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$(p, q; r)$-absolutely summing operators

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By Mitjagin and Pełczyński [7] a linear operator T from a Banach space X into another Banach space Y is said to be (p, r)-absolutely summing, $1 \leqq p, r \leqq \infty$, if there is a constant ρ such that for every finite sequence $\{x_i\}_{1 \leqq i \leqq n}$ of points in X the inequality

$$(1) \qquad (\sum_i \|Tx_i\|^p)^{1/p} \leqq \rho \sup_{\|a\| \leqq 1} (\sum_i |\langle x_i, a \rangle|^r)^{1/r}$$

holds. Here as usual, if $p = \infty$ (resp. $r = \infty$), the left (resp. right) hand side of (1) is replaced by $\sup_i \|Tx_i\|$ (resp. $\rho \sup_{\|a\| \leqq 1} (\sup_i |\langle x_i, a \rangle|)$). The notation $\sup_{\|a\| \leqq 1}$ means the supremum taken over all the elements a of the weakly compact unit ball of the dual space X^* of X. The theory of (p, r)-absolutely summing operators is a unified theory of various important classes of operators in connection with the classes of nuclear and Hilbert-Schmidt operators.

In this paper we shall define $(p, q; r)$-absolutely summing operator, generalizing (p, r)-absolutely summing operator, inspired from the theory of Lorentz space $l_{p,q}$ of sequences. The aim of this paper is to develop the theory of this operator. In § 1 we discuss the basic properties of the class of $(p, q; r)$-absolutely summing operators as a Banach ideal. In § 2 we deal with the composition of these absolutely summing operators. We give there a generalization of the classical theorem that the composition of two Hilbert-Schmidt operators is nuclear. When the Banach spaces considered as domain and range are particular, for instance Hilbert spaces, some of Banach ideals of absolutely summing operators may happen to coincide. We shall state these facts in § 3 and § 4. We also investigate there the mean spaces (Lions-Peetre [6]) of Banach ideals of absolutely summing operators.

§ 1. $(p, q; r)$-absolutely summing operator.

Let X and Y be Banach spaces and let B(X, Y) be all the bounded linear operators from X into Y.

DEFINITION 1. Let $1 \leqq p, q, r \leqq \infty$. Let $\{x_i\}_{1 \leqq i \leqq n}$ be any finite sequence of points in X, and $\{\|Tx_i\|_*\}$ be the non-increasing rearrangement of $\{\|Tx_i\|\}$.

If $T \in B(X, Y)$ satisfies the inequality

$$\{\sum_i (i^{1/p-1/q}\|Tx_i\|_*)^q\}^{1/q} \leqq \rho \sup_{\|a\|\leqq 1} (\sum_i |\langle x_i, a\rangle|^r)^{1/r}, \tag{2}$$

T is called *(p, q ; r)-absolutely summing operator.* Here, ρ is a constant. As usual the left and right hand sides in (2) are supposed to mean $\sup_i$ instead of $\{\sum_i(\cdots)^q\}^{1/q}$ and $(\sum_i|\cdots|^r)^{1/r}$ in case of $q=\infty$ and $r=\infty$ respectively.

We denote by $\pi_{p,q;r}(T)$ the least constant ρ satisfying (2) for any finite system $\{x_i\}$, and by $\Pi_{p,q;r}(X, Y)$ the set of all $(p, q ; r)$-absolutely summing operators.

REMARK 1. When $p=q$, $(p, q ; r)$-absolutely summing operator coincides with (p, r)-absolutely summing operator of Mitjagin and Pełczyński [7], [5]. Hereafter we write $\Pi_{p;r}(X, Y)$ (resp. $\Pi_p(X, Y)$) instead of $\Pi_{p,p;r}(X, Y)$ (resp. $\Pi_{p,p;p}(X, Y)$) and $\pi_{p;r}(T)$ (resp. $\pi_p(T)$) instead of $\pi_{p,p;r}(T)$ (resp. $\pi_{p,p;p}(T)$) (see [**12**]).

By the definition of $(p, q ; r)$-absolutely summing operator T, it holds

$$\|Tx\| \leqq \pi_{p,q;r}(T) \sup_{\|a\|\leqq 1} |\langle x, a\rangle| \leqq \pi_{p,q;r}(T)$$

for any $x \in X$ with $\|x\|\leqq 1$. Therefore we have

PROPOSITION 1. *Let* B(X, Y) *be the Banach space of all bounded linear operators with the norm* $\|T\|=\sup_{\|x\|\leqq 1}\|Tx\|$. *Then we have* $\Pi_{p,q;r}(X, Y) \subset B(X, Y)$ *and*

$$\|T\| \leqq \pi_{p,q;r}(T) \qquad \textit{for every } T \in \Pi_{p,q;r}(X, Y).$$

PROPOSITION 2. *With the norm* $\pi_{p,q;r}(T)$, $\Pi_{p,q;r}(X, Y)$ *becomes a Banach space.*

PROOF. Let $\{T_n\}$ be any Cauchy sequence in $\Pi_{p,q;r}(X, Y)$. Then by Proposition 1 it follows that for any $\varepsilon>0$ there exists a positive integer N such that

$$\|T_n - T_m\| \leqq \pi_{p,q;r}(T_n - T_m) < \varepsilon \qquad \text{for any } m, n > N.$$

Therefore T_n becomes a Cauchy sequence in B(X, Y) which is a Banach space under the norm $\|\cdot\|$. Hence there is an operator $T \in B(X, Y)$ such that $\lim_{m\to\infty}\|T_m - T\| = 0$. Letting $m\to\infty$ in

$$\{\sum_i (i^{1/p-1/q}\|T_n x_i - T_m x_i\|_*)^q\}^{1/q} < \varepsilon \sup_{\|a\|\leqq 1} (\sum_i |\langle x_i, a\rangle|^r)^{1/r},$$

this implies $\pi_{p,q;r}(T_n - T) \leqq \varepsilon$ for any $n > N$. This completes the proof.

PROPOSITION 3. (i) *For any* r: $1\leqq r\leqq\infty$ *we have* $\Pi_{\infty;r}(X, Y)=B(X, Y)$ *and* $\pi_{\infty;r}(T)=\|T\|$ *for every* $T\in\Pi_{\infty;r}(X, Y)$.

(ii) *If* $1\leqq p\leqq q<r\leqq\infty$, *then* $\Pi_{p,q;r}(X, Y)=\{0\}$.

PROOF. (i) is clear because for any $T\in B(X, Y)$ and any finite sequence

$\{x_i\}$ of points in X the inequality

$$\sup_i \|Tx_i\| \leqq \|T\| \sup_{\|a\|\leqq 1} (\sum_i |\langle x_i, a\rangle|^r)^{1/r}$$

holds.

If $1 \leqq p \leqq q < r$, and $T \neq 0$ be any bounded linear operator, then for any positive number N there exists a sequence $\{x_i\}$ in X such that

$$\sum_i i^{q/p-1}\|Tx_i\|^q > N \quad \text{and} \quad \sup_{\|a\|\leqq 1} \sum_i |\langle x_i, a\rangle|^r < c$$

with some constant c independent of N. This proves (ii).

Let us denote by B (resp. $\Pi_{p,q;r}$) all the bounded linear (resp. $(p, q; r)$-absolutely summing) operators which are defined between any two Banach spaces X, Y. Then $\Pi_{p,q;r}$ makes a Banach ideal of B in the following sense.

PROPOSITION 4. *Let* X, Y *and* Z *be Banach spaces.*

(i) *If* $S \in B(X, Y)$ *and* $T \in \Pi_{p,q;r}(Y, Z)$, *then* $TS \in \Pi_{p,q;r}(X, Z)$ *and the inequality*

$$\pi_{p,q;r}(TS) \leqq \pi_{p,q;r}(T)\|S\|$$

holds.

(ii) *If* $S \in \Pi_{p,q;r}(X, Y)$ *and* $T \in B(Y, Z)$, *then* $TS \in \Pi_{p,q;r}(X, Z)$ *and we obtain*

$$\pi_{p,q;r}(TS) \leqq \|T\|\pi_{p,q;r}(S).$$

PROOF. (i) For any finite sequence $\{x_i\}_{1\leqq i\leqq n}$ of points in X, by the assumptions the following inequalities are valid:

$$\begin{aligned}
&(\sum_i i^{q/p-1}\|TSx_i\|_*^q)^{1/q} \\
&\qquad \leqq \pi_{p,q;r}(T) \sup_{\|b\|\leqq 1} (\sum_i |\langle Sx_i, b\rangle|^r)^{1/r} \\
&\qquad \leqq \pi_{p,q;r}(T)\|S\| \sup_{\|b\|\leqq 1} (\sum_i |\langle x_i, \|S\|^{-1}S'b\rangle|^r)^{1/r} \\
&\qquad \leqq \pi_{p,q;r}(T)\|S\| \sup_{\|a\|\leqq 1} (\sum_i |\langle x_i, a\rangle|^r)^{1/r},
\end{aligned}$$

which proves (i). Here $\sup_{\|a\|\leqq 1}$ (resp. $\sup_{\|b\|\leqq 1}$) implies the supremum taken over all a (resp. b) of the weakly compact unit ball of X* (resp. Y*), and S′ denotes the adjoint of S.

The analogous calculation shows (ii) of the Proposition. In fact, noting that $(\sum_i i^{q/p-1}\|Sx_i\|_*^q)^{1/q}$, $q < p$, is the maximum among all the summations of this type for any possible rearrangements of $\{\|Sx_i\|\}$, it holds

$$\begin{aligned}
&(\sum_i i^{q/p-1}\|TSx_i\|_*^q)^{1/q} \\
&\qquad \leqq \|T\|(\sum_i i^{q/p-1}\|Sx_i\|_*^q)^{1/q} \\
&\qquad \leqq \|T\|\pi_{p,q;r}(S) \sup_{\|a\|\leqq 1} (\sum_i |\langle x_i, a\rangle|^r)^{1/r}.
\end{aligned}$$

Thus our assertions are proved.

We now review shortly the notion of mean space of Lions-Peetre [6] which will be utilized hereafter. For the precise notations and fundamental properties of this we may refer to [6], [3] and [8]. Let (A_0, A_1) be an interpolation couple of Banach spaces and $(A_0, A_1)_{\theta,q}$, $0<\theta<1$, $1\leqq q\leqq\infty$, be the mean space of (A_0, A_1), namely, the set of elements $a\in A_0+A_1$ such that

$$a=\sum_{n=-\infty}^{\infty} a_n:\ \{e^{\theta n}a_n\}\in l_q(A_0)\quad\text{and}\quad \{e^{(\theta-1)n}a_n\}\in l_q(A_1)$$

with the norm

$$\|a\|_{(A_0,A_1)_{\theta,q}}=\inf_{a=\Sigma a_n}\max\left(\|\{e^{\theta n}a_n\}\|_{l_q(A_0)},\ \|\{e^{(\theta-1)n}a_n\}\|_{l_q(A_1)}\right).$$

Then, as a special case of mean space, the following relation is well known:

$$(l_{p_1}, l_{p_2})_{\theta,q}\sim l_{p,q} \tag{3}$$

with p: $1/p=(1-\theta)/p_1+\theta/p_2$, $1\leqq p_i$, $q\leqq\infty$, $i=1, 2$. Here $l_{p,q}$ means the space of sequences $\{x_i\}\in c_0$ such that $\|\{x_i\}\|_{l_{p,q}}=\|\{i^{1/p-1/q}x_{i*}\}\|_{l_q}<\infty$, where $\{x_{i*}\}$ denotes the non-increasing rearrangement of $\{x_i\}$, and $\sim$ means that the both sides of (3) are set-theoretically equal and their norms are equivalent.

On account of this fact and making use of the inclusion relations of $(A_0, A_1)_{\theta,q}$ with respect to the parameters θ and q [8], we obtain

PROPOSITION 5. (i) *If* $1\leqq p_1<p_2\leqq\infty$ *and* $1\leqq q_1, q_2, r\leqq\infty$, *then we obtain* $\Pi_{p_1,q_1;r}(X, Y)\subset\Pi_{p_2,q_2;r}(X, Y)$ *and*

$$\pi_{p_2,q_2;r}(T)\leqq c\pi_{p_1,q_1;r}(T),\qquad \textit{for every } T\in\Pi_{p_1,q_1;r}(X, Y),$$

where c is a constant.

(ii) *If* $1\leqq q_1\leqq q_2\leqq\infty$ *and* $1\leqq p, r\leqq\infty$, *then we have* $\Pi_{p,q_1;r}(X, Y)\subset\Pi_{p,q_2;r}(X, Y)$ *and*

$$\pi_{p,q_2;r}(T)\leqq c\pi_{p,q_1;r}(T),\qquad \textit{for every } T\in\Pi_{p,q_1;r}(X, Y).$$

(iii) *If* $1\leqq r_1\leqq r_2\leqq\infty$ *and* $1\leqq p, q\leqq\infty$, *then we have* $\Pi_{p,q;r_2}(X, Y)\subset\Pi_{p,q;r_1}(X, Y)$ *and*

$$\pi_{p,q;r_1}(T)\leqq c\pi_{p,q;r_2}(T),\qquad \textit{for every } T\in\Pi_{p,q;r_2}(X, Y).$$

Moreover, generalizing Kwapień's result [4], (0. 7), we get the following proposition.

PROPOSITION 6. *If real numbers* $1\leqq p_i, q_i, r_i\leqq\infty$, $i=1, 2$, *with* $p_1\geqq q_1$ *or* $p_1<q_1<r_1$, *satisfy the relations* $1/p_1-1/p_2=1/q_1-1/q_2=1/r_1-1/r_2\geqq 0$, *then we have* $\Pi_{p_1,q_1;r_1}(X, Y)\subset\Pi_{p_2,q_2;r_2}(X, Y)$ *and*

$$\pi_{p_2,q_2;r_2}(T)\leqq\pi_{p_1,q_1;r_1}(T),\qquad \textit{for every } T\in\Pi_{p_1,q_1;r_1}(X, Y).$$

PROOF. In case of $p_1 < q_1 < r_1$, this is trivially true by Proposition 3, (ii). Therefore it is sufficient to prove this under the assumption $p_1 \geqq q_1$ by which it holds $q_2/p_2 - q_1/p_1 \leqq 0$. If for each $T \in \Pi_{p_1,q_1;r_1}(X, Y)$ and for each finite sequence of points $\{x_i\}_{1 \leqq i \leqq n}$ in X we put

$$\lambda_i = i^{(q_2/p_2 - q_1/p_1)/q_1} \|Tx_i\|_*^{q_2/p}, \qquad \text{with } 1/p_1 - 1/p_2 = 1/p,$$

then $\{\lambda_i\}$ is non-increasing. On account of this we have

$$(4) \qquad \begin{aligned} &(\sum_i i^{q_1/p_1 - 1} \lambda_i^{q_1} \|Tx_i\|_*^{q_1})^{1/q_1} \\ &= (\sum_i i^{q_1/p_1 - 1} \|\lambda_i T x_{j(i)}\|^{q_1})^{1/q_1} \\ &= (\sum_i i^{q_1/p_1 - 1} \|T \lambda_i x_{j(i)}\|_*^{q_1})^{1/q_1}, \end{aligned}$$

where $\{j(i)\}$ denotes the permutation of $\{i\}$ appeared when we make the non-increasing rearrangement of $\{\|Tx_i\|\}$, i.e. $\|Tx_{j(i)}\| = \|Tx_i\|_*$. By making use of the definition of T and Hölder's inequality with $1/r_1 = 1/r_2 + 1/p$, the last expression of (4) is not greater than

$$\begin{aligned} &\pi_{p_1,q_1;r_1}(T) \sup_{\|a\| \leqq 1} (\sum_i |\langle \lambda_i x_{j(i)}, a \rangle|^{r_1})^{1/r_1} \\ &\leqq \pi_{p_1,q_1;r_1}(T) \sup_{\|a\| \leqq 1} (\sum_i |\langle x_i, a \rangle|^{r_2})^{1/r_2} (\sum_i |\lambda_i|^p)^{1/p} \\ &= \pi_{p_1,q_1;r_1}(T) \sup_{\|a\| \leqq 1} (\sum_i |\langle x_i, a \rangle|^{r_2})^{1/r_2} (\sum_i i^{q_2/p_2 - 1} \|Tx_i\|_*^{q_2})^{1/p}. \end{aligned}$$

On the other hand, the left hand side of (4) is equal to $(\sum_i i^{q_2/p_2 - 1} \|Tx_i\|_*^{q_2})^{1/q_1}$. Therefore we have

$$\begin{aligned} &(\sum_i i^{q_2/p_2 - 1} \|Tx_i\|_*^{q_2})^{1/q_1 - 1/p} \\ &\leqq \pi_{p_1,q_1;r_1}(T) \sup_{\|a\| \leqq 1} (\sum_i |\langle x_i, a \rangle|^{r_2})^{1/r_2}, \end{aligned}$$

which completes the proof.

As a consequence of above two Propositions 5, 6 we get the next

COROLLARY. *If real numbers $1 \leqq p_i, q_i, r_i \leqq \infty$, $i = 1, 2$, with $p_2 \geqq q_2$ or $p_2 < q_2 < r_2$, satisfy the relations $1/p_1 - 1/p_2 \geqq 1/r_1 - 1/r_2 \geqq 0$ and $1/q_1 - 1/q_2 \geqq 1/r_1 - 1/r_2$, then we have $\Pi_{p_1,q_1;r_1}(X, Y) \subset \Pi_{p_2,q_2;r_2}(X, Y)$ and*

$$\pi_{p_2,q_2;r_2}(T) \leqq \pi_{p_1,q_1;r_1}(T), \qquad \textit{for every } T \in \Pi_{p_1,q_1;r_1}(X, Y).$$

In the final of this section, we shall give some examples of $(p, q; r)$-absolutely summing operators.

EXAMPLE 1. *If $1 \leqq q < r < p < \infty$, then the identical operator I from C[0, 1] into $L_q(0, 1)$ is $(p, q; r)$-absolutely summing. However, I is not $(q; r)$-absolutely summing.*

PROOF. Let $\{x_i(t)\}_{1 \leqq i \leqq n}$ be any finite sequence of functions of C[0, 1].

Then we have

$$\sum_i (i^{1/p-1/q}\|x_i(t)\|_{L_q})^q = \sum_i i^{q/p-1}\int_0^1 |x_i(t)|^q dt$$
$$= \sum_i i^{q/p-1}\int_0^1 |\langle x_i, \delta_t\rangle|^q dt$$
$$\leqq \sup_{\|a\|\leqq 1} (\sum_i i^{q/p-1}|\langle x_i, a\rangle|^q),$$

where δ_t is Dirac measure at t. By making use of Hölder's inequality for the right side of the above inequality, with $1/q=1/r+1/r'$, we obtain

$$\{\sum_i (i^{1/p-1/q}\|x_i(t)\|_{L_q})^q\}^{1/q}$$
$$\leqq (\sum_i i^{r'(1/p-1/q)})^{1/r'} \sup_{\|a\|\leqq 1} (\sum_i |\langle x_i, a\rangle|^r)^{1/r}.$$

In account of $r'(1/q-1/p)>1$, this shows $\mathrm{I}\in \Pi_{p,q;r}(\mathrm{C}[0, 1], \mathrm{L}_q(0, 1))$.

On the other hand, in view of Proposition 3, (ii), it is clear that $\mathrm{I}\notin \Pi_{q;r}(\mathrm{C}[0, 1], \mathrm{L}_q(0, 1))$.

EXAMPLE 2. *If* $1\leqq q<r\leqq p<\infty$, *then the identical operator* I: $\mathrm{C}[0, 1]\to \mathrm{L}_p(0, 1)$ *does belong to* $\Pi_{p;r}(\mathrm{C}[0, 1], \mathrm{L}_p(0, 1))$, *but does not belong to* $\Pi_{p,q;r}(\mathrm{C}[0, 1], \mathrm{L}_p(0, 1))$.

PROOF. Let $\{x_i(t)\}_{1\leqq i\leqq n}$ be any finite sequence of functions of $\mathrm{C}[0, 1]$. Then we have

$$(\sum_i \|x_i(t)\|_{L_p}{}^p)^{1/p} = \Big(\sum_i \int_0^1 |x_i(t)|^p dt\Big)^{1/p}$$
$$= \Big(\int_0^1 \sum_i |\langle x_i, \delta_t\rangle|^p dt\Big)^{1/p}$$
$$\leqq \sup_{\|a\|\leqq 1} (\sum_i |\langle x_i, a\rangle|^p)^{1/p}$$
$$\leqq \sup_{\|a\|\leqq 1} (\sum_i |\langle x_i, a\rangle|^r)^{1/r}.$$

This proves the first assertion of this example.

The second assertion is proved in the same way in [12] as follows. Let us consider a sequence of positive numbers τ_i such that

$$\sum_{i=1}^{\infty} \tau_i = 1 \quad \text{and} \quad \sum_{i=1}^{\infty} i^{q/p-1}\tau_i{}^{q/p} = \infty.$$

For instance we take as

$$\tau_{i-1} = \frac{1}{c(i-1)(\log i)^k}, \qquad i=2, 3, \cdots,$$

where k be a constant: $1<k<p/q$, and $c=\sum_{i=2}^{\infty} \frac{1}{(i-1)(\log i)^k}$. We put $t_0=0$, $t_j=\sum_{i=1}^{j} \tau_i$ for $j=1, 2, \cdots$, and consider a sequence of functions $\varphi_i(t)$ defined by

$$\varphi_i(t)=\begin{cases} 1-|2t-t_i-t_{i-1}|/\tau_i & \text{for } t\in[t_{i-1}, t_i] \\ 0 & \text{otherwise}. \end{cases}$$

For any element a of $(C[0, 1])'$ such that $\|a\|\leqq 1$, and for any positive integer m, we have

$$\begin{aligned}(\sum_{i=1}^{m}|\langle\varphi_i, a\rangle|^r)^{1/r} &\leqq \sum_{i=1}^{m}|\langle\varphi_i, a\rangle| \\ &=\langle\sum_{i=1}^{m}\lambda_i\varphi_i, a\rangle \\ &\leqq\|\sum_{i=1}^{m}\lambda_i\varphi_i\|_C=1,\end{aligned}$$

where the number λ_i is taken as $|\lambda_i|=1$ and $\lambda_i\langle\varphi_i, a\rangle=|\langle\varphi_i, a\rangle|$. On the other hand, we obtain

$$\begin{aligned}&(\sum_{i=1}^{m} i^{q/p-1}\|\varphi_i\|_{L_p}{}^q)^{1/q} \\ &=\Big\{\sum_{i=1}^{m} i^{q/p-1}\Big(\frac{\tau_i}{p+1}\Big)^{q/p}\Big\}^{1/q}\longrightarrow\infty \quad \text{as } m\to\infty.\end{aligned}$$

This means that $I\notin\Pi_{p,q;r}(C[0, 1], L_p(0, 1))$.

§ 2. Composition of $(p, q; r)$-absolutely summing operators.

In this section we are concerned with the classical theorem: If S and T are Hilbert-Schmidt operators on a Hilbert space H, then the composition TS becomes a nuclear operator, and the inequality

$$\nu(TS)\leqq\sigma(T)\sigma(S)$$

holds, where σ (resp. ν) stands for the norm as Hilbert-Schmidt (resp. nuclear) operator.

This theorem has been generalized in various directions. Especially, Theorem 4 of Pietsch [**12**] for p-absolutely summing operators and Proposition of Tomczak [**15**] for (p, q)-absolutely summing operators are interesting. The analogous results for $(p, q; r)$-absolutely summing operators are stated as the following two theorems. The proofs follow along the lines of [**12**] and [**15**].

THEOREM 1. *Let* X, Y *and* Z *be Banach spaces and let* $1\leqq p, p_i, q_i, r_i\leqq\infty$, $i=1, 2$, *be real numbers satisfying* $1/p_2\leqq 1/p+1/p_1\leqq 1$, $1/q_2\leqq 1/p+1/q_1\leqq 1$ *and* $1/p+1r_1\leqq 1/r_2$. *Then, for any* $T\in\Pi_p(X, Y)$ *and any* $S\in\Pi_{p_1,q_1;r_1}(Y, Z)$ *the composition* ST *belongs to* $\Pi_{p_2,q_2;r_2}(X, Z)$ *and satisfies*

$$\pi_{p_2,q_2;r_2}(ST)\leqq\pi_{p_1,q_1;r_1}(S)\pi_p(T).$$

PROOF. By virtue of Proposition 5, it will suffice to prove the assertion under the assumptions $1/p_2=1/p+1/p_1\leqq 1$, $1/q_2=1/p+1/q_1\leqq 1$ and $1/r_2=1/p+1/r_1$. Since T is p-absolutely summing operator, the following result by Pietsch [12] is well known: there exists a regular positive Borel measure μ on the weakly compact unit ball K^* of X^* such that

$$\|Tx\|\leqq \pi_p(T)\Big(\int_{K^*}|\langle x, a\rangle|^p d\mu(a)\Big)^{1/p}$$

for every $x\in X$. For any finite sequence $\{x_i\}_{1\leqq i\leqq n}$ of points in X, we put

$$x_i=x_i{}^0\xi_i \quad \text{where} \quad \xi_i=\Big(\int_{K^*}|\langle x_i, a\rangle|^{r_2}d\mu(a)\Big)^{1/p}.$$

Then, by making use of Hölder's inequality and on account of the note used in the proof of Proposition 4, (ii), it yields

$$(5)\qquad (\sum_i i^{q_2/p_2-1}\|STx_i\|_*{}^{q_2})^{1/q_2}$$
$$\leqq (\sum_i i^{q_1/p_1-1}\|STx_i{}^0\|_*{}^{q_1})^{1/q_1}(\sum_i|\xi_i|^p)^{1/p}$$
$$\leqq \pi_{p_1,q_1;r_1}(S)\sup_{\|b\|\leqq 1}(\sum_i|\langle Tx_i{}^0, b\rangle|^{r_1})^{1/r_1}\Big(\sum_i\int_{K^*}|\langle x_i, a\rangle|^{r_2}d\mu(a)\Big)^{1/p}.$$

The terms of the form $\langle Tx, b\rangle$ in the latter expression can be written as

$$\langle Tx, b\rangle=\int_{K^*}\langle x, a\rangle f(a)d\mu(a)$$

for every $x\in X$ with an $f\in L_{p'}(K^*, \mu)$, satisfying the inequality

$$(6)\qquad \Big(\int_{K^*}|f(a)|^{p'}d\mu(a)\Big)^{1/p'}\leqq \pi_p(T)\|b\|,$$

$1/p+1/p'=1$. In fact, let $E_p(K^*, \mu)$ be the subspace of $L_p(K^*, \mu)$ which is constituted by the rest classes $\hat{\varphi}_x$ for $\varphi_x(a)=\langle x, a\rangle\in C(K^*)$ with $x\in X$. Then, for each $b\in Y^*$ there exists a linear form β_b on $E_p(K^*, \mu)$ defined by

$$\langle\hat{\varphi}_x, \beta_b\rangle=\langle Tx, b\rangle$$

and it satisfies

$$|\langle\hat{\varphi}_x, \beta_b\rangle|\leqq \|Tx\|\|b\|$$
$$\leqq \pi_p(T)\Big(\int_{K^*}|\langle x, a\rangle|^p d\mu(a)\Big)^{1/p}\|b\|.$$

Therefore, there exists an $f\in L_{p'}(K^*, \mu)$, $1/p+1/p'=1$, such that

$$\langle Tx, b\rangle=\int_{K^*}\langle x, a\rangle f(a)d\mu(a)\qquad \text{for } x\in X$$

and it satisfies (6). Hence we observe that

$$|\langle Tx, b\rangle| \leqq \int_{K^*} |\langle x, a\rangle| \, |f(a)| \, d\mu(a)$$

$$= \int_{K^*} |\langle x, a\rangle|^{r_2/p} (|\langle x, a\rangle|^{r_2} |f(a)|^{p'})^{1/r_1} |f(a)|^{p'/r'} d\mu(a)$$

with $1/p+1/r_1+1/r'=1$, and by Hölder's inequality

$$\leqq \Big(\int_{K^*} |\langle x, a\rangle|^{r_2} d\mu(a)\Big)^{1/p} \Big(\int_{K^*} |\langle x, a\rangle|^{r_2} |f(a)|^{p'} d\mu(a)\Big)^{1/r_1}$$

$$\times \Big(\int_{K^*} |f(a)|^{p'} d\mu(a)\Big)^{1/r'}.$$

Replacing x by x_i^0 in the above inequality, we obtain

$$|\langle Tx_i^0, b\rangle|^{r_1}$$

$$\leqq \Big[\xi_i^{-r_2/p} \Big(\int_{K^*} |\langle x_i, a\rangle|^{r_2} d\mu(a)\Big)^{1/p}$$

$$\times \xi_i^{-r_2/r_1} \Big(\int_{K^*} |\langle x_i, a\rangle|^{r_2} |f(a)|^{p'} d\mu(a)\Big)^{1/r_1}$$

$$\times \Big(\int_{K^*} |f(a)|^{p'} d\mu(a)\Big)^{1/r'}\Big]^{r_1}$$

$$= \Big(\int_{K^*} |\langle x_i, a\rangle|^{r_2} |f(a)|^{p'} d\mu(a)\Big) \Big(\int_{K^*} |f(a)|^{p'} d\mu(a)\Big)^{r_1/r'}.$$

Summing up all these terms for $i=1, 2, \cdots, n$, we get

$$(\sum_i |\langle Tx_i^0, b\rangle|^{r_1})^{1/r_1}$$

$$\leqq \sup_{\|a\| \leqq 1} (\sum_i |\langle x_i, a\rangle|^{r_2})^{1/r_1} \Big(\int_{K^*} |f(a)|^{p'} d\mu(a)\Big)^{1/p'}.$$

Hence applying this inequality to the right hand side of (5) and in view of (6), we have

$$(\sum_i i^{q_2/p_2-1} \|STx_i\|_*^{q_2})^{1/q_2}$$

$$\leqq \pi_{p_1,q_1;r_1}(S) \pi_p(T) \sup_{\|a\| \leqq 1} (\sum_i |\langle x_i, a\rangle|^{r_2})^{1/r_2},$$

which completes the proof.

REMARK 2. This theorem in case of $p_1=q_1=r_1=p=2$, $p_2=q_2=r_2=1$, coincides with the classical theorem mentioned in the first of this section.

THEOREM 2. *Let* X, Y *and* Z *be Banach spaces and* $1 \leqq p, p_1, q_1, r_1 \leqq \infty$ *satisfy* $1/p+1/p_1 \geqq 1$, $1/p+1/q_1 \geqq 1$ *and* $1/p+1/r_1 \leqq 1$. *Then, for any* $T \in \Pi_p(X, Y)$ *and* $S \in \Pi_{p_1,q_1;r_1}(Y, Z)$ *the composition* ST *belongs to* $\Pi_1(X, Z)$ *and satisfies*

$$\pi_1(ST) \leqq \pi_{p_1,q_1;r_1}(S) \pi_p(T).$$

PROOF. In case of $p=1$, this is clear by Proposition 4. Thus, we proceed to show this in case of $p>1$. Let us put $1/p+1/p'=1$. Then it satisfies $1\leqq p_1\leqq p'$, $1\leqq q_1\leqq p'$ and $r_1\geqq p'$. By Proposition 5, $S\in\Pi_{p_1,q_1;r_1}(Y, Z)\subset\Pi_{p'}(Y, Z)$. Hence Theorem 1 is applicable to these S and T. Therefore we obtain $ST\in\Pi_1(X, Z)$ and $\pi_1(ST)\leqq\pi_{p_1,q_1;r_1}(S)\pi_p(T)$. This establishes the proof.

§ 3. $(p, q; r)$-absolutely summing operators on $\mathcal{L}_p$-spaces.

In the preceding theory of $\Pi_{p,q;r}(X, Y)$, when X is a space of type $\mathcal{L}_p$ [5], it may happen the special relation among the spaces $\Pi_{p,q;r}(X, Y)$. To see that we first prepare the next

LEMMA 1. *Let* X *be isomorphic to a subspace of* $L_1(\mu)$ *for a measure space* (K, Σ, μ) *and* Y *be any Banach space. Then,* $T\in B(X, Y)$ *is* $(p, q; 1)$*-absolutely summing if and only if for any* $S\in B(l_\infty, X)$ *the composition* $TS\in\Pi_{p,q;1}(l_\infty, Y)$.

PROOF. By virtue of Proposition 4, it is clear that if $T\in\Pi_{p,q;1}(X, Y)$ and $S\in B(l_\infty, X)$, then $TS\in\Pi_{p,q;1}(l_\infty, Y)$. We next assume that $T\in B(X, Y)$ satisfies the condition $TS\in\Pi_{p,q;1}(l_\infty, Y)$ for any $S\in B(l_\infty, X)$, but $T\notin\Pi_{p,q;1}(X, Y)$. Then there exists a sequence $\{x_i\}\subset X$ such that $\sum_i x_i$ converges unconditionally, and

$$\sum_i i^{q/p-1}\|Tx_i\|_*{}^q=\infty. \tag{7}$$

Now we define $S\in B(l_\infty, X)$ as $S(\{a_i\})=\sum_i a_ix_i$ for each $\{a_i\}\in l_\infty$. On the other hand, from (7), there exists a sequence $\{\eta_i\}\in c_0$ such that

$$\sum_i i^{q/p-1}(\eta_i\|Tx_i\|_*)^q=\infty.$$

Hence it holds

$$\begin{aligned}&\sum_i i^{q/p-1}\|TS(\eta_ie_i)\|_*{}^q\\ &\geqq\sum_i i^{q/p-1}(\eta_i\|Tx_i\|_*)^q=\infty,\end{aligned}$$

where $e_i=(\overbrace{0,\cdots,0,1}^{i},0,\ )$. Thus we have $TS\notin\Pi_{p,q;1}(l_\infty, Y)$. This contradiction leads to the completion of the proof.

We note here, by a result of Lindenstrauss and Pełczyński [5], that the operator S cited in Lemma 1 is 2-absolutely summing.

THEOREM 3. *Let* X *and* Y *be the same spaces in Lemma* 1. *If real numbers* $1\leqq p_i,\ q_i\leqq\infty$, $i=1, 2$, *satisfy the conditions* $1\leqq p_1\leqq 2$, $1\leqq q_1\leqq 2$, $p_1\geqq q_1$, $1/p_2=1/p_1-1/2$ *and* $1/q_2=1/q_1-1/2$, *then we have*

$$\Pi_{p_1,q_1;1}(X, Y)=\Pi_{p_2,q_2;2}(X, Y).$$

PROOF. Since the above conditions satisfy the assumptions of Proposition 6, it is clear

$$\Pi_{p_1,q_1;1}(X, Y) \subset \Pi_{p_2,q_2;2}(X, Y).$$

The reverse inclusion relation is proved as follows. Let $T \in \Pi_{p_2,q_2;2}(X, Y)$. Since, in view of the notice cited before Theorem 3, any $S \in B(l_\infty, X)$ is 2-absolutely summing, on account of Theorem 1 we obtain $TS \in \Pi_{p_1,q_1;1}(l_\infty, Y)$. Hence owing to Lemma 1, we have $T \in \Pi_{p_1,q_1;1}(X, Y)$, which completes the proof.

As some examples for these cases we obtain the following

COROLLARY. *Suppose p_i, q_i, $i=1, 2$, satisfy the same conditions of Theorem 3. Let $1 \leqq r \leqq 2$ and Y be any Banach space. Then we have*

$$\Pi_{p_1,q_1;1}(l_r, Y) = \Pi_{p_2,q_2;2}(l_r, Y)$$

and

$$\Pi_{p_1,q_1;1}(L_r(0, 1), Y) = \Pi_{p_2,q_2;2}(L_r(0, 1), Y).$$

This is a consequence of Theorem 3 and the result of [**5**] asserting that for $1 \leqq r \leqq 2$ the spaces l_r and $L_r(0, 1)$ are isomorphic to a subspace of $L_1(\mu)$.

§4. $(p, q; r)$-absolutely summing operators on Hilbert space.

Throughout this section, let H be a Hilbert space. When we consider the operator on Hilbert space, some particular relations may happen between $(p, q; r)$-absolutely summing operators. For instance, from [**4**], $\Pi_{2,1}(H, H) = B(H, H)$ is known. On the other hand, if $1/2+1/p \leqq 1/2+1/q \leqq 1/r$, by virtue of Corollary of Proposition 6 it holds $\Pi_{p,q;r}(H, H) \supset \Pi_{2,1}(H, H)$, by which we obtain

THEOREM 4. *Let $1 \leqq p$, q, $r < \infty$, and $1/2+1/p \leqq 1/2+1/q \leqq 1/r$, then we have*

$$\Pi_{p,q;r}(H, H) = B(H, H).$$

Extending the operator of type l_p by Pietsch [**11**], which is itself also a generalization of Hilbert-Schmidt operator and nuclear operator, we shall define the operator of type $l_{p,q}$. Let X and Y be two Banach spaces, and $\mathcal{A}_i(X, Y)$, $i=0, 1, 2, \cdots$, be the space of all the i-dimensional operators, namely the operators with range of at most i-dimension. For $T \in B(X, Y)$, the number

$$\alpha_i(T) = \inf\{\|T-A\| : A \in \mathcal{A}_i(X, Y)\}$$

is called the approximation number of T [**11**].

DEFINITION 2. If the operator $T \in B(X, Y)$ satisfies the condition $\{\alpha_i(T)\} \in l_{p,q}$ (resp. l_p), $1 \leqq p$, $q \leqq \infty$, then T is called *the operator of type $l_{p,q}$* (resp. l_p). The collection of all the operators of type $l_{p,q}$ (resp. l_p) is denoted by

$l_{p,q}(X, Y)$ (resp. $l_p(X, Y)$) and we denote $\|\{\alpha_i(T)\}\|_{l_{p,q}}$ (resp. $\|\{\alpha_i(T)\}\|_{l_p}$) by $l_{p,q}(T)$ (resp. $l_p(T)$). Then $l_{p,q}(X, Y)$ becomes a Banach space with the norm $l_{p,q}(T)$.

In particular case when X and Y are Hilbert spaces H, the operators of type $l_{p,q}$ have been treated by Dunford and Schwartz [1] (class C_p), Gohberg and Kreĭn [2] (class $\mathfrak{S}_p$) and Triebel [14] (class $\mathfrak{S}_{p,q}$) etc.

Since $l_{p,q}(H, H)$ and $\Pi_{p,q;r}(H, H)$ are both normed ideals of $B(H, H)$, by Gohberg and Kreĭn [2], Chapter III, the operator of type $l_{p,q}$ and $(p, q; r)$-absolutely summing operator on H are compact operators. Therefore, by the spectral decomposition theorem [11], 8. 3, $T \in l_{p,q}(H, H)$ can be represented as follows:

$$Tx = \sum_{i=1}^{\infty} \rho_i(x, e_i)f_i \qquad \text{for } x \in X$$

with certain two orthonormal systems $\{e_i\}$ and $\{f_i\}$ in H and $\{\rho_i\}$ such that $\|\{\rho_i\}\|_{l_{p,q}} = l_{p,q}(T) < \infty$.

In the rest of this section we shall investigate a certain relation between $(p, q; r)$-absolutely summing operators and the operators of type $l_{p,q}$. For this purpose we prepare some known results. Let $\mathfrak{N}(H, H)$ be the space of nuclear operators with the norm $\nu(T)$. Then, H. Triebel [14], Lemma 1, proved

LEMMA 2. *For any* $\theta: 0 < \theta < 1$, *putting* $1/p = 1-\theta$, *we have* $(\mathfrak{N}(H, H), l_\infty(H, H))_{\theta,q} \sim l_{p,q}(H, H)$ *with* $1 \leqq q \leqq \infty$.

Moreover, on account of the reiteration theorem for interpolation spaces [6], it yields

COROLLARY. *Let* $1 < p_i,\ p < \infty,\ 1 \leqq q_i,\ q < \infty,\ i = 1, 2$, *satisfy* $1/p = (1-\theta)/p_1 + \theta/p_2$ *with* $0 < \theta < 1$. *Then we get*

$$(l_{p_1,q_1}(H, H),\ l_{p_2,q_2}(H, H))_{\theta,q} \sim l_{p,q}(H, H)\,.$$

Concerning with the mean space for $\Pi_{p,q;r}(X, Y)$, X and Y being Banach spaces, we obtained the following lemma in the previous paper [9].

LEMMA 3. *Suppose that* $1 \leqq p_i,\ p < \infty,\ 1 \leqq q_i,\ q,\ r \leqq \infty,\ i = 1, 2$, *satisfy the conditions* $p_1 \neq p_2$, $1/p = (1-\theta)/p_1 + \theta/p_2$ *for some* $\theta: 0 < \theta < 1$. *Then it holds*

$$(\Pi_{p_1,q_1;r}(X, Y),\ \Pi_{p_2,q_2;r}(X, Y))_{\theta,r} \subset \Pi_{p,q;r}(X, Y)\,.$$

With the aid of these Lemmas we get the next two theorems.

THEOREM 5. *Let* $2 < p < \infty$, $1 \leqq q \leqq \infty$ *and* $1 < r < 2$. *Then, for any* p_1: $1/p_1 \leqq 1/p + 1/r - 1/2$, *we have*

$$l_{p,q}(H, H) \subset \Pi_{p_1,q;r}(H, H)\,.$$

PROOF. We notice first the following equations:

$$l_1(\mathrm{H}, \mathrm{H}) = \mathfrak{N}(\mathrm{H}, \mathrm{H})$$

with $l_1(\mathrm{T}) = \nu(\mathrm{T})$ for every $\mathrm{T} \in l_1(\mathrm{H}, \mathrm{H})$ [**11**], and

$$l_2(\mathrm{H}, \mathrm{H}) = \Pi_p(\mathrm{H}, \mathrm{H}) = \mathfrak{S}_2(\mathrm{H}, \mathrm{H}),\ 1 \leqq p < \infty ,$$

with $l_2(\mathrm{T}) = \pi_p(\mathrm{T}) = \sigma(\mathrm{T})$ for every $\mathrm{T} \in l_2(\mathrm{H}, \mathrm{H})$ [**11**], [**12**], [**10**], where $\mathfrak{S}_2(\mathrm{H}, \mathrm{H})$ denotes the space of all Hilbert-Schmidt operators with the norm $\sigma(\mathrm{T})$. Hence, by Lemma 2 and its corollary we have

$$\begin{aligned} (8) \qquad l_{p,q}(\mathrm{H}, \mathrm{H}) &\sim (\mathfrak{N}(\mathrm{H}, \mathrm{H}),\ l_\infty(\mathrm{H}, \mathrm{H}))_{1-1/p,q} \\ &\sim ((\mathfrak{N}(\mathrm{H}, \mathrm{H}),\ l_\infty(\mathrm{H}, \mathrm{H}))_{1/2,2},\ (\mathfrak{N}(\mathrm{H}, \mathrm{H}),\ l_\infty(\mathrm{H}, \mathrm{H}))_{1,s})_{1-2/p,q} \end{aligned}$$

with $1 < s < \infty$.

Here, since in view of the above notice we have $(\mathfrak{N}(\mathrm{H}, \mathrm{H}),\ l_\infty(\mathrm{H}, \mathrm{H}))_{1/2,2} \sim l_2(\mathrm{H}, \mathrm{H}) = \Pi_r(\mathrm{H}, \mathrm{H})$ with $1 < r < 2$, and, by Theorem 4, $(\mathfrak{N}(\mathrm{H}, \mathrm{H}),\ l_\infty(\mathrm{H}, \mathrm{H}))_{1,s} = \Pi_{r_1,r}(\mathrm{H}, \mathrm{H})$ with $1/r = 1/2 + 1/r_1$, the right hand side of (8) is equivalent to $(\Pi_r(\mathrm{H}, \mathrm{H}), \Pi_{r_1,r}(\mathrm{H}, \mathrm{H}))_{1-2/p,q}$ which is contained in $\Pi_{p_1,q\,;\,r}(\mathrm{H}, \mathrm{H})$ with $1/p + 1/r - 1/2 = 1/p_1$ by virtue of Lemma 3. This shows Theorem 5.

THEOREM 6. *Let* $1 \leqq p,\ q \leqq \infty,\ 1 \leqq r < 2$ *and let positive numbers* $p_1,\ q_1$ *satisfy* $1/p_1 \leqq 1/p + 1/2 - 1/r$ *and* $1/q_1 \leqq 1/q + 1/2 - 1/r$. *Then, we obtain*

$$\Pi_{p,q\,;\,r}(\mathrm{H}, \mathrm{H}) \subset l_{p_1,q_1}(\mathrm{H}, \mathrm{H}) .$$

PROOF. Let T be any element of $\Pi_{p,q\,;\,r}(\mathrm{H}, \mathrm{H})$. Then, for any finite sequence $\{x_i\}_{1 \leqq i \leqq n}$ of points in H, it holds

$$\begin{aligned} (9) \qquad (\sum_i i^{q/p-1} \|\mathrm{T}x_i\|^q)^{1/q} &\leqq (\sum_i i^{q/p-1} \|\mathrm{T}x_i\|_*{}^q)^{1/q} \\ &\leqq \pi_{p,q\,;\,r}(\mathrm{T}) \sup_{\|a\| \leqq 1} (\sum_i |\langle x_i, a \rangle|^r)^{1/r} . \end{aligned}$$

On the other hand, recalling the notice mentioned after Definition 2, T is expressed as follows:

$$\mathrm{T}x = \sum_{i=1}^{\infty} \rho_i(x, e_i) f_i \qquad \text{for every } x \in \mathrm{X} ,$$

where $\{e_i\}$, $\{f_i\}$ are two orthonormal systems in H and $\{\rho_i\} \in c_0$. We put

$$x_i = i^{(q_1/p_1 - 1)(1/r - 1/2)} |\rho_i|^{q_1(1/r - 1/2)} e_i .$$

Then the inequality (9), with this $\{x_i\}$, gives

$$(\sum_i i^{q_1/p_1 - 1} |\rho_i|^{q_1})^{1/q} \leqq \pi_{p,q\,;\,r}(\mathrm{T}) (\sum_i i^{q_1/p_1 - 1} |\rho_i|^{q_1})^{1/r - 1/2} ,$$

by which we have

$$(\sum_i i^{q_1/p_1 - 1} |\rho_i|^{q_1})^{1/q_1} \leqq \pi_{p,q\,;\,r}(\mathrm{T}) .$$

This implies that $\mathrm{T} \in l_{p,q}(\mathrm{H}, \mathrm{H})$, and completes the proof.

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