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Three-dimensional compact Kähler manifolds with positive holomorphic bisectional curvature

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§ 1. Introduction.

One of the most challenging problems in Riemannian geometry is to determine all compact Riemannian manifolds[1] with positive sectional curvature. As a special case, the following problem has been considered by Frankel [11].

Let M be a compact Kähler manifold of dimension n with positive sectional (or more generally, holomorphic bisectional) curvature. Is M necessarily biholomorphic to the complex projective space $P_n(C)$?

This is trivially true for $n=1$ since $P_1(C)$ is the only compact Riemann surface with positive first Chern class. The question has been answered affirmatively for $n=2$ by Frankel and Andreotti [11]; their proof depends on the classification of the rational surfaces. Recently, Howard and Smyth [18] have determined the compact Kähler surfaces of non-negative holomorphic bisectional curvature. In higher dimensions, this question has been answered affirmatively only under additional assumptions: 1) Pinching conditions (Howard [17]), or 2) Einstein-Kähler (Berger [2]) or constant scalar curvature (Bishop and Goldberg [4]).

The purpose of this paper is to answer the question above affirmatively for $n=3$, see Theorem 7.1. The proof given here leaves much to be desired, for it makes use of a difficult theorem of Aubin (see Lemma 7.3) and does not answer the following algebraic geometric question:

Let M be a compact complex manifold of dimension n with positive tangent bundle. Is M necessarily biholomorphic to $P_n(C)$?

This question, which is more general than the first one, has been answered affirmatively by Hartshorne [14] for $n=2$ by a purely algebraic method. It has been affirmatively answered also for the compact homogeneous complex manifolds [22] as well as for the complete intersection submanifolds of complex projective spaces [21]. In [21] we have shown that a 3-dimensional compact complex manifold M with positive tangent bundle admits a group

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1) All manifolds in this paper are connected unless otherwise stated.

of holomorphic transformations of dimension $\geqq 6$. In this paper we shall show that the dimension of the group is at least 7. We still do not know if for such a manifold the second Betti number is 1 and the third Betti number vanishes.

In concluding the introduction, we wish to express our thanks to Shigeru Iitaka for useful communication.

§ 2. Sufficient conditions for a manifold to be $P_n(C)$.

In this section we quote two general results which will be used in subsequent sections.

THEOREM 2.1. *Let M be an n-dimensional compact homogeneous complex manifold with positive (i.e., ample) tangent bundle. Then M is biholomorphic to the complex projective space $P_n(C)$.*

See [22] for its proof. The following theorem is proved in [23].

THEOREM 2.2. *Let M be an n-dimensional compact Kähler manifold whose first Chern class $c_1(M)$ is of the form*

$$c_1(M) = r\alpha ,$$

where α is a positive element of $H^{1,1}(M; Z)$ and r is an integer $\geqq n$. Then M is biholomorphic to either $P_n(C)$ or a hyperquadric in $P_{n+1}(C)$. If $r \geqq n+1$, then M is biholomorphic to $P_n(C)$.

We note that if $T(M) > 0$ in Theorem 2.2, then $r \geqq n$ implies that M is biholomorphic to $P_n(C)$ since a hyperquadric is a homogeneous complex manifold and cannot have positive tangent bundle by Theorem 2.1.

We remark that Theorem 2.2 is closely related to a theorem of Hirzebruch and Kodaira [16] that an n-dimensional compact Kähler manifold with positive first Chern class which is homeomorphic to $P_n(C)$ is biholomorphic to $P_n(C)$ and also to a similar theorem of Brieskorn [9] on a hyperquadric.

§ 3. Properties of 3-dimensional compact complex manifolds with positive tangent bundle.

We shall summarize main properties of compact complex manifolds M with $T(M) > 0$.

THEOREM 3.1. *Let M be a compact complex manifold with positive tangent bundle $T = T(M)$. Then*

(1) *The determinant line bundle* $\det(T)$ *is positive, i.e., $c_1(M)$ is positive;*

(2) $H^{p,0}(M; C) = H^{0,p}(M; C) = 0$ *for* $p \geqq 1$*;*

(3) $H^p(M; S^kT) = 0$ *for* $p \geqq 1$ *and* $k \geqq 0$,

where S^kT denotes the sheaf of germs of holomorphic sections of the k-th sym-

metric tensor power of T*;*

(4) *All Chern numbers of* M *are positive, in particular, the Euler number of* M *is positive;*

(5) *If* $\dim M=3$, *then*

$$c_1c_2[M]=24\,, \qquad c_1^3[M]=\textit{even}\,, \qquad c_3[M]=\textit{even}\,,$$

where $c_i=c_i(M)$ *denotes the i-th Chern class of* M*;*

(6) *If* $\dim M=3$, *then*

$$\dim H^0(M;T)=\left\{\frac{1}{2}(c_1^3-2c_1c_2+c_3)+\frac{5}{24}c_1c_2\right\}[M]\geqq 7\,.$$

We note that $H^0(M;T)$ is the space of holomorphic vector fields on M.

PROOF. (1) This is due to Hartshorne [**13**].

(2) This follows from (1) and the vanishing theorem of Kodaira.

(3) This has been proved in our previous paper [**21**; Corollary 2.5].

(4) This is due to Bloch and Gieseker [**5**].

(5) By (2), the arithmetic genus $\sum_{p=0}^{3}(-1)^p \dim H^{0,p}(M;\boldsymbol{C})$ is equal to 1. On the other hand, the Riemann-Roch theorem states that the arithmetic genus is equal to $\frac{1}{24}c_1c_2[M]$, see [**15**]. Hence, $c_1c_2[M]=24$. The Riemann-Roch theorem gives also

$$\chi(M;\det(T))=\sum(-1)^p \dim H^p(M;\det(T))=\left(\frac{1}{2}c_1^3+\frac{1}{8}c_1c_2\right)[M]\,.$$

Since $\chi(M;\det(T))$ is an integer, we may conclude that $\frac{1}{2}c_1^3$ is an integer. Another consequence of the Riemann-Roch theorem is

$$\chi(M;T)=\sum(-1)^p \dim H^p(M;T)=\left\{\frac{1}{2}(c_1^3-2c_1c_2+c_3)+\frac{5}{24}c_1c_2\right\}[M]\,.$$

This shows that $(c_1^3-2c_1c_2+c_3)[M]$ is an even integer and hence $c_3[M]$ is also even.

(6) From (3), we obtain $\chi(M;T)=\dim H^0(M;T)$. This gives the equality in (6). To prove the inequality $\dim H^0(M;T)\geqq 7$, we repeat the argument used to prove the inequality $\dim H^0(M;T)\geqq 6$ in [**21**]. Let

$$F=L(T^*)^{-1}$$

be the tautological positive line bundle over the projective bundle $P(T^*)$ associated with the cotangent bundle $T^*=T^*(M)$ as in [**21**; §4]. Let f be the first Chern class of the line bundle F. We have shown [**21**]

$$f^5[P(T^*)]=(c_1^3-2c_1c_2+c_3)[M]\,. \tag{3.1}$$

Since $F>0$ and hence f is positive, the left hand side of (3.1) is positive.

Hence, $\dim H^0(M; T) > \frac{5}{24} c_1 c_2[M] = 5$, as we have shown in [21]. Now assume that $\dim H^0(M; T) = 6$, i. e., $\frac{1}{2}(c_1^3 - 2c_1c_2 + c_3)[M] = 1$. From (3.1) we obtain

$$f^5[P(T^*)] = 2 . \tag{3.2}$$

On the other hand, we have

$$\dim H^0(P(T^*); F) = 6 . \tag{3.3}$$

This is a consequence of (see [21; Theorem 2.1])

$$H^*(M; T) = H^*(P(T^*); F) .$$

From (3.2) and (3.3) we may conclude that there are at most two common zeros (i. e., base points) of $H^0(P(T^*); F)$. This is a special case of the following general result proved in [23]:

If X is a compact complex manifold of dimension n with a positive line bundle F such that $(c_1(F))^n[X] = 2$ and $\dim H^0(X; F) = n+1$, then there are at most two common zeros of $H^0(X; F)$.

A point u of $P(T^*)$ is represented by a non-zero cotangent vector $\omega \in T^*$ (which is unique up to a non-zero constant multiple). A section $s \in H^0(P(T^*); F)$ vanishes at u if and only if the corresponding holomorphic vector field $\sigma \in H^0(M; T)$ is annihilated by ω. This means that $u \in P(T^*)$ is a common zero of $H^0(P(T^*); F)$ if and only if

$$\langle \omega, \sigma \rangle = 0 \text{ for all holomorphic vector fields } \sigma \in H^0(M; T) .$$

Let x be any point of M. There are three possibilities:

(i) No point of $P(T^*)$ over x is a common zero of $H^0(P(T^*); F)$. In this case, $H^0(M; T)$ spans the tangent space $T_x(M)$ at x.

(ii) There is exactly one common zero $u \in P(T^*)$ of $H^0(P(T^*); F)$ over x. In this case, $H^0(M; T)$ spans the hyperplane in $T_x(M)$ defined by $\omega = 0$.

(iii) There are two common zeros $u, u' \in P(T^*)$ of $H^0(P(T^*); F)$ over x. In this case, $H^0(M; T)$ spans the 1-dimensional subspace of $T_x(M)$ defined by $\omega = \omega' = 0$, (where ω and ω' are cotangent vectors representing u and u', respectively).

The set A of points x for which (ii) or (iii) holds is a finite set (with at most two points). Let G be the largest connected group of holomorphic transformations of M. Since G leaves A invariant and A is discrete, G fixes every point of A. In other words, every point of A is a common zero of $H^0(M; T)$. On the other hand, $H^0(M; T)$ spans a non-trivial subspace of $T_x(M)$ in all cases. We may conclude that A is empty, i. e., (i) holds for all $x \in M$. Then $H^0(M; T)$ spans $T_x(M)$ for all x. Hence, G is transitive on M.

By Theorem 2.1, M is biholomorphic to $P_3(C)$ and $\dim H^0(M; T)=15$, in contradiction to the assumption $\dim H^0(M; T)=6$. QED.

§ 4. Surfaces in M.

LEMMA 4.1. *Let M be a compact complex manifold of dimension 3 and $c_i=c_i(M)$ the i-th Chern class of M. Let S be a closed complex submanifold of dimension 2 and $h\in H^2(M; Z)$ its dual. Then the Euler number $\chi(S)$ of S is given by*

$$\chi(S)=(c_2-c_1h+h^2)h[M]\,.$$

PROOF. Let $d_i=c_i(S)$. Denote by j the imbedding $S\to M$. Then

$$j^*(1+c_1+c_2+c_3)=(1+d_1+d_2)(1+j^*h)\,.$$

Comparing both sides, we obtain

$$d_2=j^*(c_2-c_1h+h^2)\,.$$

Hence,

$$\chi(S)=d_2[S]=j^*(c_2-c_1h+h^2)[S]=(c_2-c_1h+h^2)h[M]\,. \qquad \text{QED.}$$

We do not know if the second Betti number of a compact complex manifold M with $T(M)>0$ is equal to 1. If M is a compact Kähler manifold with positive holomorphic bisectional curvature, then M is simply connected and $H^2(M; \boldsymbol{Z})=\boldsymbol{Z}$. In the remainder of this section, we shall assume that $T(M)$ is positive and the second Betti number of M is 1 and we shall disregard the torsion part of $H^2(M; \boldsymbol{Z})$. Thus, by a generator of $H^2(M; \boldsymbol{Z})$ we mean a generator of the Betti part of $H^2(M; \boldsymbol{Z})$ which is isomorphic to $\boldsymbol{Z}$.

LEMMA 4.2. *Let M be a 3-dimensional compact complex manifold with $T(M)>0$. Assuming that the second Betti number of M is 1, let α be the positive generator of $H^2(M; \boldsymbol{Z})$. Let S be a closed complex (hyper) surface in M.*

(1) *If $c_1(M)=\alpha$, then $\chi(S)\geqq 24$;*

(2) *If $c_1(M)=2\alpha$ and α is the dual of S, then*

$$\chi(S)=\frac{1}{4}(31-\dim H^0(M; T)) \qquad \textit{in case} \quad \chi(M)=4\,,$$

$$\chi(S)=\frac{1}{4}(30-\dim H^0(M; T)) \qquad \textit{in case} \quad \chi(M)=2\,;$$

(3) *If $c_1(M)=2\alpha$ and α is not the dual of S, then $\chi(S)\geqq 24$.*

PROOF. Let $h=s\alpha$ be the dual of S. Since $T(M)$ is positive, its restriction to S is a positive vector bundle. Since the normal bundle of S is a quotient bundle of $T(M)|_S$, it is also positive. Hence, the characteristic class of the normal bundle is positive. It folllows that $s\geqq 1$. By Lemma 4.1, we have

$$\chi(S)=(c_2-sc_1\alpha+s^2\alpha^2)s\alpha[M],$$

where $c_i=c_i(M)$.

If $c_1=\alpha$, making use of the fact that $c_1c_2[M]=24$ (see Theorem 3.1), we obtain

$$\chi(S)=(24-As+As^2)s, \quad \text{where} \quad A=\alpha^3[M].$$

Since $s\geqq 1$, we obtain

$$\chi(S)\geqq 24.$$

If $c_1=2\alpha$, then we obtain

$$\chi(S)=(12-2As+As^2)s, \quad \text{where} \quad A=\alpha^3[M].$$

Since $s\geqq 1$, we obtain

$$\begin{aligned} &\chi(S)=12-A && \text{for} \quad s=1, \\ &\chi(S)\geqq 24 && \text{otherwise}. \end{aligned}$$

We have now only to evaluate $A=\alpha^3[M]$. From (6) of Theorem 3.1, we obtain

$$A=\frac{1}{8}c_1^3[M]=\frac{1}{4}\Big(19-\frac{1}{2}c_3[M]+\dim H^0(M;T)\Big).$$

By substituting this into $\chi(S)=12-A$, we obtain the desired result. We should perhaps point out that the Euler number $c_3[M]$ of M is an even positive integer by Theorem 3.1 and does not exceed 4 by our assumption that the second Betti number of M is equal to 1. QED.

§ 5. The group of holomorphic transformations of M.

Let M be a compact complex manifold and G the largest connected group of holomorphic transformations.

LEMMA 5.1. *If the line bundle* $\det T$, *where* $T=T(M)$, *is positive, i.e., if the first Chern class* $c_1(M)$ *is positive, then* M *can be imbedded into a projective space* $P_N(C)$ *in such a way that* G *is the identity component of the group of projective linear transformations of* $P_N(C)$ *leaving the submanifold* M *invariant.*

PROOF. We shall sketch an outline of this more or less well known fact. By a result of Kodaira [**24**] there is a positive integer k such that $(\det T)^k$ is very ample, i.e., has sufficiently many holomorphic sections, say $N+1$ linearly independent sections, which induce an imbedding of M into $P_N(C)$. Every holomorphic transformation of M induces an automorphism of the bundle $(\det T)^k$ and hence a linear transformation of the space $H^0(M;(\det T)^k)$ of holomorphic sections which in turn induces a projective linear transformation of $P_N(C)$ leaving M invariant. QED.

We quote two results on algebraic groups (see Borel [**6**], [**7**]).

LEMMA 5.2. *Let M be imbedded in $P_N(\boldsymbol{C})$ and let G be the largest connected group of projective linear transformations of $P_N(\boldsymbol{C})$ leaving M invariant. Then the G-orbit of a point of M of least dimension is closed in M.*

LEMMA 5.3. *Let M be imbedded in $P_N(\boldsymbol{C})$ and let G be a connected solvable Lie group of projective linear transformations of $P_N(\boldsymbol{C})$ leaving M invariant. Then G has a common fixed point in M.*

We quote now a theorem on the zero set of a Killing vector field on a compact Kähler manifold which will be used in the next section as well as here in this section.

LEMMA 5.4. *Let M be a compact Kähler manifold and g_t a 1-parameter group of (holomorphic) isometries. Let F be the fixed point set of g_t. Let $b_i(M)$ and $b_i(F)$ denote the i-th Betti numbers of M and F, respectively. Then F is a disjoint union of closed complex submanifolds and*

(1) $\chi(M)=\sum(-1)^i b_i(M)=\sum(-1)^j b_j(F)=\chi(F)$;

(2) $\sum b_i(M)=\sum b_j(F)$ *if F is non-empty;*

(3) *The odd dimensional Betti numbers $b_{2i+1}(M)$ of M vanish if and only if those $b_{2j+1}(F)$ of F vanish, provided F is non-empty.*

If K is a compact group of holomorphic transformations of M, we can consider K as a group of isometries by averaging the metric of M by K and can apply Lemma 5.4 to the fixed point set of any 1-parameter subgroup of K.

In Lemma 5.4, (1) is valid for any Riemannian manifold (see [**19**]). For the proof of (2), see Frankel [**10**]. (3) is immediate from (1) and (2).

We shall denote by $b_i(\ )$ the i-th Betti number of the space inside the parenthesis.

LEMMA 5.5. *Let M be a 3-dimensional compact complex manifold with $T(M)>0$ and $b_2(M)=1$. Let F be the fixed point set of a 1-parameter subgroup of a compact group of holomorphic transformations of M. Then the following cases exhaust all possibilities:*

(1) $\chi(M)=4$ *and F consists of a single surface S with*

$$b_1(S)=b_3(S)=0 \quad \textit{and} \quad b_2(S)=2;$$

(2) $\chi(M)=4$ *and F consists of a surface S and a point p with*

$$b_1(S)=b_3(S)=0 \quad \textit{and} \quad b_2(S)=1;$$

(3) $\chi(M)=4$ *and F consists of two curves of genus 0;*

(4) $\chi(M)=4$ *and F consists of a curve of genus 0 and two points;*

(5) $\chi(M)=4$ *and F consists of four points;*

(6) $\chi(M)=2$ *and F consists of a single surface S with*

$$b_1(S)=b_3(S)=1 \quad \textit{and} \quad b_2(S)=2;$$

(7) $\chi(M)=2$ *and* F *consists of a surface* S *and a point* p *with*

$$b_1(S)=b_2(S)=b_3(S)=1\,;$$

(8) $\chi(M)=2$ *and* F *consists of a curve of genus* 1 *and two points.*

PROOF. Since the Euler number of M must be positive when $T(M)>0$, we have only two possibilities for the Betti numbers of M:

(a) $b_1(M)=b_3(M)=0$ and $\chi(M)=4$,

(b) $b_1(M)=b_3(M)=1$ and $\chi(M)=2$.

In case (a), the odd dimensional Betti numbers of F vanish and the sum of the (even dimensional) Betti numbers is 4 by Lemma 5.4. In case (b), the sum of the even dimensional Betti numbers of F is 4 and the sum of the odd dimensional Betti numbers of F is 2 by Lemma 5.4. Now Lemma 5.5 follows easily. QED.

LEMMA 5.6. *Let* M *and* F *be as in Lemma* 5.5. *Let* α *be the positive generator of* $H^2(M;\boldsymbol{Z})$ *as in Lemma* 4.2. *Then*

(i) *If* $c_1(M)=\alpha$, *cases* (1), (2), (6) *and* (7) *of Lemma* 5.5 *do not occur, i.e.,* F *does not contain a surface as a component;*

(ii) *If* $c_1(M)=2\alpha$, *cases* (6) *and* (7) *of Lemma* 5.5 *do not occur;*

(iii) *If* $c_1(M)=2\alpha$ *and case* (1) *of Lemma* 5.5 *occurs, then*

$$\dim H^0(M;T(M))=15\,;$$

(iv) *If* $c_1(M)=2\alpha$ *and case* (2) *of Lemma* 5.5 *occurs, then*

$$\dim H^0(M;T(M))=19\,.$$

PROOF. (i), (iii) and (iv) follow immediately from Lemma 4.2 and Lemma 5.5. To prove (ii) we have only to consider case (2) of Lemma 4.2. In this case, since $c_1(M)=2\alpha$ and the first Chern class of the normal bundle of S is equal to $\alpha|S$, it follows that $c_1(S)=\alpha|S$. In particular, $c_1(S)$ is positive. By the vanishing theorem of Kodaira, $b_1(S)=0$, which shows that cases (6) and (7) of Lemma 5.5 do not occur. QED.

LEMMA 5.7. *Let* M *be a* 3-*dimensional compact complex manifold with* $T(M)>0$ *and the second Betti number* $b_2(M)=1$. *Let* G *be the largest connected group of holomorphic transformations of* M. *If* G *has a closed orbit* S *of complex dimension* 2, *then* $\dim_{\boldsymbol{C}} G\geqq 14$ *and either* $S=P_2(\boldsymbol{C})$ *or* $S=P_1(\boldsymbol{C})\times P_1(\boldsymbol{C})$.

PROOF. Since S is a compact Kähler manifold with a transitive group of holomorphic transformations, it is a direct product of a Kähler C-space and a complex torus by a theorem of Borel and Remmert [8]. Imbed M into $P_N(\boldsymbol{C})$ as in Lemma 5.1. If S has a complex torus as a factor, consider a 1-parameter subgroup of G which induces translations on the torus factor. Such a 1-parameter group has no fixed points on S. This contradicts Lemma 5.3. Thus, S is a Kähler C-space. Since $\dim_{\boldsymbol{C}} S=2$, we have either $S=P_2(\boldsymbol{C})$

or $S=P_1(\boldsymbol{C})\times P_1(\boldsymbol{C})$. Let α be the positive generator of $H^2(M;\boldsymbol{Z})$ as in Lemma 4.2. If $c_1(M)=r\alpha$ with $r\geqq 3$, then Theorem 2.2 implies that $M=P_3(\boldsymbol{C})$ and that G is transitive on M. We have therefore only to consider the cases $c_1(M)=\alpha$ and $c_1(M)=2\alpha$. Since $\chi(S)\leqq 4$, Lemma 4.2 implies $\dim_{\boldsymbol{C}} G=\dim H^0(M;T(M))\geqq 14$. QED.

§ 6. Compact groups of holomorphic transformations.

We prove

THEOREM 6.1. *Let M be a 3-dimensional compact complex manifold with $T(M)>0$ and the second Betti number $b_2(M)=1$. Let G be the largest connected group of holomorphic transformations of M and K a maximal compact subgroup of G. Assume that $\dim K=\frac{1}{2}\dim G$ $(=\dim_{\boldsymbol{C}} G)$. Then M is biholomorphic to $P_3(\boldsymbol{C})$.*

PROOF. We prove first the following

LEMMA 6.2. *If M is a 3-dimensional compact complex manifold with $T(M)>0$ and admits a compact group K of holomorphic transformations of dimension $\geqq 10$, then M is biholomorphic to $P_3(\boldsymbol{C})$.*

PROOF. Let $K(x)$ be the K-orbit through a point x of M. Choose x such that $K(x)$ is a K-orbit of highest dimension and set

$$r=\dim K(x)\,.$$

Let K_x denote the isotropy subgroup of K at x so that $K(x)=K/K_x$ and

$$\dim K=r+\dim K_x\,. \tag{6.1}$$

Choosing a K-invariant hermitian metric on M, we consider K as a group of holomorphic isometries of M. Since $K(x)$ is a maximal dimensional K-orbit, K acts essentially effectively on $K(x)$ (see, for instance, [20]). Hence,

$$\dim K_x\leqq \dim O(r)=\frac{1}{2}r(r-1)\,. \tag{6.2}$$

Now (6.1) and (6.2) imply

$$\dim K\leqq r+\frac{1}{2}r(r-1)\,. \tag{6.3}$$

Since $\dim K\geqq 10$ by assumption, it follows that $r\geqq 4$. Let $T_x(K(x))$ be the tangent space of $K(x)$ at x; it is a real subspace of $T_x(M)$. We decompose it as follows:

$$T_x(K(x))=V+W\,,$$

where V is the largest complex subspace of $T_x(M)$ contained in $T_x(K(x))$, i.e., $V=T_x(K(x))\cap J(T_x(K(x)))$ and W is the orthogonal complement to V. (Here, J denotes the complex structure of M.) If $r=4$, then either $\dim_{\boldsymbol{C}} V=1$ or

$\dim_C V=2$ since $\dim_C M=3$. Since K_x acts on V as a unitary group and on W as an orthogonal group, we have

$$\dim K_x \leqq \dim U(1)+\dim O(2)=2 \quad \text{if} \quad \dim_C V=1, \tag{6.4}$$

$$\dim K_x \leqq \dim U(2)=4 \quad \text{if} \quad \dim_C V=2. \tag{6.5}$$

In either case, (6.1) implies $\dim K \leqq 8$, in contradiction to the assumption $\dim K \geqq 10$. If $r=5$, then $\dim_C V=2$ and

$$\dim K_x \leqq \dim U(2)=4. \tag{6.6}$$

In this case, (6.1) implies $\dim K \leqq 9$, again in contradiction to the assumption $\dim K \geqq 10$. If $r=6$, then K is transitive on M and Theorem 2.1 implies that M is biholomorphic to $P_3(\boldsymbol{C})$. QED.

LEMMA 6.3. *If M is a 3-dimensional compact complex manifold with $T(M)>0$ and admits a compact connected group K of holomorphic transformations of dimension $\geqq 6$ which has a common fixed point, then M is biholomorphic to $P_3(\boldsymbol{C})$.*

PROOF. Let x be a common fixed point of K. Under the linear isotropy representation at x, K can be considered as a subgroup of $U(3)$. Set

$$r=\dim (U(3)/K).$$

Then $r=\dim U(3)-\dim K \leqq 9-6=3$. Let N be the normal subgroup of $U(3)$ consisting of elements which act trivially on $U(3)/K$. Then N is contained in K, and $U(3)/N$ acts effectively on $U(3)/K$. Since the manifold $U(3)/K$ of dimension r cannot admit a compact group of transformations of dimension $>\frac{1}{2}r(r+1)$, we obtain

$$\dim (U(3)/N) \leqq \frac{1}{2}r(r+1) \leqq 6.$$

Hence, $\dim N \geqq 3$. Then either $N=SU(3)$ or $N=U(3)$. Since K contains N, either $K=SU(3)$ or $K=U(3)$. In either case, K acts transitively on the unit sphere in the tangent space $T_x(M)$. By a theorem of Nagano [**26**], M is C^1-diffeomorphic to a compact symmetric space of rank 1. (In [**26**], Nagano has determined all Riemannian manifolds which are isotropic at one point). Since M is a Kähler manifold, M must be C^1-diffeomorphic to $P_3(\boldsymbol{C})$. We may now use the result of Kodaira-Hirzebruch quoted in § 2 to conclude that M is biholomorphic to $P_3(\boldsymbol{C})$. But we may use Theorem 2.2 as follows. Let α be the positive generator of $H^2(M;\boldsymbol{Z})$ and write $c_1=r\alpha$. Then $\alpha^3[M]=1$, since M is homeomorphic to $P_3(\boldsymbol{C})$. As we have seen in § 3,

$$(c_1^3-2c_1c_2+c_3)[M]>0.$$

Since $c_1c_2[M]=24$ by Theorem 3.1 and $c_3[M]=\chi(M)=4$, the inequality above implies

$$r^3 - 48 + 4 > 0 .$$

Hence, $r \geqq 4$. QED.

LEMMA 6.4. *Let M be a 3-dimensional compact complex manifold with $T(M) > 0$ (more generally, $\det(T(M)) \geqq 0$). Let K be a compact group of holomorphic transformations of M. Then $\operatorname{rank} K \leqq 3$. If $\dim K \geqq 6$, then $\operatorname{rank} K \geqq 2$ and the center of K has dimension $\leqq 1$.*

PROOF. Let A be a connected maximal abelian subgroup of K. Then $\dim A = \operatorname{rank} K$. By Lemmas 5.1 and 5.3, A leaves a point x of M fixed. Then A may be considered as an abelian subgroup of $U(3)$ through the linear isotropy representation at x. Hence, $\dim A \leqq 3$. It is obvious that $\operatorname{rank} K \geqq 2$ if $\dim K \geqq 4$. Let C be the center and K_s the semi-simple part of K. Since $\operatorname{rank} K = \operatorname{rank} K_s + \dim C \leqq 3$ and $\dim K \geqq 6$, we obtain $\dim C \leqq 2$. If $\dim C = 2$, then $\operatorname{rank} K_s = 1$ and hence $\dim K_s = 3$, which implies $\dim K = 5$. Hence, $\dim C \leqq 1$. QED.

We shall now prove Theorem 6.1. Let G be the largest connected group of holomorphic transformations of M. Let m be the complex dimension of a minimal dimensional G-orbit and let $G(x)$ be such an orbit. We know (Lemma 5.2) that $G(x)$ is a closed complex submanifold of dimension m.

If $m = 3$, then G is transitive on M and, by Theorem 2.1, M is biholomorphic to $P_3(\boldsymbol{C})$. If $m = 2$, Lemma 5.7 implies that $\dim_{\boldsymbol{C}} G \geqq 14$. Since we are assuming that $\dim K = \dim_{\boldsymbol{C}} G$, Lemma 6.2 implies that M is biholomorphic to $P_3(\boldsymbol{C})$. The case $m = 1$ will be considered last. If $m = 0$, $G(x)$ is a point, i.e., G leaves the point x fixed. Since $\dim K = \dim_{\boldsymbol{C}} G \geqq 7$ by (6) of Theorem 3.1, M is biholomorphic to $P_3(\boldsymbol{C})$ by Lemma 6.3.

We shall now consider the case $m = 1$, i.e., the case where $G(x)$ is a closed curve. Let K_s denote the semi-simple part of K. Since $\dim K \geqq 7$ by (6) of Theorem 3.1 and the dimension of the center C of K is at most 1, we have $\dim K_s \geqq 6$. Since a compact group of dimension $\geqq 4$ cannot act effectively on a real 2-dimensional manifold, K_s cannot act effectively on the orbit $G(x)$ of complex dimension 1. If K_s is simple, this means that K_s acts trivially on $G(x)$. Applying Lemma 6.3 to the compact group K_s leaving x fixed, we see that M is biholomorphic to $P_3(\boldsymbol{C})$. We may now assume that K_s is not simple. By Lemma 6.2, we may also assume that $\dim K \leqq 9$. Let N denote the normal subgroup of K consisting of elements which act trivially on the curve $G(x)$. Since K/N acts effectively, $\dim(K/N) \leqq 3$. If $\dim K = 9$, then $\dim N \geqq 6$ and Lemma 6.3 applied to the compact group N implies that M is biholomorphic to $P_3(\boldsymbol{C})$. We may therefore assume that $\dim K \leqq 8$. If $\dim N \geqq 6$, M is biholomorphic to $P_3(\boldsymbol{C})$ by the same lemma. Hence, we may further assume that $\dim N \leqq 5$. If $\dim K = 8$, then $\dim N = 5$ since $\dim(K/N)$

$\leqq 3$. Since there is no simple group of dimension 4 or 5, it follows that rank $N \geqq 3$. Hence, rank $K =$ rank $N +$ rank $K/N \geqq 4$, in contradiction to Lemma 6.4. We may assume therefore that $\dim K = 7$. Again considering rank K, we see that $\dim N = 4$, rank $N = 2$ and rank $K/N = 1$. In other words,

$$K = K_1 \times K_2 \times C \qquad \text{(local direct product)},$$

where K_1 and K_2 are 3-dimensional simple compact groups and C is the 1-dimensional center, i. e., a circle group.

We shall first prove that the odd dimensional Betti numbers of M vanish and $\chi(M) = 4$. Since rank $K = 3$, we take a 3-dimensional torus subgroup A of K. The set F_A of common fixed points of A is non-empty by Lemmas 5.1 and 5.3. The linear representation of A at any point x of F_A is trivial on the tangent space $T_x(F_A)$ and hence must be faithful on the normal space $N_x(F_A)$. If $r = \dim N_x(F_A)$, then A is a subgroup of $U(r)$. But this is possible only if $r = 3$. This means that F_A consists of isolated points. A dense 1-parameter subgroup of A has the same fixed point set F_A as A. Applying Lemma 5.4 or Lemma 5.5 to this 1-parameter subgroup, we see that M has vanishing odd dimensional Betti numbers and $\chi(M) = 4$.

We consider now the set F_C of common fixed points of the center C of $K = K_1 \times K_2 \times C$. If F_C has a surface S as one of its components, then $\chi(S)$ is either 3 or 4 by Lemma 5.5. Since we can exclude the cases $c_1(M) \geqq 3\alpha$ by Theorem 2.2, we see that $\dim_C G \geqq 15$ by Lemma 4.2. Then $\dim K \geqq 15$, in contradiction to the present assumption $\dim K = 7$. Suppose F_C contains an isolated point, say x, as one of its components. Since $K_1 \times K_2$ commutes with C, it leaves F_C invariant. Since x is an isolated point of F_C, it is left fixed by $K_1 \times K_2$ also. By Lemma 6.3, M is biholomorphic to $P_3(\boldsymbol{C})$.

In view of Lemma 5.5, the only remaining case to be considered is when the fixed point set F_C of the center C consists of two curves of genus 0 (i. e., $P_1(\boldsymbol{C})$). Write

$$F_C = P \cup P', \quad \text{where } P \text{ and } P' \text{ are biholomorphic to } P_1(\boldsymbol{C}).$$

If the action of $K_1 \times K_2$ on P is trivial, Lemma 6.3 implies that M is biholomorphic to $P_3(\boldsymbol{C})$. Assume that $K_1 \times K_2$ acts non-trivially on P. Since $\dim (K_1 \times K_2) = 6$, $K_1 \times K_2$ cannot act effectively on P (of complex dimension 1). Without loss of generality, we may assume that K_1 acts effectively on P and K_2 acts trivially on P.

Take a point $a \in P$. Let T_1 be the isotropy subgroup of K_1 at a; it is a circle group and $P = K_1/T_1$. Let F_{T_1} be the fixed point set of T_1. It contains the point a.

If the component of F_{T_1} containing the point a is a surface S, we obtain a contradiction as in the case when F_C has a surface as one of its components.

(using Lemmas 5.5 and 4.2 again).

Assume that the component of F_{T_1} containing the point a is a curve, say P''. We claim that P and P'' are transversal at a, i.e., $T_a(P) \cap T_a(P'') = 0$. To see this, we consider the linear isotropy representation of the circle group T_1 at the point a. Since $P = K_1/T_1$, T_1 acts on the tangent space $T_a(P)$ as the unitary group $U(1)$. On the other hand, since T_1 leaves P'' pointwise fixed, T_1 acts trivially on $T_a(P'')$. Hence the two 1-dimensional complex subspaces $T_a(P)$ and $T_a(P'')$ of $T_a(M)$ cannot coincide and hence $T_a(P) \cap T_a(P'') = 0$. We consider now the linear isotropy representation of $K_2 \times C$ at the point a. We write

$$T_a(M) = T_a(P) + T_a(P'') + N_a ,$$

where N_a is the 1-dimensional complex subspace of $T_a(M)$ which is normal to $T_a(P) + T_a(P'')$. Since $K_2 \times C$ leaves P pointwise fixed, it acts trivially on $T_a(P)$. Since $K_2 \times C$ commutes with the circle group $T_1 \subset K_1$, it leaves the fixed point set F_{T_1} invariant and hence it leaves P'' invariant. It follows that $K_2 \times C$ leaves $T_a(P'')$ invariant. Consequently, $K_2 \times C$ must leave N_a invariant. This means that with respect to the decomposition $T_a(M) = T_a(P) + T_a(P'') + N_a$ the linear isotropy representation of $K_2 \times C$ at a must look like the following:

$$K_2 \times C \subset \begin{pmatrix} 1 & 0 & 0 \\ 0 & U(1) & 0 \\ 0 & 0 & U(1) \end{pmatrix}.$$

But this is impossible since $\dim(K_2 \times C) = 4$. This shows that the component of F_{T_1} containing the point a cannot be a curve.

Assume that the component of F_{T_1} containing the point a is of dimension 0, i.e., the point a is an isolated fixed point of T_1. Let N_a denote the normal space to P at a; it is a 2-dimensional complex subspace of $T_a(M)$ and

$$T_a(M) = T_a(P) + N_a .$$

Since $K_2 \times C$ acts trivially on P and hence on $T_a(P)$ and since $\dim(K_a \times C) = 4$, the linear isotropy representation of $K_2 \times C$ at a is of the following form with respect to the decomposition $T_a(M) = T_a(P) + N_a$:

$$K_2 \times C = \begin{pmatrix} 1 & 0 \\ 0 & U(2) \end{pmatrix}.$$

Since T_1 leaves P and hence $T_a(P)$ invariant, the linear isotropy representation of T_1 at a is of the form

$$T_1 \subset \begin{pmatrix} U(1) & 0 \\ 0 & U(2) \end{pmatrix}.$$

Write the circle group T_1 as a 1-parameter group f_t. Then the linear isotropy representation of f_t at a is of the form:

$$\begin{pmatrix} * & 0 \\ 0 & A(t) \end{pmatrix},$$

where $A(t)$ is a 1-parameter subgroup of $U(2)$. Take a 1-parameter subgroup g_t of $K_2 \times C$ whose linear isotropy representation at a is of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & A(t) \end{pmatrix}.$$

We define a non-trivial 1-parameter subgroup h_t of $T_1 \times K_2 \times C$ by

$$h_t = f_t \circ g_t^{-1}.$$

Then h_t leaves the point a fixed and its linear isotropy representation at a leaves the space N_a pointwise fixed. Let F_{h_t} be the fixed point set of the 1-parameter group h_t. Its component containing the point a is a surface whose tangent space at a coincides with N_a. Using Lemmas 5.5 and 4.2 again, we obtain a contradiction as in the case when F_C has a surface as one of its components. This completes the proof of Theorem 6.1.

§ 7. Positive holomorphic bisectional curvature.

We shall prove the main theorem of this paper.

THEOREM 7.1. *Let M be a 3-dimensional compact Kähler manifold with positive holomorphic bisectional curvature. Then it is biholomorphic to $P_3(\boldsymbol{C})$.*

PROOF. We have shown in our previous paper [**21**] that a compact Kähler manifold with positive holomorphic bisectional curvature has a positive tangent bundle. On the other hand, such a manifold M has the second Betti number $b_2(M)=1$ by a result of Bishop and Goldberg [**3**] (see also [**12**]). Let G be the largest connected group of holomorphic transformations of M and K a maximal compact subgroup of G. Our theorem will follow from Theorem 6.1 if we show that $\dim K = \frac{1}{2} \dim G$ $(= \dim_C G)$. But this is a consequence of the following two results.

LEMMA 7.2. (Matsushima [**25**]). *If M is a compact Einstein-Kähler manifold, then the Lie algebra $\mathfrak{g}$ of holomorphic vector fields is the complexification of the Lie algebra $\mathfrak{k}$ of infinitesimal isometries (i. e., Killing vector fields).*

Thus, for a compact Einstein-Kähler manifold M, the largest connected group K of isometries is a maximal compact subgroup of G and $\dim K = \dim \mathfrak{k} = \frac{1}{2} \dim \mathfrak{g} = \frac{1}{2} \dim G$.

LEMMA 7.3. (Aubin [**1**]). *Let M be a compact Kähler manifold with non-negative holomorphic bisectional curvature. If the Kähler 2-form represents the first Chern class $c_1(M)$, then M admits a Kähler-Einstein metric.*

We note that if $b_2(M)=1$ as in the present case, a suitable constant multiple of the given Kähler 2-form represents $c_1(M)$. Since $c_1(M)$ is positive in the present case, this constant is positive. Hence the result of Aubin can be applied to the manifold M. QED.

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