(G(\tau'))},\quad \mu_{A,2}^{(i)},\ \lambda_2^{(i)}\in\overline{\sum_{\tau'\in S_i}M(G(\tau'))}^{\perp}\end{aligned} \tag{3.17}$$

Since $\tau_i\in S_i$ $(i=1,\cdots,m)$, we have

$$\mu_{A,1}^{(i)}=\mu_A\,,\qquad \lambda_A*\mu_{A,1}^{(i)}\in\overline{\sum_{\tau'\in S_i}M(G(\tau'))}\qquad (A\ni\tau_i)\,. \tag{3.18}$$

Here remembering that $\overline{\sum_{\tau'\in S_i}M(G(\tau'))}^{\perp}$ is an ideal and using (3.10), (3.14), (3.16), (3.17) and (3.18) we obtain

$$\begin{aligned}\|\lambda_1^{(i)}-P_{\tau_i}(\lambda)\|&\leqq\|(\sum_{A\not\ni\tau_i}\lambda_A*\mu_{A,1}^{(i)}+\sum_{A\ni\tau_i}\lambda_A*\mu_A)-\sum_{A\ni\tau_i}\lambda_A*\mu_A\|\\ &\leqq\sum_{A\not\ni\tau_i}\|\lambda_A\|\|\mu_{A,1}^{(i)}\|\leqq(1+\varepsilon)\|\mu_1^{(i)}-P_{\tau_i}(\mu)\|\\ &\leqq\varepsilon/2m\max[|a_k|\,;\ k=1,\cdots,m]\quad (i=1,\cdots,m)\,.\end{aligned} \tag{3.19}$$

From (3.19) we get at once

$$|\hat{\lambda}_1^{(i)}(\alpha(r_i))-\widehat{P_{\tau_i}(\lambda)}(\alpha(r_i))|\leqq\varepsilon/2m\max[|a_k|\,;\ k=1,\cdots,m]\,. \tag{3.20}$$

Using (3.11), (3.14), (3.16), (3.17) and (3.20) we obtain

$$(3.21)\qquad |\beta_\mu(r_i)-\hat{\hat{\lambda}}(\alpha(r_i))|=|\beta_\mu(r_i)-\hat{\hat{\lambda}}_1^{(i)}(\alpha(r_i))|$$
$$\leqq|\widehat{P_{\tau_i}(\mu)}(\alpha(r_i))-\widehat{P_{\tau_i}(\lambda)}(\alpha(r_i))|+\varepsilon/m\max[|a_k|\,;\,k=1,\cdots,m]$$
$$=|\sum_{A\supset\tau_i}\hat{\mu}_A(\alpha(r_i))-\sum_{A\ \tau_i}\hat{\mu}_A(\alpha(r_i))\hat{\lambda}_A(\varphi_{\tau_A}^{S_i}(\alpha(r_i)))|$$
$$+\varepsilon/m\max[|a_k|\,;\,k=1,\cdots,m]$$
$$=\varepsilon/m\max[|a_k|\,;\,k=1,\cdots,m]\qquad(i=1,\cdots,m)\,.$$

From (3.21) we have

$$(3.22)\qquad |a_i\beta_\mu(r_i)-a_i\hat{\hat{\lambda}}(\alpha(r_i))|\leqq\varepsilon/m\qquad(i=1,\cdots,m)\,.$$

Summing (3.22) from $i=1$ to m, and using the fact that $\beta_\mu(r_i)=\hat{h}(\lambda)(r_i)=0$ $(i=m+1,\cdots,n)$, we have

$$(3.23)\qquad |\sum_{i=1}^n a_i\beta_\mu(r_i)-\sum_{i=1}^n a_i\widehat{h(\lambda)}(r_i)|\leqq\sum_{i=1}^m|a_i\beta_\mu(r_i)-a_i\hat{\hat{\lambda}}(\alpha(r_i))|\leqq\varepsilon\,.$$

From (3.15) applying the Bochner-Eberlein's theorem we obtain

$$(3.24)\qquad |\sum_{i=1}^n a_i\widehat{h(\lambda)}(r_i)|\leqq\|h(\lambda)\|\|P\|_\infty\leqq\|h\|\|\lambda\|\|P\|_\infty\leqq\|h\|(1+\varepsilon)\|\mu\|\|P\|_\infty\,.$$

Combining (3.23) and (3.24) and letting $\varepsilon\to0$ to obtain

$$(3.25)\qquad |\sum_{i=1}^n a_i\beta_\mu(r_i)|\leqq\|h\|\|\mu\|\|P\|_\infty\,,$$

and again by the Bochner-Eberlein's theorem we obtain $\beta_\mu\in B(\Lambda_\sigma)$, $\|\beta_\mu\|\leqq\|h\|\|\mu\|$. This completes the proof of Proposition 3.4.

§ 4. Some results on the topology of the maximal ideal space of $M(G(\tau))$.

THEOREM 4.1. *Let τ_0 be an element of $\mathfrak{T}(G(\tau))$, then we have*

(a) *If $\mu\in M(G(\tau))$ such that there exists $\delta>0$ and a non-empty open set $\tilde{U}$ in Γ_{τ_0} with*

$$(4.1)\qquad |\hat{\mu}(r)|>\delta>0\qquad(r\in\varphi_{\tau_0}^{\tau\,-1}(\tilde{U}))\,,$$

then μ is not an element of $M(G(\tau_0))^\perp$.

(b) $\|\hat{\mu}\|_\infty\geqq\|\widehat{P_{\tau_0}(\mu)}\|_\infty$ $(\mu\in M(G(\tau)))$.

(c) $\bar{\Gamma}_\tau\supset\Gamma_{\tau_0}$ *($\bar{\Gamma}_\tau$ denotes the closure of Γ_τ in $\mathfrak{M}$).*

(c) was proved by T. Shimizu [6] for a special class of elements in $\mathfrak{T}(G(\tau))$ which contains the discrete topology on G. Also (b) and (c) were proved independently by C. Dunkl and D. Ramirez [2], [3] for each element of $\mathfrak{T}(G(\tau))$. Since (a) is easily led from (b), Theorem 4.1 is essentially contained in [2] and [3]. But, for the completeness, we give here the proof of

Theorem 4.1 which is somewhat different from their proof.

We denote by $P_c(G(\tau))$ (resp. $P_c(G(\tau_0))$) the set of all continuous positive-definite functions of $G(\tau)$ (resp. $G(\tau_0)$) with compact support, and by spt. p the support of $p \in P_c(G(\tau))$ (resp. $P_c(G(\tau_0))$) in $G(\tau)$ (resp. $G(\tau_0)$). We denote by m the Haar measure on $G(\tau_0)$. If $p \in P_c(G(\tau))$ and $f \in P_c(G(\tau_0))$, we define $p*f(x)=\int_{G(\tau_0)} p(y)f(x-y)dm(y)$ $(x \in G)$. Since $p*f$ is $G(\tau)$-continuous and has a compact support in $G(\tau)$, $p*f$ belongs to $L^1(G(\tau))$. $p*f \in L^1(G(\tau))$ is just the convolution of $p \in L^1(G(\tau))$ and $f \in L^1(G(\tau_0))$ in $M(G(\tau))$, and thus $p*f \in P_c(G(\tau))$ by the inversion theorem.

The following lemma is due to C. Dunkl and D. Ramirez [2].

LEMMA 4.2. *Let $p_0 \in P_c(G(\tau_0))$, and let W be an open set in $G(\tau)$ such that $W \supset \mathrm{spt.}\,p_0$. Then, for each $\varepsilon > 0$, there exists $p \in P_c(G(\tau))$ such that*

$$\mathrm{spt.}\,p \subset W\,; \quad |p(x)-p_0(x)| < \varepsilon \quad (x \in \mathrm{spt.}\,p_0)\,. \tag{4.2}$$

PROOF. Let $\varepsilon > 0$ and put $K=\mathrm{spt.}\,p_0$. Since p_0 is uniformly continuous on $G(\tau_0)$, there exists $(0 \in)U \in \tau_0$ such that

$$|p_0(x+y)-p_0(x)| < \varepsilon \quad (x \in G\,;\ y \in U)\,; \quad U=-U,\ m(U)<\infty\,. \tag{4.3}$$

$K-K$ is $G(\tau_0)$-compact, and the induced topology on $K-K$ from $G(\tau)$ agree with $G(\tau_0)$-topology on $K-K$. Thus we can choose $(0 \in)V \in \tau$ such that

$$V \cap (K-K) \subset U \cap (K-K)\,; \quad V+K \subset W\,. \tag{4.4}$$

Let $g \in P_c(G(\tau))$ such that

$$\mathrm{spt.}\,g \subset V\,; \quad \int_U g\,dm=1\,; \quad g \geqq 0\,. \tag{4.5}$$

We have from (4.3), (4.4) and (4.5) that

$$\begin{aligned} |g*p_0(x)-p_0(x)| &= \left|\int_V g(y)p_0(x-y)dm(y)-p_0(x)\right| \\ &= \left|\int_U g(y)(p_0(x-y)-p_0(x))dm(y)\right| < \varepsilon \quad (x \in K)\,. \end{aligned} \tag{4.6}$$

If we put $p=g*p_0 \in P_c(G(\tau))$, we have

$$\mathrm{spt.}\,p \subset \mathrm{spt.}\,g+K \subset V+K \subset W\,. \tag{4.7}$$

This completes the proof.

THE PROOF OF THEOREM 4.1. (a) Suppose $\mu \in M(G(\tau_0))^{\perp}$. Let $0 \neq g \in L^1(G(\tau_0))$ be a continuous positive-definite function on $G(\tau_0)$ such that

$$\begin{aligned} &0 \leqq \hat{g}(r) < |\hat{\mu}(r)|^2 \qquad (r \in \varphi_{\tau_0}^{\tau\,-1}(\tilde{U}))\,, \\ &\hat{g}(r)=0 \qquad (r \notin \varphi_{\tau_0}^{\tau\,-1}(\tilde{U}))\,, \end{aligned} \tag{4.8}$$

and let $p_0 \in P_c(G(\tau_0))$ with $p_0 * g \neq 0$. If we put $K = \text{spt.}\, p_0$, then $|\tilde{\mu} * \mu|(K) = 0$ by Proposition 2.1 of [4], and we can choose $\varepsilon > 0$ and $W \in \tau$ such that

$$(4.9) \qquad p_0 * g(0) > \varepsilon > 0 ; \qquad W \supset K ,$$

$$|\tilde{\mu} * \mu|(W) < (p_0 * g(0) - \varepsilon)/2p_0(0) ,$$

$$2p_0(0) \cdot \int_{W-K} |g(x)| \, dm(x) < \varepsilon/2 .$$

By Lemma 4.2, there exists $p \in P_c(G(\tau))$ such that

$$(4.10) \qquad |p(x) - p_0(x)| < \theta \qquad (x \in K) ; \quad \text{spt.}\, p \subset W ,$$

where $\theta = \min \left\{ p_0(0),\ \varepsilon/2\int_K |g(x)| \, dm(x) \right\}$. We have from (4.10)

$$(4.11) \qquad \max \{|p(x)|,\ |p(x) - p_0(x)|\} < 2p_0(0) \qquad (x \in G) ,$$

and from (4.9), (4.10) and (4.11) we get

$$(4.12) \qquad p * \tilde{\mu} * \mu(0) = \int_W p(-x) d\tilde{\mu} * \mu(x) < 2p_0(0) |\tilde{\mu} * \mu|(W) < p_0 * g(0) - \varepsilon$$

$$\leqq p * g(0) + \int_K |(p_0 - p)g|(x) dm(x) + \int_{W-K} |(p_0 - p)g|(x) dm(x) - \varepsilon$$

$$< p * g(0) + \theta \int_K |g(x)| \, dm(x) + 2p_0(0) \int_{W-K} |g(x)| \, dm(x) - \varepsilon$$

$$< p * g(0) .$$

On the other hand we have by the inversion theorem and from (4.8) that

$$p * \tilde{\mu} * \mu(0) = \int_{\Gamma_\tau} \hat{p}(r) |\hat{\mu}(r)|^2 dr$$

$$\geqq \int_{\Gamma_\tau} \hat{p}(r) \hat{g}(r) dr = p * g(0) .$$

This contradicts (4.12) and thus (a) follows.

(b) Suppose that there exists $\mu \in M(G(\tau))$ such that $\|\hat{\mu}\|_\infty < \|\widehat{P_{\tau 0}(\mu)}\|_\infty$. Then there exists $\varepsilon > 0$ and a non-empty open set U in $\Gamma_{\tau 0}$ such that

$$(4.13) \qquad |\hat{\mu}(r)| < |\widehat{P_{\tau 0}(\mu)}(r)| - \varepsilon \qquad (r \in \varphi_{\tau_0}^{\tau -1}(U)) .$$

Since $|\hat{\mu}(r)| \geqq |\widehat{P_{\tau 0}(\mu)}(r)| - |\widehat{\mu - P_{\tau 0}(\mu)}(r)|$, we have from (4.13)

$$|\widehat{\mu - P_{\tau 0}(\mu)}(r)| > \varepsilon \qquad (r \in \varphi_{\tau_0}^{\tau -1}(U)) ,$$

and this contradicts (a).

(c) Suppose that there exists $r_0 \in \Gamma_{\tau 0}$ such that r_0 is not in Γ_τ, then there exists $\varepsilon > 0$ and $\mu_1, \cdots, \mu_n \in M(G(\tau))$ such that

$$\bigcap_{i=1}^{n} \{r \in \mathfrak{M} : |\hat{\mu}_i(r) - \hat{\mu}_i(r_0)| < \varepsilon\} \cap \Gamma_\tau = \emptyset .$$

Put $\lambda_i = \mu_i - \hat{\mu}_i(r_0)\delta_0$ $(i=1, \cdots, n)$, $\lambda = \sum_{i=1}^{n} \tilde{\lambda}_i * \lambda_i$, where δ_0 denotes the unit mass at $0 \in G$, then we have

$$\{r \in \mathfrak{M} : |\hat{\hat{\lambda}}(r)| < \varepsilon^2\} \cap \Gamma_\tau \subset \bigcap_{i=1}^{n} \{r \in \mathfrak{M} : |\hat{\hat{\lambda}}_i(r)| < \varepsilon\} \cap \Gamma_\tau = \emptyset . \tag{4.14}$$

Since $\hat{\hat{\lambda}}(r_0) = \sum_{i=1}^{n} |\hat{\hat{\lambda}}_i(r_0)|^2 = 0$, we can choose a neighborhood W of r_0 in Γ_{τ_0} such that

$$|\widehat{P_{\tau_0}(\lambda)}(r)| (= \widehat{\widehat{P_{\tau_0}(\lambda)}}(r) = \hat{\hat{\lambda}}(r)) < \varepsilon^2/2 \qquad (r \in W),$$

and we have

$$\begin{aligned} |\hat{\lambda}(r)| &\leqq |\widehat{P_{\tau_0}(\lambda)}(r)| + |\widehat{\lambda - P_{\tau_0}(\lambda)}(r)| \\ &< \varepsilon^2/2 + |\widehat{\lambda - P_{\tau_0}(\lambda)}(r)| \qquad (r \in \varphi_{\tau_0}^{\tau^{-1}}(W)) . \end{aligned} \tag{4.15}$$

From (4.14) we get $|\hat{\hat{\lambda}}(r)| = |\hat{\lambda}(r)| \geqq \varepsilon^2$ $(r \in \Gamma_\tau)$, and if we combine this with (4.15) we have

$$\varepsilon^2/2 < |\widehat{\lambda - P_{\tau_0}(\lambda)}(r)| \qquad (r \in \varphi_{\tau_0}^{\tau^{-1}}(W)) .$$

Since $\lambda - P_{\tau_0}(\lambda) \in M(G(\tau_0))^\perp$, this contradicts (a).

§ 5. Consideration of the general case.

LEMMA 5.1. *Suppose that $H(\sigma)$ is an open subgroup of a LCA group H'. We consider $M(H(\sigma))$ as a closed subalgebra of $M(H')$ in the natural way. We denote by Λ' and Λ_σ the dual groups of H' and $H(\sigma)$ respectively, and by φ we denote the natural open continuous homomorphism of Λ' onto Λ_σ such that*

$$(x, \varphi(r)) = (x, r) \qquad (x \in H(\sigma),\ r \in \Lambda') .$$

Suppose that there exist $r_0 \in \Lambda_\sigma$, a sequence $[W_n]_{n=1}^{\infty}$ of elements of the coset ring of Λ_σ, and sequences $[\lambda'_n \in M(H(\sigma))]_{n=1}^{\infty}$, $[\lambda_n \in M(H') \cap M(H(\sigma))^\perp]_{n=1}^{\infty}$ such that

$$\begin{aligned} & W_1 \supset W_2 \supset W_3 \supset \cdots \ni r_0 , \\ & \left.\begin{aligned} &\hat{\lambda}'_n(r) = \hat{\lambda}_n(r) = 0 \quad (r \notin \varphi^{-1}(W_n)) \\ &\hat{\lambda}_n(r) = \hat{\lambda}_{n+1}(r) \qquad (r \in \varphi^{-1}(W_{n+1})) \end{aligned}\right\} \quad (n = 1, 2, \cdots) \\ & \lim_{n\to\infty} \sup_{r \in \varphi^{-1}(W_n)} |\widehat{\lambda_n + \lambda'_n}(r)| = 0 , \end{aligned} \tag{5.1}$$

then we have

$$\hat{\lambda}_n(r) = 0 \qquad (r \in \varphi^{-1}(r_0 + F)) , \tag{5.2}$$

where F denotes the connected component of $0\in\Lambda_{\sigma}$.

PROOF. Since $\{\hat{\lambda}': \lambda'\in M(H(\sigma))\}$ and $\{\hat{\lambda}: \lambda\in M(H')\cap M(H(\sigma))^{\perp}\}$ are translation invariant and Λ_{σ}/F is totally disconnected, we may assume without loss of generality that $r_0=0$ and each W_n $(n=1, 2, \cdots)$ is an open compact subgroup of Λ_{σ}.

First we consider the case that F is compact. We assume that (5.2) does not hold, that is, there exists $r_0'\in\varphi^{-1}(r_0+F)$ such that $|\hat{\lambda}_n(r_0')|=\delta>0$, and derive a contradiction. Also we may assume $r_0'=0$ since each W_n contains F. Choose neighborhoods U and V of $r_0'=0$ in Λ' such that

$$\varphi^{-1}(W_1)\supset U\supset V+V\,, \tag{5.3}$$

$$|\hat{\lambda}_n(r)|>\delta/2 \qquad (r\in U\cap\varphi^{-1}(W_n)\,;\ n=1, 2, \cdots)\,.$$

Since $\operatorname{Ker}\varphi$ is equal to the annihilator of $H(\sigma)$ in Λ', we have that $\operatorname{Ker}\varphi$ is compact. Combining this with the assumption that F is compact, we get that $\varphi^{-1}(F)$ is compact. From this it follows that there exists a finite number of elements $r_1, r_2, \cdots, r_l\in\varphi^{-1}(F)$ such that

$$\bigcup_{i=1}^{l}(V+r_i)\supset\varphi^{-1}(F)\,. \tag{5.4}$$

From (5.3) and (5.4) we have

$$\bigcup_{i=1}^{l}(U+r_i)\supset\bigcup_{i=1}^{l}(V+V+r_i)\supset\varphi^{-1}(F)+V\,. \tag{5.5}$$

Since $(\varphi^{-1}(F)+V)/\varphi^{-1}(F)$ is a neighborhood of 0 of the totally disconnected group $\Lambda'/\varphi^{-1}(F)$, we can choose an open compact subgroup $\widetilde{W}$ of Λ' such that

$$\bigcup_{i=1}^{l}(U+r_i)\supset\widetilde{W}\supset\varphi^{-1}(F)\,. \tag{5.6}$$

Again observe that we can assume without loss of generality that $\varphi(\widetilde{W})=W_1$. For each element $r\in\varphi^{-1}(W_n)$ $(\subset\varphi^{-1}(W_1)=\widetilde{W}\subset\bigcup_{i=1}^{l}(U+r_i))$ there exists a positive integer i such that $U+r_i\ni r$, that is $U\ni r-r_i$. Since $\varphi^{-1}(W_n)\supset\varphi^{-1}(F)\ni r_i$, we have $\varphi^{-1}(W_n)\ni r-r_i$ and thus $U\cap\varphi^{-1}(W_n)\ni r-r_i$. This shows $U\cap\varphi^{-1}(W_n)+r_i\ni r$ and hence

$$\bigcup_{i=1}^{l}(U\cap\varphi^{-1}(W_n)+r_i)\supset\varphi^{-1}(W_n)\,. \tag{5.7}$$

If we denote by t_n the number of elements of the finite group W_1/W_n $(n=1, 2, \cdots)$, then we have from (5.3), (5.6) and (5.7) that

$$\begin{aligned}(5.8)\qquad \sqrt{t_n}\,\|\hat{\lambda}_n\|_2 &\geqq (\sqrt{t_n}/l)\|\sum_{i=1}^{l}|\hat{\lambda}_n(r-r_i)|\,\|_2\\ &\geqq (\sqrt{t_n}/l)\|(\delta/2)\sum_{i=1}^{l}\chi_{(U\cap\varphi^{-1}(W_n)+r_i)}\|_2\\ &\geqq (\delta\sqrt{t_n}/2l)\|\chi_{\varphi^{-1}(W_n)}\|_2=(\delta/2l)\|\chi_{\tilde{W}}\|_2>0 \quad (n=1,2,\cdots),\end{aligned}$$

where $\chi_{\varphi^{-1}(W_n)}$ denotes the characteristic function of $\varphi^{-1}(W_n)$ and etc.

On the other hand, since $\varphi^{-1}(W_n)$ is compact we have

$$\widehat{\lambda_n+\lambda'_n}\in L^1(\Lambda')\cap L^2(\Lambda') \qquad (n=1,2,\cdots),$$

and by the inversion theorem, we have λ_n, $\lambda'_n\in L^1(H')$. If we choose Borel functions f_n and f'_n on H' such that

$$f_n dm=d\lambda_n\,,\qquad f'_n dm=d\lambda'_n \qquad (n=1,2,\cdots),$$

where m denotes the Haar measure on H', then we have f_n, $f'_n\in L^1(H')\cap L^2(H')$, and from the last condition of (5.1) using the Plancherel theorem we get

$$(5.9)\qquad \lim_{n\to\infty}\sqrt{t_n}\|f_n+f'_n\|_2=\lim_{n\to\infty}\sqrt{t_n}\|\widehat{\lambda_n+\lambda'_n}\|_2=0\,.$$

Since $\lambda_n\perp\lambda'_n$ $(n=1,2,\cdots)$, we have from (5.9), using the Plancherel theorem again that

$$\lim_{n\to\infty}\sqrt{t_n}\|\hat{\lambda}_n\|_2=\lim_{n\to\infty}\sqrt{t_n}\|f_n\|_2=0\,,$$

and this contradicts (5.8). Hence we conclude that (5.2) holds in the case that F is compact.

Suppose next that F is not compact and that (5.2) does not hold, that is there exists $r'_0\in\varphi^{-1}(r_0+F)$ such that $|\hat{\lambda}_n(r'_0)|=\delta>0$. As before we can assume without loss of generality that $r'_0=0$ and that $W_1=R^m\times K$, where K is a compact subgroup of Λ_σ and R^m is a closed subgroup of Λ_σ isomorphic with the $m(>0)$-dimensional real vector group.

Let H_1 be the annihilator of $\varphi^{-1}(K)$ and let $\bar{\lambda}_n$ and $\bar{\lambda}'_n$ be elements of $M(H'/H_1)$ such that (cf. p. 53 [5])

$$(5.10)\qquad \left.\begin{aligned}\int_{H'}f(\phi(x))d\lambda_n(x)&=\int_{H'/H_1}fd\bar{\lambda}_n\\ \int_{H'}f(\phi(x))d\lambda'_n(x)&=\int_{H'/H_1}fd\bar{\lambda}'_n\end{aligned}\right\}\quad (f\in C_0(H'/H_1))\,,$$

where ϕ is the natural continuous homomorphism of H' onto H'/H_1. Since (5.10) also holds for any bounded Borel function on H'/H_1, we have

$$(5.11)\qquad \hat{\lambda}_n(r)=\hat{\bar{\lambda}}_n(r)\,,\qquad \hat{\lambda}'_n(r)=\hat{\bar{\lambda}}'_n(r)\qquad (r\in\varphi^{-1}(K),\ n=1,2,\cdots),$$

and it is easy to see from (5.10) that

$$\bar{\lambda}'_n \in M(H(\sigma)/H_1), \qquad \bar{\lambda}_n \in M(H(\sigma)/H_1)^{\perp}. \tag{5.12}$$

From (5.11) and (5.12), and by the discussion of the compact case, we get

$$\hat{\lambda}_n(r) = \hat{\bar{\lambda}}_n(r) = 0 \qquad (r \in \varphi^{-1}(K')),$$

where K' is the connected component of 0 in K_1. Since $\varphi^{-1}(K') \supset \mathrm{Ker}\,\varphi \ni r'_0 = 0$, this contradicts the first assumption and thus (5.2) holds in this case and Lemma 5.1 is proved.

THEOREM 5.2. *Let h be a homomorphism of $L^*(G(\tau))$ into $M(H(\sigma))$, then there exists a natural norm-preserving extension of h to a homomorphism of $M(G(\tau))$ into $M(H(\sigma))$.*

PROOF. Let $\bar{H}$ be the Bohr compactification of $H(\sigma)$, let Λ_σ be the dual group of $H(\sigma)$, and let σ_d be the discrete topology on Λ_σ. We can find an element σ_0 of $\mathfrak{T}(\bar{H})$ such that $\eta' = \eta_{\sigma_0}^{-1} \circ \eta$ is an open continuous map of $H(\sigma)$ into $\bar{H}(\sigma_0)$, where η is the natural continuous isomorphism of $H(\sigma)$ onto a dense subgroup of $\bar{H}$, and η_{σ_0} is the natural continuous isomorphism of $\bar{H}(\sigma_0)$ onto $\bar{H}$.

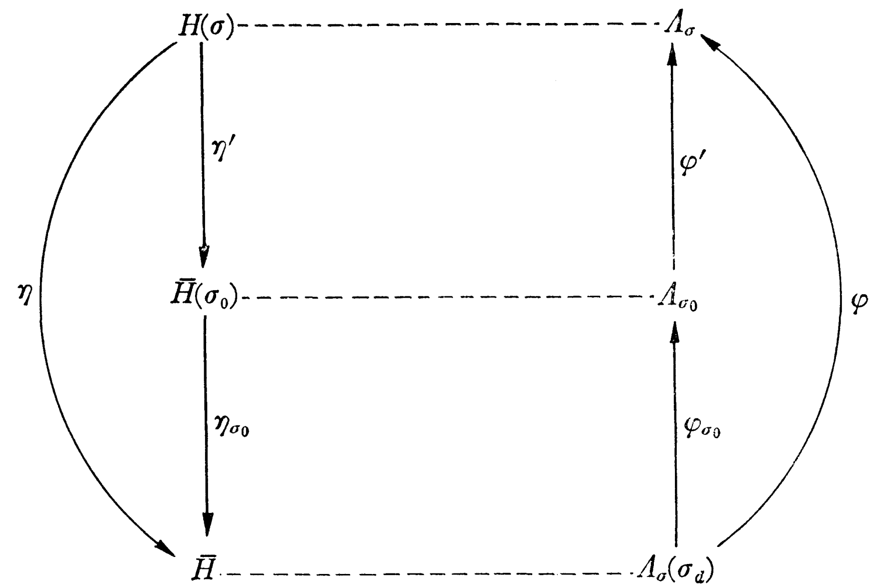

Fig. 2.

If we denote by Λ_{σ_0} the dual group of $\bar{H}(\sigma_0)$, and if we put $\varphi'(r) = r \circ \eta'$ $(r \in \Lambda_{\sigma_0})$, then φ' is an open continuous homomorphism of Λ_{σ_0} onto Λ_σ. If we denote by φ and φ_{σ_0} the natural continuous isomorphism of $\Lambda_\sigma(\sigma_d)$ onto Λ_σ and the natural continuous isomorphism of $\Lambda_\sigma(\sigma_d)$ onto a dense subgroup of Λ_{σ_0} respectively, then we have $\varphi' \circ \varphi_{\sigma_0} = \varphi$. Clearly the annihilator of $\eta'(H(\sigma))$ is the kernel of φ' and that $\Lambda_\sigma \cong \Lambda_{\sigma_0}/\mathrm{Ker}\,\varphi'$.

By Theorem 4.1 of [4], there exist a subset Y of Λ_σ and a map α of Y

into Γ^* such that $[Y, \alpha]$ and the corresponding $[\alpha_{\tau'}, Y_{\tau'}, h_{\tau'}]_{\tau'\in\mathfrak{T}(G(\tau))}$ satisfies (4.1) of [4] and conditions 1), 2) of Theorem 4.1 of [4]. We put $Y'=\varphi^{-1}(Y)$, $\alpha'=\alpha\circ\varphi$. Then it is easy to see that the corresponding $[\alpha'_{\tau'}, Y'_{\tau'}, h'_{\tau'}]_{\tau'\in\mathfrak{T}(G(\tau))}$ satisfies the conditions 1), 2) of Theorem 4.1 of [4], and there exists a homomorphism h' of $L^*(G(\tau))$ into $M(\bar{H})$ such that for each $\mu\in L^*(G(\tau))$ we have

$$\widehat{h'(\mu)}(r)=\begin{cases}\hat{\mu}(\alpha'(r)): r\in Y'\\ 0 \quad : r\in \Lambda_\sigma(\sigma_d)-Y'.\end{cases}$$

By Proposition 3.4, there exists a norm-preserving extension of h' to a homomorphism $\bar{h}'$ of $M(G(\tau))$ into $M(\bar{H})$ such that

$$\widehat{\bar{h}'(\mu)}(r)=\begin{cases}\hat{\mu}(\alpha'(r))=\hat{\hat{\mu}}(\alpha'(r)): r\in Y'\\ 0 \quad : r\in \Lambda_\sigma(\sigma_d)-Y'\end{cases} \qquad (\mu\in M(G(\tau))).$$

Let P_{σ_0} be the projection of $M(\bar{H})$ onto $M(\bar{H}(\sigma_0))$, and put $\bar{h}=P_{\tau_0}\circ\bar{h}'$. We identify $M(H(\sigma))$ with the closed subalgebra $\{\mu\in M(\bar{H}): \mu$ is concentrated on $\eta(H(\sigma))\}$ of $M(\bar{H})$. In this point of view h' is a homomorphism of $L^*(G(\tau))$ into $M(H(\sigma))$ $(\subset M(\bar{H}))$, and each $h'_{\tau'}$ $(\tau'\in\mathfrak{T}(G(\tau)))$ is a homomorphism of $L^1(G(\tau'))$ into $M(H(\sigma))$ $(\subset M(\bar{H}))$, and that $h'=h$, $h'_{\tau'}=h_{\tau'}$. Therefore if we show that $\bar{h}(\mu)$ $(\mu\in M(G(\tau)))$ is concentrated in $\eta(H(\sigma))$, this will complete the proof of Theorem 5.2. We extend $h'_{\tau'}$ $(\tau'\in\mathfrak{T}(G(\tau)))$ to the natural homomorphism of $M(G(\tau'))$ into $M(H(\sigma))$ (cf. Theorem 1, ii of [4]) and denote this extension by the same symbol $h'_{\tau'}$.

Let μ be an element of $M(G(\tau))$ and write $\bar{h}'(\mu)=\mu_1+\mu_2+\mu_3$, where $\mu_1\in M(H(\sigma))$, $\mu_2\in M(\bar{H}(\sigma_0))\cap M(H(\sigma))^\perp$ and $\mu_3\in M(\bar{H}(\sigma_0))^\perp$. Our task is to show that $\mu_2=0$.

Suppose that $\mu_2\neq 0$, then there exists $r^*\in\Lambda_\sigma(\sigma_d)$ such that $\hat{\mu}_2(r^*)\neq 0$. Choose a neighborhood W of $\varphi_{\sigma_0}(r^*)$ such that $\hat{\mu}_2(r)\neq 0$ $(r\in\varphi_{\sigma_0}^{-1}(W))$, and choose an open coset W_0 in Λ_σ such that $\varphi(r^*)\in W_0\subset\varphi'(W)+F$ and W_0/F is compact, where F is the connected component of $0\in\Lambda_\sigma$. Therefore for each $r\in W_0$ there exists $r'\in\varphi'^{-1}(r+F)$ such that

$$\hat{\mu}_2(r')\neq 0. \tag{5.13}$$

Put $M_1=\sup\limits_{r\in W_0}\sup\limits_{\alpha(r)\in_{\tau'\in S} S}[\|P_{\tau'}(\mu)\|]$, then there exists $r_1\in W_0$, $S_1\subset\mathfrak{T}(G(\tau))$ and $\tau_1\in S_1$ such that

$$\alpha(r_1)\in\Gamma_{S_1}, \qquad M_1\leqq\|P_{\tau_1}(\mu)\|+1/2. \tag{5.14}$$

We have from (5.14)

$$|(\widehat{\widehat{\mu-P_{\tau_1}(\mu)}})(\alpha(r))|\leqq 1/2 \qquad (r\in W_0\cap Y_{\tau_1}).$$

Put $W_1 = Y_{\tau_1} \cap W_0$, and put $M_2 = \sup_{r \in W_1} \sup_{\alpha(r) \in \Gamma_S,\, S \ni \tau'} [\|P_{\tau'}(\mu)\|]$, then there exist $r_2 \in W_1$, $S_2 \subset \mathfrak{T}(G(\tau))$ and $\tau_2 \in S_2$ such that

$$(5.15) \qquad \alpha(r_2) \in \Gamma_{S_2}, \qquad M_2 \leqq \|P_{\tau_2}(\mu)\| + 1/2^2 .$$

From (5.15) we get

$$|(\widehat{\mu - P_{\tau_2}(\mu)})(\alpha(r))| \leqq 1/2^2 \qquad (r \in W_1 \cap Y_{\tau_2}) .$$

If we put $W_2 = W_1 \cap Y$, and continue in the same way, we get a sequence $[W_n, r_n, \tau_n, S_n]_{n=1}^{\infty}$ such that

(5.16) $[W_n]_{n=1}^{\infty}$ is a decreasing sequence of elements of the coset ring of Λ_σ.

$$r_n \in W_{n-1},\ \alpha(r_n) \in \Gamma_{S_n},\ \tau_n \in S_n,\ W_n = W_{n-1} \cap Y_{\tau_n} \quad (n = 1, 2, \cdots),$$

$$|(\widehat{\mu - P_{\tau_n}(\mu)})(\alpha(r))| \leqq 1/2^n \qquad (r \in W_n) .$$

Since W_0/F is compact, $\bigcap_{n=1}^{\infty} W_n$ is not empty.

Let $r_0 \in \bigcap_{n=1}^{\infty} W_n$ and let $\nu_n \in M(H(\sigma))$ such that $\hat{\nu}_n$ is the characteristic function of W_n, and put

$$(5.17) \qquad \bar{h}'(\mu - P_{\tau_n}(\mu)) = \mu_1^{(n)} + \mu_2^{(n)} + \mu_3^{(n)} ,$$

where $\mu_1^{(n)} \in M(H(\sigma))$, $\mu_2^{(n)} \in M(\bar{H}(\sigma_0)) \cap M(H(\sigma))^{\perp}$, and $\mu_3^{(n)} \in M(\bar{H}(\sigma_0))^{\perp}$. We have from the definition of ν_n and $\bar{h}'$ that

$$(5.18) \qquad \nu_n * \bar{h}'(P_{\tau_n}(\mu)) = \nu_n * h'_{\tau_n}(P_{\tau_n}(\mu)) \in M(H(\sigma)) \qquad (n = 1, 2, \cdots) .$$

From (5.17) and (5.18) we can lead the relations

$$(5.19) \qquad \nu_n * \mu_2^{(n)} = \nu_n * \mu_2 , \qquad \nu_n * \mu_3^{(n)} = \nu_n * \mu_3 \qquad (n = 1, 2, \cdots) .$$

From the last condition of (5.16), we have by Theorem 4.1 (b) that

$$(5.20) \qquad |\hat{\mu}_1^{(n)}(r) + \hat{\mu}_2^{(n)}(r)| \leqq 1/2^n \qquad (r \in \varphi^{-1}(W_n),\ n = 1, 2, \cdots) .$$

Since $\varphi_{\sigma_0}(\varphi^{-1}(W_n))$ is dense in $\varphi'^{-1}(W_n)$, we have from (5.19)

$$(5.21) \qquad |\hat{\mu}_1^{(n)}(r) + \hat{\mu}_2^{(n)}(r)| \leqq 1/2^n \qquad (r \in \varphi'^{-1}(W_n),\ n = 1, 2, \cdots) .$$

If we put

$$(5.22) \qquad \nu_n * \mu_1^{(n)} = \lambda'_n , \qquad \nu_n * \mu_2^{(n)} = \lambda_n \qquad (n = 1, 2, \cdots) ,$$

it follows from (5.21) that we can apply Lemma 5.1 to $[\lambda'_n]_{n=1}^{\infty}$ and $[\lambda_n]_{n=1}^{\infty}$ of (5.22), and we have

(5.23) $$\hat{\mu}_2^{(n)}(r)=\hat{\lambda}_n(r)=0 \qquad (r\in\varphi'^{-1}(r_0+F)).$$

On the other hand, since r_0 is an element of W_0, we have from (5.13) that there exists $r_0'\in\varphi'^{-1}(r_0+F)$ such that $\hat{\mu}_2(r_0')=\hat{\mu}_2^{(n)}(r_0')\neq 0$ and this contradicts (5.23). This shows that $\mu_2=0$, and thus we have $\bar{h}(\mu)=\mu_1\in M(H(\sigma))$. Q.E.D.

REMARK. The following example shows the reason why the discussion in § 3 is not enough to solve our problem in the general case.

EXAMPLE. Let $H(\sigma)$ be a discrete group such that its dual group Λ_σ is an infinite totally disconnected group, and let $G(\tau)$ be a LCA group such that there exists an open subgroup G' of $G(\tau)$ isomorphic with $\prod_{n=1}^{\infty}T^{(n)}$ and $G(\tau)/G'\cong H(\sigma)$, where $T^{(n)}$ $(n=1,2,\cdots)$ is the circle group. We identify G' with $\prod_{n=1}^{\infty}T^{(n)}$. We choose open compact subgroups $\{W_n: n=1,2,\cdots\}$ of Λ_σ such that

(5.24) $$W_1\supset W_2\supset W_3\supset\cdots,$$
$$\bigcap_{n=1}^{\infty}W_n \text{ is not an open subgroup of } \Lambda_\sigma.$$

Let τ_n $(n=1,2,\cdots)$ be an element of $\mathfrak{T}(G(\tau))$ such that $\eta_{\tau_n}^{\tau\,-1}(\prod_{k=n+1}^{\infty}T^{(k)})$ is an open compact subgroup of $G(\tau_n)$. If we put $S=\{\tau'\in\mathfrak{T}(G(\tau)): \tau'\leqq\tau_n, n=1,2,\cdots\}$, then S is a directed subset of $\mathfrak{T}(G(\tau))$ (cf. p. 288 [4]). Since the annihilator of G' is an open compact subgroup of Γ_τ isomorphic with Λ_σ, there exists an open continuous isomorphism $\tilde{\alpha}$ of W_1 into Γ_τ. Put $Y=W_1$. For each $r\in Y-\bigcap_{n=1}^{\infty}W_n$, there exists a positive integer m such that $r\in W_m$, $r\notin W_{m+1}$ and put $\alpha(r)=\varphi_{\tau_m}^{\tau}(\tilde{\alpha}(r))$. If $r\in\bigcap_{n=1}^{\infty}W_n$, we put $\alpha(r)=(\varphi_{\tau'}^{\tau}(\tilde{\alpha}(r)))_{\tau'\in S}\in\Gamma_S$. Then α is a map of Y into Γ^* and the corresponding $\alpha_{\tau'}$, $Y_{\tau'}$ and $h_{\tau'}$ $(\tau'\in\mathfrak{T}(G(\tau)))$ satisfies the conditions 1), 2) of Theorem 4.1 of [4], and there exists a homomorphism h of $L^*(G(\tau))$ into $M(H(\sigma))$ such that

$$\widehat{h(\mu)}(r)=\begin{cases}\hat{\mu}(\alpha(r)): & r\in Y\\ 0 & : r\in\Lambda_\sigma-Y\end{cases}\qquad (\mu\in L^*(G(\tau))).$$

The fact that $[\alpha_{\tau'}, Y_{\tau'}, h_{\tau'}]_{\tau'\in\mathfrak{T}(G(\tau))}$ satisfies the condition 2) of Theorem 4.1 of [4] is shown as follows. For each $\tau'\in\mathfrak{T}(G(\tau))$ let $h_{\tau'}^*$ be a homomorphism of $L^1(G(\tau'))$ into $M(H(\sigma))$ such that

$$\widehat{h_{\tau'}^*(\mu)}(r)=\begin{cases}\hat{\mu}(\varphi_{\tau'}^{\tau}(\tilde{\alpha}(r))): & r\in Y\\ 0 & : r\in\Lambda_\sigma-Y\end{cases}\qquad (\mu\in L^1(G(\tau')))$$

and for each positive integer n let ν_n be an idempotent measure in $M(H(\sigma))$ such that $\hat{\nu}_n$ is the characteristic function of W_1-W_n, then we have $\|h_{\tau'}^*\|=1$

and $\|\nu_n\| \leqq 2$ (cf. p. 79, 42.1 of [5]). If $\tau' \notin S$, there exists a positive integer m such that $\tau' \leqq \tau_{m-1}$, $\tau' \nleqq \tau_m$, and thus we have $h_{\tau'}(\mu) = \nu_m * h^*_{\tau'}(\mu)$ $(\mu \in L^1(G(\tau')))$. If $\tau' \in S$, $h_{\tau'} = h^*_{\tau'}$, and consequently we get $\sup_{\tau' \in \mathfrak{T}(G(\tau))} \|h_{\tau'}\| \leqq 2$, and thus the condition 2) holds.

Next we choose a sequence $\{x^{(n)} \in T^{(n)},\ n = 1, 2, \cdots\}$ such that

$$\{x^{(1)}, x^{(2)}, \cdots, x^{(m)}\} \qquad (m = 1, 2, \cdots) \tag{5.25}$$

is independent as a subset of a circle group, and that each $x^{(k)}$ $(k = 1, 2, \cdots)$ has an infinite order (we use addition as group operation of $T^{(n)}$; cf. p. 97 [5]). We put $C = \prod_{n=1}^{\infty} \{0, x^{(n)}\}$, then C is a compact subset of G', and there exists a continuous measure $\mu \in M(G(\tau))$ such that

$$|\mu|(C) = |\mu|(G(\tau)), \qquad \hat{\mu}(0) \neq 0. \tag{5.26}$$

Let τ_0 be an arbitrary element of S and let F be the connected component of 0 in $G(\tau_0)$. Since $\tau_0 \leqq \tau_n$ $(n = 1, 2, \cdots)$, F is contained in $\eta^{\tau -1}_{\tau_0}(\prod_{k=n+1}^{\infty} T^{(k)})$ $(n = 1, 2, \cdots)$, that is

$$\eta^{\tau}_{\tau_0}(F) \subset \bigcap_{n=1}^{\infty} (\prod_{k=n+1}^{\infty} T^{(k)}) = \{0\},$$

and thus $G(\tau_0)$ is totally disconnected.

Let G'' be an open compact subgroup of $G(\tau_0)$, and let $x_1 = (x_1^{(k)})_{k=1}^{\infty}$, $x_2 = (x_2^{(k)})_{k=1}^{\infty}$ be elements of $\eta^{\tau -1}_{\tau_0}(C)$ such that $x_1 - x_2 \in G''$. If $x_1 \neq x_2$, there exists a non-empty set N of positive integers such that $x_1^{(k)} \neq x_2^{(k)}$ if and only if $k \in N$. By virtue of (5.25), 0 is the only character on $\prod_{k \in N} T^{(k)}$ which annihilates $x_1 - x_2$, and thus $\{n(x_1 - x_2) : n = 0, \pm 1, \pm 2, \cdots\}$ is a dense subgroup of $\prod_{k \in N} T^{(k)}$. Since G'' is compact, $\eta^{\tau}_{\tau_0}|_{G''}$ is a homeomorphism, and hence G'' contains $\eta^{\tau -1}_{\tau_0}(\prod_{k \in N} T^{(k)})$ as a compact subgroup of G''. This contradicts the fact that G'' is totally disconnected. Consequently we conclude that each coset of G'' contains at most one element of $\eta^{\tau -1}_{\tau_0}(C)$. Combining this with the fact that C is contained in the finite union of cosets of $\prod_{k=n+1}^{\infty} T^{(k)}$ $(n = 1, 2, \cdots)$, we get

$$\mu \in M(G(\tau_i)) \qquad (i = 1, 2, \cdots);\ \mu \notin M(G(\tau_0)), \tag{5.27}$$

and this shows that Theorem 3.1 is not valid for infinite n.

Let β_μ be a function on Λ_σ such that

$$\beta_\mu(r) = \begin{cases} \hat{\mu}(\alpha(r)) = \hat{\hat{\mu}}(\alpha(r)) : & r \in Y \\ 0 & : r \in \Lambda_\sigma - Y. \end{cases}$$

Using (5.27) we can lead another expression of β_μ, namely

$$\beta_\mu(r)=\begin{cases}(\hat{\mu}\circ\alpha-(\hat{\mu}\circ\alpha)\chi_{(\bigcap_{n=1}^{\infty}W_n)})(r): & r\in Y\\ 0 & : r\in \Lambda_\sigma - Y,\end{cases}$$

where $\chi_{(\bigcap_{n=1}^{\infty}W_n)}$ denotes the characteristic function of $\bigcap_{n=1}^{\infty}W_n$. Since $\bigcap_{n=1}^{\infty}W_n$ is not a neighborhood of 0 in Λ_σ, $\hat{\mu}(\tilde{\alpha}(0))=\hat{\mu}(0)\neq 0$ and $\hat{\mu}\circ\tilde{\alpha}$ is continuous at 0, we can see that β_μ is not continuous at $0\in\Lambda_\sigma$ (this is the reason why the discussion in § 3 is not enough to solve our problem in the general case).

Next we show that $\{\tau_n\}_{n=1}^{\infty}$ has no l. u. b. in $\mathfrak{T}(G(\tau))$, that is, $\mathfrak{T}(G(\tau))$ is not a σ-complete lattice with respect to the partial ordering $\leqq$. For this it is enough to show that there exists $\tau_0'\in S$ such that $\tau_0\lneqq\tau_0'$. Since G'' is totally disconnected and compact, $G''\cap\eta_{\tau_0}^{\tau^{-1}}(T^{(1)})$ is a finite group. Choose a finite subgroup D_1 of $T^{(1)}$ such that $D_1\supsetneqq\eta_{\tau_0}^{\tau}(G'')\cap T^{(1)}$. By the same way choose a finite subgroup D_2 of $T^{(2)}$ such that $D_2\supsetneqq(\eta_{\tau_0}^{\tau}(G'')+D_1)\cap T^{(2)}$. If we continue this process, we get a sequence of subgroups $\{D_n\subset T^{(n)}: n=1, 2, \cdots\}$ such that

$$D_{n+1}\supsetneqq(\eta_{\tau_0}^{\tau}(G'')+\sum_{k=1}^{n}D_k)\cap T^{(n+1)} \qquad (n=0, 1, 2, \cdots). \tag{5.28}$$

Put $D=\prod_{k=1}^{\infty}(D_k)$. Given an arbitrary positive integer n, there exists a finite number of elements $x_1, x_2, \cdots, x_l\in G(\tau)$ such that

$$\bigcup_{i=1}^{l}(x_i+(\prod_{k=n+1}^{\infty}T^{(k)}))\supset D. \tag{5.29}$$

Again choose a finite number of elements $y_1, y_2, \cdots, y_m\in G(\tau)$ such that

$$\bigcup_{i=1}^{m}(y_i+(\prod_{k=n+1}^{\infty}T^{(k)}))\supset\eta_{\tau_0}^{\tau}(G''). \tag{5.30}$$

This choice is possible since $\tau_0\leqq\tau_n$. Summing (5.29) and (5.30) we get

$$\bigcup_{\substack{j=1,\cdots,m\\ i=1,\cdots,l}}(x_i+y_j+(\prod_{k=n+1}^{\infty}T^{(k)}))\supset D+\eta_{\tau_0}^{\tau}(G''). \tag{5.31}$$

Let τ_0' be an element of $\mathfrak{T}(G(\tau))$ such that $\Omega=\eta_{\tau_0'}^{\tau^{-1}}(D+\eta_{\tau_0}^{\tau}(G''))$ is an open compact subgroup of $G(\tau_0')$. From (5.28), $(\eta_{\tau_0}^{\tau^{-1}}(D)+G'')/G''$ is an infinite group and this shows that $\tau_0\lneqq\tau_0'$.

For each positive integer n, the restriction of $\eta_{\tau_n}^{\tau^{-1}}$ on $\bigcup_{\substack{i=1,\cdots,l\\ j=1,\cdots,m}}(x_i+y_j+(\prod_{k=n+1}^{\infty}T^{(k)}))$ is continuous and by (5.31) we have that $\eta_{\tau_n}^{\tau^{-1}}\circ\eta_{\tau_0'}^{\tau}|_{\Omega}$ is a continuous function. Since Ω is open in $G(\tau_0')$, we have $\tau_0'\leqq\tau_n$, and hence $\tau_0'\in S$. This was what we wanted to show.

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