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Remarks on codimension one foliations of spheres

By Tadayoshi MIZUTANI

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§ 1. Introduction.

In [1], B. Lawson constructed codimension one foliations of S^{2^k+3}, $k=1, 2, \cdots$. Recently, I. Tamura succeeded in proving that every odd dimensional homotopy sphere has a codimension one foliation [2]. In both cases, it was important that S^5 has a codimension one foliation. In this article, we shall show that Lawson's examples are obtained by a reduction theorem of S^1-bundles and that there exist other examples of foliations of S^5. These examples of S^5 are S^1-invariant, especially, Z_k-invariant for any positive integer k. Thus, we obtain also new types of foliations of five dimensional lens spaces.

All foliations considered are differentiable codimension one foliations unless otherwise stated.

§ 2. Fibrations over a circle.

Let η be the standard S^1-principal bundle over CP^n with total space S^{2n+1} and projection map η defined by $\eta(z_0, \cdots, z_n)=[z_0, \cdots, z_n]$, where $S^{2n+1}=\{(z_0, \cdots, z_n)\in C^{n+1}; |z_0|^2+\cdots+|z_n|^2=1\}$ and $[z_0, \cdots, z_n]$ denotes the homogeneous coordinate.

PROPOSITION 1. *Let d be a positive integer and let M^{2n-2} be a $(2n-2)$-dimensional connected closed differentiable submanifold of CP^n such that the fundamental class of M^{2n-2} represents d-times the generator of $H_{2n-2}(CP^n, Z)\cong Z$. Let $\nu(M)$ denote the closed tubular neighbourhood of M^{2n-2} in CP^n. Then η restricted to $W^{2n}=CP^n-\mathrm{int}\,\nu(M)$ has a Z_d-reduction.*

PROOF. Let α be the canonical generator of $H^2(CP^n, Z)$ and let i be the inclusion map $W^{2n}\to CP^n$. To prove the proposition, it is sufficient to show that $d\cdot(i^*(\alpha))=0$ in $H^2(W^{2n}, Z)$. This follows from the following observation.

Consider the exact sequence of groups; $Z_d\to S^1\to S^1$, here the first map is a natural injection and the second map is multiplication by d. Passing to classifying spaces of bundles, we have a fibration; $BZ_d\to BS^1\to BS^1$, or $K(Z_d, 1)\to K(Z, 2)\to K(Z, 2)$. Hence, for a CW-complex X, we have an exact sequence $[X, K(Z_d, 1)] \to [X, K(Z, 2)] \to [X, K(Z, 2)]$, or $H^1(X, Z_d) \xrightarrow{\delta} H^2(X, Z) \xrightarrow{d} H^2(X, Z)$, where δ is a Bockstein map associated to $0 \to Z \to Z \to Z_d \to 0$ and d_* is a map such that $d_*(x) = dx$ for each element x of $H^2(X, Z)$. Therefore, for an element of order d, say $\tilde{\alpha}$, in $H^2(X, Z)$, there is an element $\tilde{\beta}$ in $H^1(X, Z_d)$ such that $\delta(\tilde{\beta}) = \tilde{\alpha}$. Clearly, $\tilde{\beta}$ represents a Z_d-bundle over X. Thus, the S^1-bundle corresponding to $\tilde{\alpha}$ has a Z_d-reduction.

Now, by the cohomology exact sequence of a pair with coefficient in Z, we have the following exact sequence,

$$\longrightarrow H^1(W) \xrightarrow{\delta} H^2(CP^n, W) \xrightarrow{j^*} H^2(CP^n) \xrightarrow{i^*} H^2(W) \longrightarrow .$$

By excision isomorphism and Poincaré duality, we have isomorphisms,

$$\varphi : H^2(CP^n, W) \longrightarrow H_{2n-2}(M), \qquad \phi : H^2(CP^n) \longrightarrow H_{2n-2}(CP^n).$$

Thus, we have the following diagram which is commutative up to sign.

$$\begin{tikzcd}
\longrightarrow H^1(W) \arrow[r, "\delta"] \arrow[d, equal] & H^2(CP^n, W) \arrow[r, "j^*"] \arrow[d, "\varphi"] & H^2(CP^n) \arrow[r, "i^*"] \arrow[d, "\phi"] & H^2(W) \longrightarrow \arrow[d, equal] \\
\longrightarrow H^1(W) \arrow[r] & H_{2n-2}(M) \arrow[r, "l_*"] & H_{2n-2}(CP^n) \arrow[r] & H^2(W) \longrightarrow
\end{tikzcd}$$

where l_* is the map induced by the inclusion $l : M \to CP^n$.

For the generator α of $H^2(CP^n)$, by the assumption, there exists an element β in $H_{2n-2}(M)$ such that $l_*(\beta) = d \cdot \phi(\alpha)$. Hence, $d \cdot i^*(\alpha) = i^* \circ \phi^{-1}(d \cdot \phi \alpha) = i^* \circ \phi^{-1} \circ l_*(\beta) = i^* \circ j^* \circ \varphi^{-1}(\beta) = 0$. This completes the proof of the Proposition 1.

PROPOSITION 2. *Let W^{2n} be as in Proposition* 1, *then $\eta^{-1}(W)$ is a fibration over S^1. The fibre is diffeomorphic to a covering space of W.*

PROOF. Let $\eta^{-1}(W) = E$. By Proposition 1, S^1-bundle $\eta|_E : E \to W$ has a Z_d-reduction. Let $\widetilde{W}$ be the Z_d-principal bundle associated to this bundle. Then, E is bundle equivalent to $S^1 \underset{Z_d}{\times} \widetilde{W} = \{(t, w) \in S^1 \times \widetilde{W}\}/\sim$, where $\sim$ denotes an equivalence relation such that $(t, w) \sim (t', w')$, if and only if, $t = t' \cdot g$, $w = w' \cdot g$, for some g in Z_d. Let $\pi_1 : S^1 \times \widetilde{W} \to S^1$ be the projection to the first factor. Passing to the quotient, we have a map $\pi : S^1 \underset{Z_d}{\times} \widetilde{W} \to S^1/Z_d$. It can be easily checked that π is a bundle projection over S^1 with fibre $\widetilde{W}$. This completes the proof.

§ 3. Construction of foliations.

We prove the following fundamental lemma.

LEMMA. *Let E be an orientable differentiable manifold with boundary and let $p : E \to S^1$ be a differentiable fibration. Then E has a foliation with each* *connected component of ∂E as a leaf.*

PROOF. By a collar, we identify $E \bigcup_{\partial E} \partial E \times [0, 1]$ with E, which is a union of manifolds E and $\partial E \times [0, 1]$ identified ∂E with $\partial E \times \{0\}$. Define the fibration $q = p|_{\partial E} \times id: \partial E \times [0, 1] \to S^1 \times [0, 1]$. There exists on $S^1 \times [0, 1]$ a non-zero smooth vector field $\mathcal{F}$ with the following properties; (1) $S^1 \times \{1\}$ is an orbit of $\mathcal{F}$. (2) The orbits of $\mathcal{F}$ intersect normally to $S^1 \times \{0\}$. (3) The natural S^1-action on $S^1 \times [0, 1]$ preserves the orbits of $\mathcal{F}$.

Then, $\{q^{-1}(\text{orbits of } \mathcal{F})\}$ and $\{\text{fibres of } p\}$ give a differentiable foliation of $E \bigcup_{\partial E} \partial E \times [0, 1]$ with $\partial E \times \{1\}$ as a union of leaves. This completes the proof.

Let M^{2n-2} be a submanifold of $\boldsymbol{CP}^n$ satisfying the conditions of Proposition 1, and let L^{2n-1} be a submanifold of S^{2n+1} which is the total space of η restricted over M^{2n-2}. The normal bundle of L^{2n-1} in S^{2n+1} is always trivial, so we have a decomposition:

$$S^{2n+1} = L^{2n-1} \times D^2 \cup E, \ E = S^{2n+1} - \operatorname{int}(L^{2n-1} \times D^2).$$

By Proposition 2, E is a fibre bundle over S^1, hence, by the above lemma, E has a foliation with ∂E as a compact leaf.

Thus, we have,

PROPOSITION 3. *In the above notation, if $L^{2n-1} \times D^2$ has a foliation with boundary as a compact leaf, then S^{2n+1} has a foliation.*

PROOF. Since both $L^{2n-1} \times D^2$ and E have foliations with boundaries as leaves, glueing them along the boundaries we have a foliation on S^{2n+1}.

Using this proposition, we are now going to construct foliations on spheres.

Let $M^2(d)$ be the non-singular curve (real dimension $=2$) in $\boldsymbol{CP}^2$ of degree d. The genus of $M(d)$ is given by $g = (d-1)(d-2)/2$. Thus $M(3)$ is diffeomorphic to $T^2 = S^1 \times S^1$ and the fundamental cycle $[T^2]$ is homologous to 3-times of $[\boldsymbol{CP}^1]$ which is a generator of $H_2(\boldsymbol{CP}^2, Z)$ (because the intersection $[T^2] \cdot [\boldsymbol{CP}^1] = 3$). Corresponding submanifold $L(3)$ of S^5 (see above proposition) is a fibre bundle over T^2, in particular, is a fibre bundle over S^1. Hence, according to preceding lemma, $L(3) \times D^2$ has a foliation with boundary as a compact leaf. Therefore, by Proposition 3, S^5 has a foliation. This is just the example of Lawson [**1**].

For $d = 1$, $M(1)$ is diffeomorphic to $\boldsymbol{CP}^1 = S^2$. Imbed a torus T^2 into a small disc D^4 contained in $\boldsymbol{CP}^2$ so that T^2 and S^2 do not intersect. Connecting T^2 and S^2 by a small tube, we can make a connected sum of T^2 and S^2 in $\boldsymbol{CP}^2$. It is apparent that the obtained submanifold of $\boldsymbol{CP}^2$ is diffeomorphic to T^2 and homologous to $M(1)$. Then the similar argument as above shows that we have another foliation of S^5. Since $M(2)$ is also diffeomorphic to S^2, the same argument as in the case $d=1$ holds for $d=2$. These are new types of foliations of S^5. We can easily check that all these example of foliations are S^1-invariant with respect to the natural S^1-action on S^5. That is for any $g \in S^1$, $g \cdot F_1$ is contained in some F_2, where F_1, F_2 are leaves. This shows that we have foliations on each five dimensional lens spaces.

We can now prove Lawson's result without using Milnor's fibration theorem.

THEOREM (Lawson [**1**]). *(2^k+3)-dimensional spheres have codimension one foliations, for $k=1, 2, \cdots$.*

PROOF. First we remark that if S^{n+2} has a foliation, then $S^n \times D^2$ has a foliation with boundary as a compact leaf. This can be proved as follows. Take a closed curve transversal to the leaves of S^{n+2} (such a curve always exists since S^{n+2} is compact). Taking away small tubular neighbourhood of this curve, we have a manifold diffeomorphic to $S^n \times D^2$. The leaves of S^{n+2} restricted to $S^n \times D^2$ are transversal to the boundary. As in lemma, we can modify the leaves in $S^n \times D^2$ so that the boundary is a leaf.

Let $M^{2n-2}(2)$ be a non-singular complex hypersurface in $\boldsymbol{CP}^n$ of degree 2. Then $M^{2n-2}(2)$ satisfies the conditions of Proposition 1. The corresponding S^1-bundle $L^{2n-1}(2)$ is known to be diffeomorphic to the tangent sphere bundle of S^n. Let π be the projection of this bundle. We have a fibration, $\pi \times id$: $L^{2n-1} \times D^2 \to S^n \times D^2$. By the above remark, if S^{n+2} has a foliation, then $L^{2n-1}(2) \times D^2$ has a foliation with boundary as a leaf which is the pull-back of the foliation on $S^n \times D^2$ by $\pi \times id$. Thus, by Proposition 3, we have a foliation on S^{2n+1}. But we have already constructed foliations on S^5. So, starting from $n=3$, we can inductively obtain foliations on 2^k+3 dimensional spheres, $k=1, 2, \cdots$. This completes the proof.

University of Tokyo

References

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[2] I. Tamura, Every odd dimensional homotopy sphere has a foliation of codimension one, Comm. Math. Helv. (to appear).