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On zeta-theta functions

(To the memory of Professor M. Sugawara)

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§ 0. Introduction.

In our previous paper [**1**], we treated certain zeta-functions attached to symmetric tensor representations of odd degrees of the group $G=SL(2, \boldsymbol{R})$. In the present paper, we deal with analogous functions connected with symmetric tensor representations of *even* degrees of the same group G.

Let M_ν^* be the "modified" symmetric tensor representation of even degree $\nu \geqq 2$ of G (Cf., 1.3). Then $M_\nu^*(\sigma)$, $\sigma \in G$, leaves an indefinite symmetric matrix S_ν invariant and so $M_\nu^*(G)$ is contained in the orthogonal group $\tilde{G}_\nu$ of S_ν. Let K be the orthogonal subgroup of G (which is a maximal compact subgroup of G) and $\tilde{K}_\nu$ be a maximal compact subgroup of $\tilde{G}_\nu$ containing $M_\nu^*(K)$. To determine $\tilde{K}_\nu$, we take and fix a definite symmetric matrix P_ν which is a "majorant" for S_ν (Cf., 1.2). Now $\tilde{H}_\nu=\tilde{K}_\nu\backslash\tilde{G}_\nu$ has a structure of Riemannian symmetric space, called the representation space of $\tilde{G}_\nu$ by Siegel [**3**]. Let $H=\{z \in \boldsymbol{C} \mid \operatorname{Im} z>0\}$ be the usual upper half plane. Then $H=K\backslash G$. Using P_ν and M_ν^*, we can define an imbedding φ_ν of H into $\tilde{H}_\nu$ (Cf., 1.3).

Let $\tilde{f}_\mathfrak{a}(\omega, \mathfrak{H})$ be the Siegel's theta-function defined on $H \times \tilde{H}_\nu \ni (\omega, \mathfrak{H})$, attached to our indefinite S_ν, where $\mathfrak{a}$ is a rational vector such that $2S_\nu\mathfrak{a}$ is integral (Cf., 2.1 or 2.3). Let $\mathfrak{a}_0, \cdots, \mathfrak{a}_t$ be a complete set of representatives mod 1 of rational vectors $\mathfrak{a}$ with integral $2S_\nu\mathfrak{a}$ and $\tilde{f}(\omega, \mathfrak{H})$ be the vector with components $\tilde{f}_{\mathfrak{a}_0}, \cdots, \tilde{f}_{\mathfrak{a}_t}$. We denote with $f_\mathfrak{a}(\nu; \omega, z)$ and $f(\nu; \omega, z)$ the pull-back of $\tilde{f}_\mathfrak{a}(\omega, \mathfrak{H})$ and $\tilde{f}(\omega, \mathfrak{H})$ to H by φ_ν, respectively. Then $f_\mathfrak{a}(\nu; \omega, z)$ is a non-holomorphic function defined on $H \times H \ni (\omega, z)$. Let $\mathfrak{F}_\mathfrak{a}(\nu)$ be a fundamental domain on H for the group $\Gamma_\mathfrak{a}(\nu)=\{\sigma \in SL(2, \boldsymbol{Z}) \mid M_\nu^*(\sigma)\mathfrak{a} \equiv \mathfrak{a} \pmod 1\}$. Then $f_\mathfrak{a}$, as a function of the second argument z, is invariant by $\Gamma_\mathfrak{a}(\nu)$ and so can be viewed as a function on $\mathfrak{F}_\mathfrak{a}(\nu)$. Using the fact that $\varphi_\nu(\mathfrak{F}_\mathfrak{a}(\nu))$, for integral $\mathfrak{a}$, is contained in the so-called Siegel domain in $\tilde{H}_\nu$ (Proposition 4), we can prove that the integral of $f_\mathfrak{a}$, with the modified factor $y^{-3\nu(\nu+1)/8}$, on $\mathfrak{F}_\mathfrak{a}(\nu)$ with respect to the invariant volume element of H is convergent (Theorem 1). Though our integral does not give a direct analogy to Siegel-Eisenstein's formula, it is conjectured that in finding the true nature of the value of this integral, we could obtain some results analogous to that formula. Thus we propose Problem 1. (Cf., Remark at the end of § 2.)

In § 3, we define zeta-functions $\zeta_{\mathfrak{a}}(\nu\,;\,\omega,\,s)$ by means of the modified Mellin integral of $f_{\mathfrak{a}}(\nu\,;\,\omega,\,iy)$ and prove the convergence of the integral for sufficiently large Re s (Theorem 2). Also we prove the functional equation satisfied by $\zeta_{\mathfrak{a}}$ and the analyticity of $\zeta_{\mathfrak{a}}$ and determine poles and residues at the poles (Theorem 3).

In § 4, ν being 2, we derive the infinite series expression for $\zeta_{\mathfrak{a}}$ (Theorem 4). Now for general even ν, $f(\nu\,;\,\omega,\,z)$ satisfies the transformation formula, which is the pull-back by the imbedding φ_ν of the transformation formula for F (Proposition 6). Let ν be 2. As is seen in Theorem 4, $\zeta_{\mathfrak{a}}(\omega,\,s)=\zeta_{\mathfrak{a}}(2\,;\,\omega,\,s)$ has the series expression by modified Bessel functions $K_{\frac{1}{2}s}(\;)$. Therefore $\zeta_{\mathfrak{a}}$, as a function of ω, is to be called the " Bessel theta-function ", which is different from the usual " exponential theta-function ". Now it is interesting that

$$\mathcal{Z}(\omega,\,s)=\begin{pmatrix}\zeta_{\mathfrak{a}_0}(\omega,\,s)\\ \zeta_{\mathfrak{a}_1}(\omega,\,s)\end{pmatrix}$$

satisfies the same transformation formula as that for $f(2\,;\,\omega,\,z)$, where $\{\mathfrak{a}_0,\,\mathfrak{a}_1\}$ is a complete set of representatives mod 1 of rational vectors $\mathfrak{a}$ with integral $2S_2\mathfrak{a}$ (Theorem 5)[1]. Therefore the multiplicator in that formula is given as the matrix $\Lambda(\sigma)$ with a usual type of Gaussian sums as coefficients (even for our Bessel theta-functions !). (Cf., 2.1 and 4.3.) Thus we call $\mathcal{Z}(\omega,\,s)$ the *zeta-theta function.* But our definition of zeta-function for general $\nu\geqq 4$ does not yield " zeta-theta " function in our sense. It would be interesting to find a definition of zeta-function having a certain transformation formula and to determine the multiplicator in that formula by means of " Gaussian sums ". (Problem 2.)

NOTATION. As usual, we denote by $\boldsymbol{C}, \boldsymbol{R}, \boldsymbol{Q}, \boldsymbol{Z}$ the fields of complex numbers, real numbers and rational numbers and the ring of rational integers. We mean by $\langle a_1,\,a_2,\,\cdots,\,a_n\rangle$ the diagonal matrix

$$\begin{pmatrix}a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n\end{pmatrix}.$$

For a symmetric matrix S of size n, the signature of S is $(p,\,q)$, $p+q=n$, if

1) As an application of Theorems 4, 5, we can prove a formula of Ramanujan concerning $\zeta(s)$ for the case $s=3,\,5$. Cf., our paper "On Ramanujan's formula for values of Riemann zeta-function at positive odd integers" to appear in Acta Arithmetica.

S is transformed to the matrix

$$\langle \underbrace{+1, \cdots, +1}_{p}, \underbrace{-1, \cdots, -1}_{q} \rangle .$$

For a vector or a matrix x of size n, we use Siegel's notation $S[x] = {}^t xSx$. For a ring R, we mean by $M(n, R)$ the total matrix ring of size n with coefficients in R. Numbers $c_1, c_2, \cdots$ mean some positive constants.

§1. Imbedding of H into $\tilde{H}_\nu$.

1.1. *Tensor representation.*

Let M_ν be the symmetric tensor representation of $G = SL(2, \boldsymbol{R})$ of degree ν; namely for $\sigma \in G$, $M_\nu(\sigma)$ is defined by

$$\left[\sigma\begin{pmatrix} u \\ v \end{pmatrix}\right]^\nu = M_\nu(\sigma)\begin{pmatrix} u \\ v \end{pmatrix}^\nu ,$$

where

$$\begin{pmatrix} u \\ v \end{pmatrix}^\nu = {}^t(u^\nu, u^{\nu-1}v, \cdots, uv^{\nu-1}, v^\nu) \qquad u, v \in \boldsymbol{C}.$$

$M_\nu(\sigma)$ belongs to $SL(\nu+1, \boldsymbol{R})$. It is known that $M_\nu(\sigma)$ leaves the following matrix invariant;

$$2S_\nu = \begin{pmatrix} & & & & 1 \\ & & & -\binom{\nu}{1} & \\ & & \binom{\nu}{2} & & \\ & \cdot^{\cdot^{\cdot}} & & & \\ (-1)^\nu\binom{\nu}{\nu} & & & & \end{pmatrix} .$$

Hereafter we assume throughout that ν is even. Therefore S_ν is a symmetric matrix of size $\nu+1$;

$$2S_\nu = \begin{pmatrix} & & & 1 \\ & & -\binom{\nu}{1} & \\ & \cdot^{\cdot^{\cdot}} & & \\ -\binom{\nu}{\nu-1} & & & \\ \binom{\nu}{\nu} & & & \end{pmatrix} . \tag{1}$$

LEMMA 1. *Let S_ν be the symmetric matrix in* (1). *Then the signature of S_ν is* $((\nu+2)/2, \nu/2)$ *for* $\nu \equiv 0 \pmod 4$ *and* $(\nu/2, (\nu+2)/2)$ *for* $\nu \equiv 2 \pmod 4$.

PROOF. It is easy to see that the signature of S_ν is $((\nu+2)/2, \nu/2)$ or $(\nu/2, (\nu+2)/2)$ if the entry at the center of S_ν is positive or negative, respectively. It is the $(\nu/2+1)$-th entry on the subdiagonal of S_ν and the entries of S_ν have alternate signs with the positive right top. Therefore, the entry at the center is positive if $\nu/2+1-1$ is even and negative if $\nu/2+1-1$ is odd.

More explicitly, if we define the matrix $A_\nu=(a_{ij}^\nu)$ by

$$a_{i,i}^\nu=\left(2\binom{\nu}{i-1}\right)^{-\frac{1}{2}} \qquad i\neq \nu/2+1\,,$$

$$a_{i,\nu+1-i+1}^\nu=(-1)^{\nu+1}\left(2\binom{\nu}{i-1}\right)^{-\frac{1}{2}} \qquad i\neq \nu/2+1\,,$$

and

$$a_{\nu/2+1,\,\nu/2+1}^\nu=\binom{\nu}{\nu/2}^{-\frac{1}{2}},$$

then we have

$$2S_\nu[A_\nu]=\langle +1, \cdots, +1, \pm 1, -1, \cdots, -1\rangle .$$

Here at the center, $+1$ stands for $\nu\equiv 0 \pmod 4$ and -1 for $\nu\equiv 2 \pmod 4$.

1.2. *Positive matrices P_ν.*

Let K be the group $\left\{\begin{pmatrix}\cos\theta, & -\sin\theta\\ \sin\theta, & \cos\theta\end{pmatrix} \,\middle|\, \theta\in \boldsymbol{R}\right\}$, which is a maximal compact subgroup of G. Let $\tilde{G}_\nu$ be the orthogonal group of S_ν. We shall determine a maximal compact subgroup $\tilde{K}_\nu$ of $\tilde{G}_\nu$ containing $M_\nu(K)$. For this purpose, we take a positive definite matrix P_ν satisfying the following conditions (P_ν is a majorant matrix for S_ν in the sense of Hermite);

(i) $S_\nu^{-1}[P_\nu]=S_\nu$,

(ii) ${}^tP_\nu=P_\nu$,

(iii) $P_\nu[M_\nu(k)]=P_\nu$ for any $k\in K$.

For such P_ν, we put

$$\tilde{K}_\nu=\{\tilde{\sigma}\in\tilde{G}_\nu \mid P_\nu[\tilde{\sigma}]=P_\nu\}\,. \tag{2}$$

Then $\tilde{K}_\nu$ will be a maximal compact subgroup of $\tilde{G}_\nu$ containing $M_\nu(K)$ (Cf. Siegel [**3**]).

Now we define P_ν by

$$2P_\nu=\left\langle 1, \binom{\nu}{1}, \binom{\nu}{2}, \cdots, \binom{\nu}{\nu-1}, 1\right\rangle \qquad \text{(size } \nu+1). \tag{3}$$

Then it is obvious that P_ν satisfies (i), (ii). To prove that P_ν satisfies (iii), we need the following

LEMMA 2. *Put*

$$i_\nu=\begin{pmatrix} & & & 1\\ & & -1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ (-1)^{\nu/2-1} & & & \end{pmatrix} \qquad (\textit{size } \nu/2)\,.$$

For $k \in K$, we put

$$(\sharp) \qquad M_\nu(k) = \begin{pmatrix} P & \mathfrak{p} & Q \\ {}^t\mathfrak{r} & u & {}^t\mathfrak{q} \\ R & \mathfrak{s} & S \end{pmatrix}$$

with $P, Q, R, S \in M(\nu/2, \boldsymbol{R})$ and column vectors $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{s}$ of size $\nu/2$. Then we have

$$R = j_\nu Q j_\nu, \qquad S = (-1)^{\nu/2+1} j_\nu P j_\nu,$$

$${}^t\mathfrak{q} = (-1)^{\nu/2}\, {}^t\mathfrak{r} j_\nu, \qquad \mathfrak{s} = -j_\nu \mathfrak{p}.$$

Conversely, if $P, Q, R, S, \mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{s}$ are given by these formulas, the matrix in the right hand side of ($\sharp$) *is contained in $M_\nu(K)$.*

The proof goes in the same way as in the proof of Lemma 3.3 of Katayama [**1**].

PROPOSITION 1. *The P_ν, defined in* (3), *satisfies* (iii); *namely $P_\nu[M_\nu(k)] = P_\nu$ holds for any $k \in K$.*

PROOF. Put

$$J_\nu = \begin{pmatrix} & & j_\nu \\ & (-1)^{\nu/2} & \\ {}^tj_\nu & & \end{pmatrix} \qquad (\text{size } \nu+1).$$

Then we have

$$P_\nu = J_\nu S_\nu = S_\nu J_\nu, \qquad {}^tJ_\nu = J_\nu \quad \text{and} \quad J_\nu^{-1} = J_\nu.$$

Also,

$$j_\nu j_\nu = (-1)^{\nu/2-1} = (-1)^{\nu/2+1} \quad \text{and} \quad j_\nu = (-1)^{\nu/2+1} j_\nu.$$

We have $S_\nu[M_\nu(k)] = S_\nu$ for any $k \in K$. Hence

$${}^tM_\nu(k) P_\nu J_\nu M_\nu(k) J_\nu = P_\nu,$$

$${}^tM_\nu(k) S_\nu J_\nu J_\nu M_\nu(k) J_\nu = S_\nu J_\nu.$$

Thus for the proof, it is sufficient to prove

$$(*) \qquad M_\nu(k) = J_\nu M_\nu(k) J_\nu.$$

But (*) can be seen by the straight-forward calculation by Lemma 2.

1.3. *Imbedding φ_ν.*

We define

$$M_\nu^*(\sigma) = I_\nu M_\nu(\sigma) I_\nu \qquad \text{for} \quad \sigma \in G$$

with

$$I_\nu = \begin{pmatrix} & & & 1 \\ & & \cdot^{\cdot^{\cdot}} & \\ & 1 & & \\ 1 & & & \end{pmatrix} \qquad (\text{size } \nu+1).$$

Then we have

$$S_\nu[I_\nu]=S_\nu\,, \qquad S_\nu^{-1}[I_\nu]=S_\nu^{-1}\,,$$

$$S_\nu[M_\nu^*(\sigma)]=S_\nu \qquad \text{for} \quad \sigma\in G$$

and

$$M_\nu^*(k)=M_\nu(k) \qquad \text{for} \quad k\in K\,.$$

Since M_ν^* satisfies (i), (ii), (iii) with the same P_ν, S_ν as for M_ν, we can take the same maximal compact subgroup $\tilde{K}_\nu$ for M_ν^* as for M_ν. $M_\nu^*(K)$ is contained in $\tilde{K}_\nu$. Therefore M_ν^* induces the imbedding φ_ν of $H=K\backslash G$ into $\tilde{H}_\nu=\tilde{K}_\nu\backslash\tilde{G}_\nu$; namely

$$\varphi_\nu(z)={}^tM_\nu^*(\tau)P_\nu M_\nu^*(\tau) \tag{4}$$

for $z=x+iy={}^t\tau(i)\in H$ with $\tau=\begin{pmatrix} y^{\frac{1}{2}} & 0 \\ x/y^{\frac{1}{2}} & 1/y^{\frac{1}{2}}\end{pmatrix}\in G$. Note that by the properties of S_ν mentioned above, $S_\nu=S_\nu^{-1}[\varphi_\nu(z)]$ holds for any $z\in H$.

We shall prove the following

PROPOSITION 2.

$$\tilde{K}_\nu\cap M_\nu^*(G)=M_\nu^*(K)\,.$$

PROOF. It is sufficient to prove the above for M_ν instead of M_ν^*. $\tilde{K}_\nu\cap M_\nu(G)\supset M_\nu(K)$ is obvious. For $\tilde{\sigma}\in\tilde{K}_\nu\cap M_\nu(G)$, there exists $\sigma\in G$ such that $\tilde{\sigma}=M_\nu(\sigma)$. Put

$$M_\nu(\sigma)=\begin{pmatrix} P & \mathfrak{p} & Q \\ {}^t\mathfrak{r} & u & {}^t\mathfrak{q} \\ R & \mathfrak{s} & S\end{pmatrix}$$

and check the conditions for $M_\nu(\sigma)\in\tilde{K}_\nu$. Then we see that $M_\nu(\sigma)$ satisfies the conditions of Lemma 2. Hence by the converse part of the Lemma, we have $\sigma\in K$.

Let Γ be the elliptic modular group. Let $\mathfrak{F}$ be the standard fundamental domain in H for Γ; namely

$$\mathfrak{F}=\left\{z\in H \;\middle|\; -\frac{1}{2}\leqq \operatorname{Re} z<\frac{1}{2},\ |z|>1 \text{ for } \operatorname{Re} z>0 \text{ and } |z|\geqq 1 \text{ for } \operatorname{Re} z\leqq 0\right\}.$$

Following Siegel [3], we consider two types of domains in the space of positive definite symmetric matrices of size n (so-called Siegel domains);

1. For a given positive constant μ, we define $R^*(\mu)$ to be the set of all $\mathfrak{S}=(s_{kl})$ such that

1) $\mathfrak{S}>0$,

2) $s_1/s_2,\ s_2/s_3,\ \cdots,\ s_{n-1}/s_n<\mu$,

3) $-\mu<2s_{kl}/s_k<\mu \quad (k<l)$,

4) $s_1s_2\cdots s_n/\det\mathfrak{S}<\gamma_n\mu$,

with a constant γ_n depending only on n.

2. We write $\mathfrak{S}=\mathfrak{T}[\mathfrak{D}]$ with

$$\mathfrak{D}=\begin{pmatrix} 1 & & d_{kl} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \qquad \text{(upper-triangular)},$$

$$\mathfrak{T}=\langle t_1, t_2, \cdots, t_n\rangle .$$

For a given constant μ, we define $R^{**}(\mu)$ to be the set of all $\mathfrak{S}=\mathfrak{T}[\mathfrak{D}]$ such that

1) $\mathfrak{T}>0$,

2) $t_1/t_2, \cdots, t_{n-1}/t_n<\mu$,

3) $-\mu<d_{kl}<\mu \quad (1\leqq k<l\leqq n)$.

Siegel proved the following Lemmas ([3]).

LEMMA 3. *For* $\mathfrak{S}=(s_{kl})\in R^*(\mu)$, *put* $\mathfrak{S}_0=\langle s_1, \cdots, s_n\rangle$. *Then there exists* $\mu_1>0$ *such that*

$$\mathfrak{S}_0[\mathfrak{x}]/\mu_1\leqq \mathfrak{S}[\mathfrak{x}]\leqq \mu_1\mathfrak{S}_0[\mathfrak{x}]$$

holds for any real vector $\mathfrak{x}$ *of size* n.

LEMMA 4. 1) *For a given* $\mu>0$, *there exists* $\mu_0>0$ *such that*

$$R^{**}(\mu)\subset R^*(\mu_0).$$

2) *For a given* $\mu>0$, *there exists* $\mu_0>0$ *such that*

$$R^*(\mu)\subset R^{**}(\mu_0).$$

Taking $n=\nu+1$ in our case, we consider $R^*(\mu)$, $R^{**}(\mu)$ and so we denote them by $R_\nu^*(\mu)$, $R_\nu^{**}(\mu)$, respectively.

PROPOSITION 4. *Put* $\mathfrak{F}^*(c_1, c_2)=\{z\in H \mid \operatorname{Im} z>c_1, -c_2<\operatorname{Re} z<c_2\}$. *Then there exists* $\mu>0$, *independent of* $z\in\mathfrak{F}^*(c_1, c_2)$, *such that*

$$\varphi_\nu(\mathfrak{F}^*(c_1, c_2))\subset R_\nu^*(\mu) .$$

PROOF. To prove our Proposition, it is sufficient, by Lemma 4, to see that there exists $\mu>0$ such that

$$\varphi_\nu(z)\in R_\nu^{**}(\mu) \qquad \text{for} \quad z\in\mathfrak{F}^*(c_1, c_2).$$

For $z=x+iy={}^t\tau(i)$, with $\tau=\begin{pmatrix} y^{\frac{1}{2}} & 0 \\ x/y^{\frac{1}{2}} & 1/y^{\frac{1}{2}} \end{pmatrix}$, we have

$$\begin{aligned} M_\nu^*(\tau) &= I_\nu M_\nu(\tau) I_\nu \\ &= M_\nu\left(\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\tau\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\right) \\ &= M_\nu\left(\begin{pmatrix} 1/y^{\frac{1}{2}} & x/y^{\frac{1}{2}} \\ & y^{\frac{1}{2}} \end{pmatrix}\right) \end{aligned}$$

$$=M_\nu\left(\begin{pmatrix}1/y^{\frac{1}{2}} & \\ & y^{\frac{1}{2}}\end{pmatrix}\right)\cdot M_\nu\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right).$$

Then

$$\mathfrak{T}={}^tM_\nu\left(\begin{pmatrix}1/y^{\frac{1}{2}} & \\ & y^{\frac{1}{2}}\end{pmatrix}\right)P_\nu M_\nu\left(\begin{pmatrix}1/y^{\frac{1}{2}} & \\ & y^{\frac{1}{2}}\end{pmatrix}\right)$$

is diagonal and

$$\mathfrak{D}=M_\nu\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)$$

is triangular with 1 on the diagonal. We have

$$2\mathfrak{T}=\left\langle y^{-\nu}, \binom{\nu}{1}y^{2-\nu}, \cdots, \binom{\nu}{\nu/2}, \cdots, \binom{\nu}{\nu-1}y^{\nu-2}, y^{\nu}\right\rangle.$$

Hence if we put

$$\mathfrak{T}=\langle t_1, t_2, \cdots, t_{\nu+1}\rangle,$$

then

$$t_k=\frac{1}{2}\binom{\nu}{k-1}y^{2k-2-\nu}, \qquad k=1, \cdots, \nu+1,$$

and

$$t_k/t_{k+1}=\left(\binom{\nu}{k-1}\Big/\binom{\nu}{k}\right)y^{-2}.$$

Since $y>c_1$, by the assumption $z\in\mathfrak{F}^*(c_1, c_2)$, we have

$$t_k/t_{k+1}<c_3=\left(\binom{\nu}{k-1}\Big/\binom{\nu}{k}\right)c_1^{-2}. \tag{5}$$

If we put $\mathfrak{D}=(d_{kl})$, then every coefficient d_{kl} is some power of x up to constant factor. By the assumption $z\in\mathfrak{F}^*(c_1, c_2)$, $-c_2<x<c_2$. Hence

$$-c_4<d_{kl}<c_4 \tag{6}$$

with some positive constant c_4. By (5), (6), Proposition follows.

COROLLARY. *There exists $\mu>0$ such that*

$$\varphi_\nu(\mathfrak{F})\subset R^*(\mu).$$

§ 2. Theta-functions.

2.1. *Siegel's theta-functions.*

We shall quote some results of Siegel from [3], [4]. Let $\mathfrak{S}$ be a half integral indefinite symmetric matrix of signature (p, q) and of size n. Let H be the usual upper half plane and $\tilde{H}$ the representation space of the orthogonal group of $\mathfrak{S}$;

$$\tilde{H}=\{\mathfrak{H}>0 \mid \mathfrak{S}^{-1}[\mathfrak{H}]=\mathfrak{S}\}\,.$$

Put $\mathfrak{R}=\xi\mathfrak{S}+i\eta\mathfrak{H}$ with $\omega=\xi+i\eta\in H$. Let $\mathfrak{a}_0, \cdots, \mathfrak{a}_t$ be a complete set of representatives modulo 1 of rational vectors $\mathfrak{a}$ with integral $2\mathfrak{S}\mathfrak{a}$. For every $\mathfrak{a}_j$, Siegel defined the theta-series by

$$\begin{aligned}\tilde{f}_j(\omega, \mathfrak{H}) &= \tilde{f}_{\mathfrak{a}_j}(\omega, \mathfrak{H}) \\ &= \sum_{\mathfrak{m}\in \mathbf{Z}^n} \exp(2\pi i \mathfrak{R}[\mathfrak{m}+\mathfrak{a}_j])\,.\end{aligned}$$

Put

$$\tilde{f}(\omega)=\tilde{f}(\omega, \mathfrak{H})={}^t(\tilde{f}_0(\omega, \mathfrak{H}), \cdots, \tilde{f}_t(\omega, \mathfrak{H}))\,.$$

Then Siegel proved the following transformation formula;

$$(c\omega+d)^{-p/2}(c\bar{\omega}+d)^{-q/2}\tilde{f}(\sigma(\omega))=\Lambda(\sigma)\tilde{f}(\omega)\,, \qquad \sigma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma\,, \tag{7}$$

where the radical in the left-hand side of (7) means that of the principal branch and $\Lambda(\sigma)$ is the unitary matrix defined as follows:

(i) The case $c\neq 0$.

$$\Lambda(\sigma)=\varepsilon D^{-\frac{1}{2}}|c|^{-\frac{1}{2}n}(\lambda_{\mathfrak{a}\mathfrak{b}}(\sigma))\,, \tag{8}$$

$$D=|\det 2\mathfrak{S}|\,, \tag{9}$$

$$\varepsilon=\exp((q-p)/4\cdot\pi i)\,, \tag{10}$$

$$\lambda_{\mathfrak{a}\mathfrak{b}}(\sigma)=\sum_{\mathfrak{g} \bmod c}\exp(2\pi i/c)\{a\mathfrak{S}[\mathfrak{g}+\mathfrak{a}]-2{}^t\mathfrak{b}\mathfrak{S}(\mathfrak{g}+\mathfrak{a})+d\mathfrak{S}[\mathfrak{b}]\}\,. \tag{11}$$

(ii) The case $c=0$.

$$\Lambda(\sigma)=(e_{\mathfrak{a},a\mathfrak{a}}\exp(2\pi i ab\mathfrak{S}[\mathfrak{a}]))\,, \qquad e_{\mathfrak{a},\mathfrak{b}}=\begin{cases}1 & \mathfrak{a}\equiv\mathfrak{b} \pmod 1\\ 0 & \text{otherwise}\,.\end{cases} \tag{12}$$

Also he proved that

$$G(\sigma\sigma', \omega)=G(\sigma, \sigma'(\omega))G(\sigma', \omega) \tag{12'}$$

holds with $G(\sigma, \omega)=(c\omega+d)^{p/2}(c\bar{\omega}+d)^{q/2}\Lambda(\sigma)$.

We define

$$\tilde{\Gamma}_{\mathfrak{a}}=\{\tilde{\sigma} \text{ integral} \mid \mathfrak{S}[\tilde{\sigma}]=\mathfrak{S},\ \tilde{\sigma}\mathfrak{a}\equiv\mathfrak{a} \pmod 1)\}\,.$$

Then $\tilde{\Gamma}_{\mathfrak{a}}$ is of finite index in the unit group $\tilde{\Gamma}$ of $\mathfrak{S}$. For $\tilde{\sigma}\in\tilde{\Gamma}_{\mathfrak{a}}$, we have $\tilde{f}_{\mathfrak{a}}(\omega, \mathfrak{H})=\tilde{f}_{\mathfrak{a}}(\omega, \mathfrak{H}[\tilde{\sigma}])$. Therefore $\tilde{f}_{\mathfrak{a}}$ can be viewed as a function on a fundamental domain $\tilde{\mathfrak{F}}_{\mathfrak{a}}$ of $\tilde{\Gamma}_{\mathfrak{a}}$ on $\tilde{H}$. Siegel considered the integral

$$V_{\mathfrak{a}}^{-1}\int_{\tilde{\mathfrak{F}}_{\mathfrak{a}}}\tilde{f}_{\mathfrak{a}}(\omega, \mathfrak{H})dv \tag{13}$$

and proved that the integral, as a function of ω, is essentially equal to the Eisenstein series of ω $(n\geqq 4)$. In the above, $V_{\mathfrak{a}}$ is the volume of $\tilde{\mathfrak{F}}_{\mathfrak{a}}$ with respect to the invariant measure dv.

2.2. *Theta-series attached to* φ_ν.

Coming back to our case, we take $\mathfrak{S}=S_\nu$ and $\mathfrak{H}=\varphi_\nu(z)$. Also we consider the representation space $\tilde{H}_\nu=\tilde{K}_\nu\backslash\tilde{G}_\nu$ as $\tilde{H}$ in 2.1. Note that for $\sigma\in\Gamma$, $M_\nu^*(\sigma)$ belongs to the unit group $\tilde{\Gamma}_\nu$ of S_ν.

Let $\mathfrak{a}_0, \cdots, \mathfrak{a}_t$ be a complete set of representatives modulo 1 of rational vectors $\mathfrak{a}$ of size $\nu+1$ with integral $2S_\nu\mathfrak{a}$. Put

$$R_\nu(z)=R_\nu(\omega, z)=\xi S_\nu+i\eta\,\varphi_\nu(z)$$

with $(\omega, z)\in H\times H$, $\omega=\xi+i\eta$, $z=x+iy$. We define

$$f_j(\nu\,;\,\omega, z)=f_{\mathfrak{a}_j}(\nu\,;\,\omega, z)=\sum_{\mathfrak{m}\in\mathbf{Z}^{\nu+1}}\exp 2\pi i R_\nu(\omega, z)[\mathfrak{m}+\mathfrak{a}_j] \tag{14}$$

and

$$f(\nu\,;\,\omega, z)={}^t(f_0(\nu\,;\,\omega, z), \cdots, f_t(\nu\,;\,\omega, z))\,.$$

Define

$$\Gamma_{\mathfrak{a}}(\nu)=\{\sigma\in\Gamma\,|\,M_\nu^*(\sigma)\mathfrak{a}\equiv\mathfrak{a}\pmod 1\}\,.$$

That σ belongs to $\Gamma_{\mathfrak{a}}(\nu)$ gives congruence conditions to the entries of σ, hence $\Gamma_{\mathfrak{a}}(\nu)$ is of finite index in Γ.

PROPOSITION 5.

$$f_{\mathfrak{a}}(\nu\,;\,\omega, z)=f_{\mathfrak{a}}(\nu\,;\,\omega, {}^t\sigma(z)) \qquad \textit{holds for} \quad \sigma\in\Gamma_{\mathfrak{a}}(\nu)\,.$$

PROOF. z can be written as $z={}^t\tau(i)$ with

$$\tau=\begin{pmatrix} y^{\frac{1}{2}} & 0 \\ x/y^{\frac{1}{2}} & 1/y^{\frac{1}{2}} \end{pmatrix}.$$

Then we have

$$\begin{aligned}\varphi_\nu({}^t\sigma(z))&=\varphi_\nu({}^t\sigma{}^t\tau(i))\\ &=P_\nu[M_\nu^*(\tau)][M_\nu^*(\sigma)]=\varphi_\nu(z)[M_\nu^*(\sigma)]\,.\end{aligned}$$

Now $M_\nu^*(\sigma)$ is unimodular and $M_\nu^*(\sigma)\mathfrak{a}\equiv\mathfrak{a}\pmod 1$ by the definition. From these, Proposition follows.

The signature of S_ν is given by Lemma 1. Therefore ε, defined in (10), is given as follows;

$$\varepsilon=\begin{cases} e^{\frac{1}{4}\pi i}, & \nu\equiv 2\pmod 4\\ e^{-\frac{1}{4}\pi i}, & \nu\equiv 0\pmod 4)\,. \end{cases} \tag{15}$$

The determinant D defined in (9) is given by

$$|\det 2S_\nu|=\binom{\nu}{0}^2\binom{\nu}{1}^2\cdots\binom{\nu}{\nu/2-1}^2\binom{\nu}{\nu/2}\,. \tag{16}$$

Now let $\Lambda_\nu(\sigma)$ be the unitary matrix defined by (8), (12) for our S_ν, σ. Then by (7), we have the following transformation formula for our f;

PROPOSITION 6. *For* $\sigma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma$, *we take* $c>0$ *or* $c=0$, $d=1$. *Then*

(i) *If* $\nu\equiv 2 \pmod 4$,

$$(c\omega+d)^{-\frac{1}{4}\nu}(c\bar{\omega}+d)^{-\frac{1}{4}(\nu+2)}f(\nu\,;\sigma(\omega),z)=\Lambda_\nu(\sigma)f(\nu\,;\omega,z)\,,$$

(ii) *If* $\nu\equiv 0 \pmod 4$,

$$(c\omega+d)^{-\frac{1}{4}(\nu+2)}(c\bar{\omega}+d)^{-\frac{1}{4}\nu}f(\nu\,;\sigma(\omega),z)=\Lambda_\nu(\sigma)f(\nu\,;\omega,z)\,.$$

2.3. *"Modified" integrals.*

Let $\mathfrak{F}_a(\nu)$ be the fundamental domain of $\Gamma_a(\nu)$ on H as the space of z. It is known that $dv(z)=dxdy/y^2$ is an invariant volume element. Since $f_a(\nu\,;\omega,z)$, as a function of z, is $\Gamma_a(\nu)$-invariant, it can be viewed as a function on $\mathfrak{F}_a(\nu)$. Then an analogue to Siegel's integral (13) would have the form:

$$\int_{\mathfrak{F}_a(\nu)}f_a(\nu\,;\omega,z)dv(z)\,. \tag{16'}$$

Here we should multiply the integrand by some additional factor so that the integral converges.

THEOREM 1. (i) *If* a *is integral,*

$$\Phi_a(\nu\,;\omega)=\int_{\mathfrak{F}}f_a(\nu\,;\omega,z)y^{-\nu(\nu+2)/8}dv(z)\ ^{2)}$$

is convergent.

(ii) *If* a *is not integral,*

$$\Phi_a(\nu\,;\omega)=\int_{\mathfrak{F}_a(\nu)}f_a(\nu\,;\omega,z)y^{-3\nu(\nu+2)/8}dv(z)$$

is convergent.

We call the above integrals the modified integrals.

PROOF. First consider the case (ii). If we put

$$\Gamma=\bigcup_k\Gamma_a(\nu)\sigma_k$$

and

$${}^t\sigma_k(\mathfrak{F})=\mathfrak{F}_k\,,$$

then we have

$$\mathfrak{F}_a(\nu)=\bigcup_k\mathfrak{F}_k\,.$$

Therefore we have

$$\int_{\mathfrak{F}_a(\nu)}=\sum_k\int_{\mathfrak{F}_k}f_a(\nu\,;\omega,z)y^{-3\nu(\nu+2)/8}dv(z)$$

2) $\mathfrak{F}=\mathfrak{F}_a(\nu)$, if a is integral.

$$= \sum_k \int_{\mathfrak{F}} f_{\mathfrak{a}}(\nu\,;\,\omega,\,{}^t\sigma_k(w))(\mathrm{Im}\,{}^t\sigma_k(w))^{-3\nu(\nu+2)/8} dv({}^t\sigma_k(w))\,.$$

Take a rational vector $\mathfrak{b}$ such that $\mathfrak{b} \equiv M_\nu^*(\sigma_k)\mathfrak{a}$ (mod 1), then by definition

$$f_{\mathfrak{a}}(\nu\,;\,\omega,\,{}^t\sigma_k(w)) = f_{\mathfrak{b}}(\nu\,;\,\omega,\,w)\,.$$

For ${}^t\sigma_k = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$ and $w = x+iy$, we have $\mathrm{Im}\,{}^t\sigma_k(w) = y/|cw+d|^2$. Hence it is sufficient to prove that

$$(16'') \qquad \int_{\mathfrak{F}} f_{\mathfrak{b}}(\nu\,;\,\omega,\,w)|cw+d|^{\frac{1}{4}\nu(\nu+2)} y^{-3\nu(\nu+2)/8} dx\,dy/y^2$$

is convergent. Now

$$|cw+d|^2 = (cx+d)^2 + c^2y^2$$

and x is bounded. Since σ_k is finite in number, $(cx+d)^2$ is bounded, too, and we have

$$|cw+d|^2 \leqq l + c^2y^2$$

with some positive l. Therefore the absolute value of the integral (16″) can be majorized by

$$\int_{\mathfrak{F}} |f_{\mathfrak{b}}(\nu\,;\,\omega,\,w)|(l+c^2y^2)^{\nu(\nu+2)/8} y^{-3\nu(\nu+2)/8} dx\,dy/y^2\,.$$

To show the convergence of this integral, it is sufficient to prove the convergence of the integral, in which the factor $(l+c^2y^2)^{\nu(\nu+2)/8}$ is replaced by the maximal power of y in the expansion of it (note that $\nu(\nu+2)/8 \in \boldsymbol{Z}$); namely we have only to prove the convergence of

$$\int_{\mathfrak{F}} |f_{\mathfrak{b}}(\nu\,;\,\omega,\,w)| y^{-\nu(\nu+2)/8} dx\,dy/y^2\,.$$

Now we shall prove the convergence of this integral, which will imply the proof of case (i).

Here, we recall the theta-inversion formula. Let $\mathfrak{S}$ be a complex symmetric matrix of size n with positive definite real part. Taking a complex vector $\mathfrak{u}$, we consider the series

$$\sum_{\mathfrak{m} \in \boldsymbol{Z}^n} \exp(-\pi \mathfrak{S}[\mathfrak{m}+\mathfrak{u}])\,.$$

Then it is well-known that the following formula, called the theta-inversion formula, holds;

$$(17) \qquad \sum_{\mathfrak{m} \in \boldsymbol{Z}^n} \exp(-\pi\mathfrak{S}[\mathfrak{m}+\mathfrak{u}]) = (\det \mathfrak{S})^{-\frac{1}{2}} \sum_{\mathfrak{m} \in \boldsymbol{Z}^n} \exp(-\pi\mathfrak{S}^{-1}[\mathfrak{m}] + 2\pi i\,{}^t\mathfrak{m}\mathfrak{u})\,,$$

where the radical in the right hand side means that of the positive-for-real-positive branch.

Let us come back to our case. From definition, we have

$$|f_{\mathfrak{b}}(\nu\,;\,\omega,\,w)|\leqq\sum_{\mathfrak{m}}\exp\left(-2\pi\eta\,\varphi_\nu(w)[\mathfrak{m}+\mathfrak{b}]\right).$$

By the corollary to Proposition 4, we have $\varphi_\nu(w)\in R_\nu^*(\mu)$. Therefore, by Lemma 3, there exists a positive constant c_5 such that

$$\varphi_\nu(w)[\mathfrak{m}+\mathfrak{b}]\geqq c_5\varphi_\nu(w)_0[\mathfrak{m}+\mathfrak{b}]\,.$$ [3]

We have

$$\varphi_\nu(w)_0=\Big\langle y^{-\nu},\ \binom{\nu}{1}^2x^2y^{-\nu}+\binom{\nu}{1}y^{-\nu+2},$$
$$\binom{\nu}{2}^2x^4y^{-\nu}+\binom{\nu-1}{1}^2\binom{\nu}{1}x^2y^{-\nu+2}+\binom{\nu}{2}y^{-\nu+4},\ \cdots\Big\rangle.$$

Note that in $\varphi_\nu(w)_0$, x appears with even powers. Now, x is bounded, hence replacing x by 0, we get the following;

$$\varphi_\nu(w)_0[\mathfrak{m}+\mathfrak{b}]\geqq\Big\langle y^{-\nu},\ \binom{\nu}{1}y^{-\nu+2},\ \cdots,\ \binom{\nu}{\nu/2},\ \cdots,\ \binom{\nu}{\nu-1}y^{\nu-2},\ y^{\nu}\Big\rangle[\mathfrak{m}+\mathfrak{b}]\,.$$

For simplicity, we write $\mathfrak{m}+\mathfrak{b}={}^t(m_0,\,m_1,\,\cdots,\,m_\nu)$. Then

$$|f_{\mathfrak{b}}(\nu\,;\,\omega,\,w)|\leqq\Big(\sum_{m_{\nu/2}}\exp\Big(-\pi c_5\binom{\nu}{\nu/2}m_{\nu/2}^2\Big)\Big)$$
$$\cdot\prod_{j=1}^{\nu/2}\sum_{m_j}\exp\Big(-\pi\eta c_5\binom{\nu}{j}m_j^2y^{-\nu+2j}\Big)$$
$$\cdot\prod_{j=1}^{\nu/2}\sum_{m_{\nu/2+j}}\exp\Big(-\pi\eta c_5\binom{\nu}{\nu/2+j}m_{\nu/2+j}^2y^{2j}\Big)\,.$$

Applying the theta-inversion formula (17) to the second factor, we have

(18) the second factor

$$=y^{\nu(\nu+2)/8}(c_5\eta)^{-\frac{1}{4}\nu}\Big(\prod_{j=1}^{\nu/2}\binom{\nu}{j}\Big)^{-\frac{1}{2}}\prod_{j=1}^{\nu/2}\sum_{m_j}\exp\Big(-\frac{\pi y^{\nu-2j}}{c_5\eta\binom{\nu}{j}}m_j^2\Big)\,.$$

Thus

$$\Big|\int_{\mathfrak{F}}f_{\mathfrak{b}}(\nu\,;\,\omega,\,w)\,y^{-\nu(\nu+2)/8}dx\,dy/y^2\Big|$$
$$\leqq\int_{\mathfrak{F}}|f_{\mathfrak{b}}(\nu\,;\,\omega,\,w)|\,y^{-\nu(\nu+2)/8}dx\,dy/y^2$$
$$\leqq c_6\int_{y>c_1}y^{\nu(\nu+2)/8}\sum_{m_{\nu/2}}\exp\Big(-\pi\eta c_5\binom{\nu}{\nu/2}m_{\nu/2}^2\Big)$$
$$\cdot\prod\sum\exp\Big(-(\pi/c_5\eta)\Big(y^{\nu-2j}\Big/\binom{\nu}{j}\Big)m_j^2\Big)$$

3) For the meaning of $\varphi_\nu(w)_0$, see Lemma 3.

$$\cdot \prod \sum \exp\left(-\pi\eta c_5\binom{\nu}{\nu/2+j} y^{\nu-2j} m_{\nu/2+j}^2\right) y^{-\nu(\nu+2)/8} dy/y^2$$

$$(19) \quad = c_6 \sum \exp\left(-\pi\eta\binom{\nu}{\nu/2} m_{\nu/2}^2\right)$$

$$\cdot\int_{y>c_1} \prod \sum \exp\left(-(\pi/c_5\eta)\left(y^{\nu-2j}\Big/\binom{\nu}{j}\right)m_j^2 - \pi\eta c_5\binom{\nu}{\nu/2+j} y^{\nu-2j} m_{\nu/2+j}^2\right) dy/y^2.$$

Since $y^{\nu-2j} > c_7 y$, we have

$$(19) \leqq c_8 \int_{y>c_1} \prod \sum \exp(-c_9\pi y m_j^2 - c_{10}\pi y m_{\nu/2+j}^2) dy/y^2$$

$$\leqq c_{11} \int_{y>c_1} \prod (1+y^{-\frac{1}{2}}) dy/y^2,$$

because of

$$\sum_m \exp(-c_{12} y m^2) \leqq c_{13}(1+y^{-\frac{1}{2}}).$$

Thus we have proved our Theorem.

In the course of the proof, we have seen that the integral is convergent absolutely and uniformly on every compact subset of $\mathfrak{F}$. Hence we can change the order of $\int$ and $\sum$.

COROLLARY TO THE PROOF. (i) *If* $\mathfrak{a}$ *is integral,*

$$\int_{\mathfrak{F}} f_{\mathfrak{a}}(\nu; \omega, z) y^{-\nu(\nu+2)/8} dv(z)$$

$$= \sum_{\mathfrak{m}} \exp 2\pi i \xi S_\nu[\mathfrak{m}+\mathfrak{a}] \int_{\mathfrak{F}} \exp(-2\pi\varphi_\nu(z)[\mathfrak{m}+\mathfrak{a}]) y^{-\nu(\nu+2)/8} dv(z).$$

(ii) *If* $\mathfrak{a}$ *is not integral, the analogue to the above holds when* $y^{-\nu(\nu+2)/8}$ *is replaced by* $y^{-3\nu(\nu+2)/8}$ *and* $\mathfrak{F}$ *by* $\mathfrak{F}_{\mathfrak{a}}(\nu)$.

Here we propose

PROBLEM 1. What is the nature of $\Phi_{\mathfrak{a}}(\nu; \omega)$?

REMARK. (Concerning this problem) In general, let V be a vector space and $G \subset GL(V)$ an algebraic group defined over $\boldsymbol{Q}$. Let χ be a character of G, φ a Schwartz function on $V_{\boldsymbol{R}}$ and dg the $G_{\boldsymbol{Z}}$-invariant measure on $G_{\boldsymbol{R}}$. Then the integral

$$\int_{G_{\boldsymbol{R}}/G_{\boldsymbol{Z}}} |\chi(g)|^s \sum_{\mathfrak{m}\in V_{\boldsymbol{Z}}} \varphi(g\cdot\mathfrak{m}) dg$$

is called the Tate integral, which often appears and is important in number theory.

Let K be a maximal compact subgroup of $G_{\boldsymbol{R}}$ and χ a character of $G_{\boldsymbol{R}}$ with value 1 on K. Assume that φ is left-invariant by K. We take measures dg, $d\dot{g}$ and dk on $G_{\boldsymbol{R}}$, $K\backslash G_{\boldsymbol{R}}$ and K so that the total volume of K is 1 and

$dg = d\dot{g}dk$. Then the above Tate integral is of the form

$$(*) \qquad \int_{K\backslash G_{\boldsymbol{R}}/G_{\boldsymbol{Z}}} |\chi(\dot{g})|^s \sum_{\mathfrak{m}\in V_{\boldsymbol{Z}}} \varphi(\dot{g}\mathfrak{m})d\dot{g}\,.$$

Now we consider Siegel's case, where $\mathfrak{S}$ is a half-integral indefinite symmetric matrix of size n, G is the orthogonal group of $\mathfrak{S}$ and $K\backslash G_{\boldsymbol{R}} = \tilde{H}$ is the representation space of $G_{\boldsymbol{R}}$. For $\omega = \xi + i\eta \in H$, $\mathfrak{H} \in \tilde{H}$, we put $\mathfrak{R} = \xi\mathfrak{S} + i\eta\mathfrak{H}$ and consider

$$\sum_{\mathfrak{m}\in \boldsymbol{Z}^n} \exp 2\pi i \mathfrak{R}[\mathfrak{m}]$$

which stands for $\sum_{\mathfrak{m}\in V_{\boldsymbol{Z}}} \varphi(\dot{g}\mathfrak{m})$ in (*).

Let $G_{\boldsymbol{Z}}$ be the unit group of $\mathfrak{S}$ and $\mathfrak{F}$ the fundamental domain of $G_{\boldsymbol{Z}}$ on $\tilde{H}$. χ being a character of $G_{\boldsymbol{R}}$ with $\chi = 1$ on K, the integral (*) will be of the form

$$\int_{\mathfrak{F}} |\chi(\mathfrak{H})|^s \sum_{\mathfrak{m}\in \boldsymbol{Z}^n} \exp 2\pi i \mathfrak{R}[\mathfrak{m}]\, d\mathfrak{H} = \Phi(\omega, s)\,.$$

When $s = 0$, $\Phi(\omega, s)$ is nothing but the integral considered by Siegel. Consequently the so-called Siegel formula can be viewed as an aspect for $s = 0$, at which the integrand is $G_{\boldsymbol{Z}}$-invariant, of the infinite series representation of the zeta-function $\Phi(\omega, s)$ (by Siegel, $\Phi(\omega, s)$ has a meaning at least for $s = 0$).

Coming back to our situation (for simplicity we take $\mathfrak{a} = 0$), we consider

$$\sum_{\mathfrak{m}\in \boldsymbol{Z}^{\nu+1}} \exp 2\pi i R_\nu(\omega, {}^t\tau(i))[\mathfrak{m}]$$

with

$$R_\nu(\omega, {}^t\tau(i)) = \xi S_\nu + i\eta\, \varphi_\nu({}^t\tau(i))\,,$$

$$\tau = \begin{pmatrix} y^{\frac{1}{2}} & 0 \\ x/y^{\frac{1}{2}} & 1/y^{\frac{1}{2}} \end{pmatrix}.$$

Put

$$\chi(\tau) = y \qquad \text{for} \quad \tau = \begin{pmatrix} y^{\frac{1}{2}} & \\ x/y^{\frac{1}{2}} & 1/y^{\frac{1}{2}} \end{pmatrix}.$$

Then χ is a character of the subgroup of $G = SL(2, \boldsymbol{R})$ formed by elements $\begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix}$. We extend χ to χ^* on G so that

$$\chi^*(k\tau) = \chi(\tau) \qquad \text{for} \quad k \in K\,.$$

χ^* is a character of G. Then the integral (*) will be of the form

$$\Phi(\nu\,;\, \omega, s) = \int_{\mathfrak{F}} |\chi^*(\tau)|^s \sum_{\mathfrak{m}\in \boldsymbol{Z}^{\nu+1}} \exp 2\pi i R_\nu(\omega, {}^t\tau(i))[\mathfrak{m}] dv\,,$$

where $\mathfrak{F}$ is the fundamental domain for $G_{\boldsymbol{Z}} = \Gamma$ on H and dv is the Γ-invariant volume element on H. Our Theorem asserts that $\Phi(\nu;\omega,s)$ is convergent absolutely and uniformly for $\mathrm{Re}\, s \leqq -\nu(\nu+2)/8$. Thus in our case, the infinite series representation of the zeta-function $\Phi(\nu;\omega,s)$ would give the "weak Siegel formula", and if $\Phi(\nu;\omega,s)$ can be continued analytically to the whole s-plane and does not have a pole at $s=0$, we would have the "exact formula".

§ 3. Zeta-functions.

3.1. *Definition.* We put

$$\mathfrak{m}=\begin{pmatrix}\mu_0\\ m_{\nu/2}\\ \mu_1\end{pmatrix},\qquad \mathfrak{a}=\begin{pmatrix}\alpha_0\\ a_{\nu/2}\\ \alpha_1\end{pmatrix}$$

with column vectors μ_j, α_j of size $\nu/2$ and

$$S_\nu=\begin{pmatrix} & & T_\nu\\ & \frac{1}{2}\varepsilon_\nu\binom{\nu}{\nu/2} & \\ {}^tT_\nu & & \end{pmatrix}$$

with the matrix T_ν of size $\nu/2$, where we define $\varepsilon_\nu=1$ or -1 according as $\nu\equiv 0$ or $2 \pmod 4$. Then

$$S_\nu[\mathfrak{m}+\mathfrak{a}]=2\,{}^t(\mu_0+\alpha_0)T_\nu(\mu_1+\alpha_1)+\frac{1}{2}\varepsilon_\nu\binom{\nu}{\nu/2}(m_{\nu/2}+a_{\nu/2})^2\,.$$

Further, putting

$$\varphi_\nu(iy)=\begin{pmatrix} Y_0 & & \\ & \frac{1}{2}\binom{\nu}{\nu/2} & \\ & & Y_1\end{pmatrix}$$

with matrices Y_j of size $\nu/2$, we have

$$\varphi_\nu(iy)[\mathfrak{m}+\mathfrak{a}]=Y_0[\mu_0+\alpha_0]+\frac{1}{2}\binom{\nu}{\nu/2}(m_{\nu/2}+a_{\nu/2})^2+Y_1[\mu_1+\alpha_1]\,.$$

Let $f_{\mathfrak{a}}(\nu;\omega,z)$ be the function defined in (14). We define

$$\theta_{\mathfrak{a}}(\nu;\omega)=\sum_{m_{\nu/2}\in \mathbf{Z}}\exp\left\{\left(\varepsilon_\nu\pi i\xi\binom{\nu}{\nu/2}-\pi\eta\binom{\nu}{\nu/2}\right)(m_{\nu/2}+a_{\nu/2})^2\right\} \tag{20}$$

and $\delta(\beta)$, for a vector β of any size, by

$$\delta(\beta)=\begin{cases}1 & \text{if } \beta \text{ is integral}\\ 0 & \text{otherwise}\,.\end{cases}$$

Then we have

(21) $f_{\mathfrak{a}}(\nu\,;\,\omega,\,iy)$

$$=\exp\left\{2\pi i\xi\left(2^{t}(\mu_0+\alpha_0)T_\nu(\mu_1+\alpha_1)+\frac{1}{2}\varepsilon_\nu\binom{\nu}{\nu/2}\right)(m_{\nu/2}+a_{\nu/2})^2\right.$$

$$\left.-2\pi\eta\left(Y_0[\mu_0+\alpha_0]+\frac{1}{2}\binom{\nu}{\nu/2}(m_{\nu/2}+a_{\nu/2})^2+Y_1[\mu_1+\alpha_1]\right)\right\}$$

$$=\theta_{\mathfrak{a}}(\nu\,;\,\omega)\Big[\delta(\alpha_0)\delta(\alpha_1)+\delta(\alpha_1)\sum_{\mu_0+\alpha_0\neq 0}\exp(-2\pi\eta Y_0[\mu_0+\alpha_0])$$

$$+\delta(\alpha_0)\sum_{\mu_1+\alpha_1\neq 0}\exp(-2\pi\eta Y_1[\mu_1+\alpha_1])$$

$$+\sum_{\mu_0+\alpha_0\neq 0,\,\mu_1+\alpha_1\neq 0}\exp\{4\pi i\xi^{t}(\mu_0+\alpha_0)T_\nu(\mu_1+\alpha_1)$$

$$-2\pi\eta(Y_0[\mu_0+\alpha_0]+Y_1[\mu_1+\alpha_1])\}\Big].$$

Applying the Theta-inversion formula (17) to

$$(*)=\delta(\alpha_0)+\sum_{\mu_0+\alpha_0\neq 0}\exp(-2\pi\eta Y_0[\mu_0+\alpha_0])$$

$$=\sum_{\mu_0\in\boldsymbol{Z}^{\nu/2}}\exp(-2\pi\eta Y_0[\mu_0+\alpha_0])\,,$$

we get

(22) $$(*)=(\det 2\eta Y_0)^{-\frac{1}{2}}\sum_{\mu_0}\exp(-(\pi/2\eta)Y_0^{-1}[\mu_0]+2\pi i\,{}^{t}\mu_0\alpha_0)$$

with

$$\det(2\eta Y_0)=\eta^{\nu/2}y^{-\frac{1}{4}\nu(\nu+2)}\binom{\nu}{0}\binom{\nu}{1}\cdots\binom{\nu}{\nu/2-1}.$$

We put

(23) $$c_\nu(\eta)=\eta^{\frac{1}{4}\nu}\binom{\nu}{0}^{\frac{1}{2}}\binom{\nu}{1}^{\frac{1}{2}}\cdots\binom{\nu}{\nu/2-1}^{\frac{1}{2}}.$$

On the basis of the above consideration, we give the following

DEFINITION (ν: even)

$$\zeta_{\mathfrak{a}}(\nu\,;\,\omega,\,s)=\int_0^\infty\left\{f_{\mathfrak{a}}(\nu\,;\,\omega,\,iy)-\frac{\delta(\alpha_1)\theta_{\mathfrak{a}}(\nu\,;\,\omega)y^{\nu(\nu+2)/8}}{c_\nu(\eta)}\right\}y^{s-1}dy\,.$$

$\zeta_{\mathfrak{a}}(\nu\,;\,\omega,\,s)$ is called the zeta-function (= the zeta-theta function for the case $\nu=2$) attached to $\mathfrak{a}$, M_ν^*.

THEOREM 2. *There exists $\sigma_0>0$ such that the integral of the $\zeta_{\mathfrak{a}}$ is convergent absolutely and uniformly for* $\operatorname{Re}s>\sigma_0$.

PROOF. From (21), we have

$$f_{\mathfrak{a}}(\nu\,;\,\omega,\,iy)-\frac{\delta(\alpha_1)\theta_{\mathfrak{a}}(\nu\,;\,\omega)}{c_\nu(\eta)}y^{\nu(\nu+2)/8}$$

$$=\theta_a(\nu\,;\,\omega)\Big[\delta(\alpha_1)\Big\{\delta(\alpha_0)+\sum_{\alpha_0+\mu_0\neq0}\cdots\Big\}$$

$$+\delta(\alpha_0)\sum_{\alpha_1+\mu_1\neq0}\cdots+\sum_{\alpha_0+\mu_0\neq0,\,\alpha_1+\mu_1\neq0}\cdots-\frac{\delta(\alpha_1)y^{\nu(\nu+2)/8}}{c_\nu(\eta)}\Big]$$

$$=\theta_a(\nu\,;\,\omega)\Big[\frac{\delta(\alpha_1)y^{\nu(\nu+2)/8}}{c_\nu(\eta)}\sum_{\mu_0}\exp(-(\pi/2\eta)Y_0^{-1}[\mu_0]+2\pi i\,{}^t\mu_0\alpha_0)$$

$$+\delta(\alpha_0)\sum_{\alpha_1+\mu_1\neq0}\cdots+\sum_{\alpha_0+\mu_0\neq0,\,\alpha_1+\mu_1\neq0}\cdots-\frac{\delta(\alpha_1)y^{\nu(\nu+2)/8}}{c_\nu(\eta)}\Big]$$

$$=\theta_a(\nu\,;\,\omega)\Big[\frac{\delta(\alpha_1)y^{\nu(\nu+2)/8}}{c_\nu(\eta)}\sum_{\mu_0\neq0}\exp(-(\pi/2\eta)Y_0^{-1}[\mu_0]+2\pi i\,{}^t\mu_0\alpha_0)$$

$$+\delta(\alpha_0)\sum_{\alpha_1+\mu_1\neq0}\cdots+\sum_{\alpha_0+\mu_0\neq0,\,\alpha_1+\mu_1\neq0}\cdots\Big].$$

Therefore, with $\mathrm{Re}\,s=\sigma$, we have

$$\Big|\int_0^\infty\Big\{f_a(\nu\,;\,\omega,\,iy)-\frac{\delta(\alpha_1)\theta_a(\nu\,;\,\omega)y^{\nu(\nu+2)/8}}{c_\nu(\eta)}\Big\}y^{s-1}dy\Big|$$

$$\leqq|\theta_a(\nu\,;\,\omega)|\int_0^\infty\Big\{\frac{\delta(\alpha_1)y^{\nu(\nu+2)/8}}{c_\nu(\eta)}\sum_{\mu_0\neq0}\exp(-(\pi/2\eta)Y_0^{-1}[\mu_0])$$

$$+\delta(\alpha_0)\sum_{\alpha_1+\mu_1\neq0}\exp(-2\pi\eta Y_1[\mu_1+\alpha_1])+\sum_{\alpha_0+\mu_0\neq0,\,\alpha_1+\mu_1\neq0}\cdots\Big\}y^{\sigma-1}dy$$

$$=|\theta_a(\nu\,;\,\omega)|\int_0^\infty\Big\{\frac{\delta(\alpha_1)y^{\nu(\nu+2)/8}}{c_\nu(\eta)}\sum_{\mu_0\neq0}\cdots+\delta(\alpha_0)\sum_{\alpha_1+\mu_1\neq0}\cdots$$

$$+\sum_{\alpha_1+\mu_1\neq0}\exp(-2\pi\eta Y_1[\mu_1+\alpha_1])\Big(\sum_{\mu_0}\exp(-2\pi\eta Y_0[\mu_0+\alpha_0])-\delta(\alpha_0)\Big)\Big\}y^{\sigma-1}dy.$$

Since it suffices to consider only the absolute values, we have

$$=|\theta_a(\nu\,;\,\omega)|\int_0^\infty\Big\{\frac{\delta(\alpha_1)y^{\nu(\nu+2)/8}}{c_\nu(\eta)}\sum_{\mu_0\neq0}\exp(-(\pi/2\eta)Y_0^{-1}[\mu_0])$$

$$+\frac{y^{\nu(\nu+2)/8}}{c_\nu(\eta)}\sum_{\alpha_1+\mu_1\neq0}\exp(-2\pi\eta Y_1[\mu_1+\alpha_1])\sum_{\mu_0}\exp\Big(-\frac{\pi}{2\eta}Y_0^{-1}[\mu_0]\Big)\Big\}y^{\sigma-1}dy$$

$$=\frac{|\theta_a(\nu\,;\,\omega)|}{c_\nu(\eta)}\int_0^\infty\Big\{\delta(\alpha_1)\sum_{\mu_0\neq0}\exp\Big(-\frac{\pi}{2\eta}Y_0^{-1}[\mu_0]\Big)$$

$$+\sum_{\substack{\alpha_1+\mu_1\neq0\\ \mu_0\neq0}}\exp\Big(-2\pi\eta Y_1[\mu_1+\alpha_1]-\frac{\pi}{2\eta}Y_0^{-1}[\mu_0]\Big)$$

$$+\sum_{\alpha_1+\mu_1\neq0}\exp(-2\pi\eta Y_1[\mu_1+\alpha_1])\Big\}y^{\nu(\nu+2)/8+\sigma-1}dy.$$

We decompose this last integral as follows:

$$=\frac{|\theta_a(\nu\,;\,\omega)|}{c_\nu(\eta)}\Big\{\int_0^1+\int_1^\infty\Big\}. \tag{24}$$

The matrices Y_0^{-1} and Y_1 are diagonal and their entries are $y^2, y^4, \cdots, y^\nu$ up to constant factors. Observing that

$$y^\nu \leqq y^{\nu-2} \leqq \cdots \leqq y^2 \quad \text{for} \quad 0<y\leqq 1$$

and

$$y^\nu \geqq y^{\nu-2} \geqq \cdots \geqq y^2 \quad \text{for} \quad y\geqq 1,$$

we replace $y^{\nu-2}, \cdots, y^2$ by y^ν in the part $\int_0^1$ of (24) and $y^\nu, \cdots, y^4$ by y^2 in the part $\int_1^\infty$ of (24) in order to majorize (24). Then $\int_1^\infty$ so changed is convergent for any s. As for $\int_0^1$, we majorize $\int_0^1$ by $\int_0^\infty$. Thus the convergence of the part $\int_0^1$ is reduced to the ordinary case of defining zeta-function by Mellin transformation of theta-function attached to the positive definite quadratic form.

3.2. *Functional equations.*

For $\mathfrak{a}=\begin{pmatrix}\alpha_0\\ a_{\nu/2}\\ \alpha_1\end{pmatrix}$, we have

$$j\mathfrak{a}=\begin{pmatrix}\alpha_1\\ a_{\nu/2}\\ \alpha_0\end{pmatrix}, \tag{25}$$

with

$$j=\begin{pmatrix} & & 1_{\nu/2}\\ & 1 & \\ 1_{\nu/2} & & \end{pmatrix}.$$

PROPOSITION 7.

$$f_{\mathfrak{a}}(\nu\,;\omega, iy)=f_{j\mathfrak{a}}(\nu\,;\omega, i/y).$$

PROOF. If y is changed to y^{-1} in $f_{j\mathfrak{a}}(\nu\,;\omega, iy)$, Y_0 and Y_1 are changed mutually. Then if we interchange α_0 and α_1 and write μ_1, μ_0 instead of μ_0, μ_1 in the summation condition on $\mathfrak{m}={}^t({}^t\mu_0, m_{\nu/2}, {}^t\mu_1)$, we get $f_{\mathfrak{a}}(\nu\,;\omega, i/y)$.

THEOREM 3. (i) *$\zeta_{\mathfrak{a}}$ satisfies the following functional equation;*

$$\zeta_{\mathfrak{a}}(\nu\,;\omega, s)=\zeta_{j\mathfrak{a}}(\nu\,;\omega, -s).$$

(ii) *$\zeta_{\mathfrak{a}}$ can be continued analytically to the whole s-plane except for its possible poles, which are at $s=\nu(\nu+2)/8$ and $-\nu(\nu+2)/8$ with the respective residues given by*

$$\frac{\delta(\alpha_0)\theta_{\mathfrak{a}}(\nu\,;\omega)}{c_\nu(\eta)} \quad \text{and} \quad -\frac{\delta(\alpha_1)\theta_{\mathfrak{a}}(\nu\,;\omega)}{c_\nu(\eta)}.$$

PROOF. We can prove the Theorem by a well-known method. We have

$$\zeta_{a}(\nu\,;\,\omega,\,s)=\int_{0}^{1}+\int_{1}^{\infty}$$

$$(26)\qquad =\int_{0}^{1}f_{a}(\nu\,;\,\omega,\,iy)y^{s-1}dy-\frac{\delta(\alpha_{1})\theta_{a}(\nu\,;\,\omega)}{c_{\nu}(\eta)}\int_{0}^{1}y^{\nu(\nu+2)/8+s-1}dy+\int_{1}^{\infty}.$$

In the above, we see that

$$\text{the second term}=-\frac{\delta(\alpha_{0})\theta_{a}(\nu\,;\,\omega)}{c_{\nu}(\eta)}\frac{1}{s+\nu(\nu+2)/8}$$

and

$$\text{the first integrand}=f_{ja}(\nu\,;\,\omega,\,i/y)$$

by Proposition 7. Therefore by (25),

$$(26)=\int_{0}^{1}\left\{f_{ja}(\nu\,;\,\omega,\,i/y)-\frac{\delta(\alpha_{0})\theta_{a}(\nu\,;\,\omega)}{c_{\nu}(\eta)}y^{-\nu(\nu+2)/8}\right\}y^{s-1}dy$$

$$+\frac{\delta(\alpha_{0})\theta_{a}(\nu\,;\,\omega)}{c_{\nu}(\eta)}\int_{0}^{1}y^{-\nu(\nu+2)/8+s-1}dy-\frac{\delta(\alpha_{1})\theta_{a}(\nu\,;\,\omega)}{c_{\nu}(\eta)}\frac{1}{s+\nu(\nu+2)/8}+\int_{1}^{\infty}$$

$$=\int_{1}^{\infty}\left\{f_{ja}(\nu\,;\,\omega,\,iy)-\frac{\delta(\alpha_{0})\theta_{a}(\nu\,;\,\omega)}{c_{\nu}(\eta)}y^{\nu(\nu+2)/8}\right\}y^{-s-1}dy$$

$$+\frac{\delta(\alpha_{0})\theta_{a}(\nu\,;\,\omega)}{c_{\nu}(\eta)}\frac{1}{s-\nu(\nu+2)/8}-\frac{\delta(\alpha_{1})\theta_{a}(\nu\,;\,\omega)}{c_{\nu}(\eta)}\frac{1}{s+\nu(\nu+2)/8}$$

$$+\int_{1}^{\infty}\left\{f_{a}(\nu\,;\,\omega,\,iy)-\frac{\delta(\alpha_{1})\theta_{a}(\nu\,;\,\omega)}{c_{\nu}(\eta)}y^{\nu(\nu+2)/8}\right\}y^{s-1}dy.$$

Theorem 3 easily follows from the last formula.

§ 4. The case $\nu=2$. Zeta-theta functions.

In this section, we always assume $\nu=2$. Hence we omit $\nu=2$ from the notation; we write, for example, $S_{\nu}=S$, $\zeta_{a}(\nu\,;\,\omega,\,s)=\zeta_{a}(\omega,\,s)$, $\cdots$ etc.

4.1. *A review of Theorem* 3.

First we note that the complete set of representatives mod 1 of rational vectors a with integral $2Sa$ are given by only two vectors; $a_{0}={}^{t}(0,\,0,\,0)$ and $a_{1}={}^{t}\left(0,\,\frac{1}{2},\,0\right)$.

Put

$$f_{a_{i}}(\omega,\,z)=f_{i}(\omega,\,z)\,,$$

$$\zeta_{a_{i}}(\omega,\,s)=\zeta_{i}(\omega,\,s)\,,$$

$$\theta_{a_{i}}(\omega)=\theta_{i}(\omega)\,,\ \cdots\ \text{etc}\,.$$

Consider now the function vector

$$\mathcal{Z}(\omega,\,s)=\begin{pmatrix}\zeta_{0}(\omega,\,s)\\ \zeta_{1}(\omega,\,s)\end{pmatrix}.$$

Since we have $j\mathfrak{a}_i=\mathfrak{a}_i$, the following functional equations hold by Theorem 3;

$$\zeta_i(\omega, s)=\zeta_i(\omega, -s) \qquad \text{for} \quad i=0, 1$$

or

$$\mathcal{Z}(\omega, s)=\mathcal{Z}(\omega, -s).$$

In (25), we see $c_\nu(\eta)=\eta^{\frac{1}{2}}$. By the shape of $\mathfrak{a}_0$, $\mathfrak{a}_1$, ζ_i has always two poles at $s=1, -1$ with residues

$$\theta_i(\omega)/\eta^{\frac{1}{2}}, \qquad -\theta_i(\omega)/\eta^{\frac{1}{2}}$$

respectively. We have by (20),

$$(27) \qquad \begin{cases} \theta_0(\omega)=\sum_{m_1\in\boldsymbol{Z}}\exp(-2\pi i\xi m_1^2-2\pi\eta m_1^2) \\ \qquad\;\; =\sum_{m_1\in\boldsymbol{Z}}\exp(-2\pi i\bar{\omega}m_1^2) \\ \theta_1(\omega)=\sum_{m_1\in\boldsymbol{Z}}\exp\left(-2\pi i\bar{\omega}\left(m_1+\frac{1}{2}\right)^2\right). \end{cases}$$

Thus we have the following

THEOREM 3′ (Theorem 3 in the case $\nu=2$). (i) *$\mathcal{Z}$ satisfies the following functional equation;*

$$\mathcal{Z}(\omega, s)=\mathcal{Z}(\omega, -s).$$

(ii) *$\mathcal{Z}(\omega, s)$ can be continued analytically to the whole s-plane, except for two poles at $s=1, -1$. The residues at $s=1, -1$ are*

$$\eta^{-\frac{1}{2}}\begin{pmatrix}\theta_0(\omega)\\ \theta_1(\omega)\end{pmatrix}, \qquad -\eta^{-\frac{1}{2}}\begin{pmatrix}\theta_0(\omega)\\ \theta_1(\omega)\end{pmatrix}$$

respectively.

4.2. *Series expansion.*

We shall determine the infinite series expansion of ζ_i. First, note that for any non-zero real numbers a, b, we have

$$(28) \qquad 2\left|\frac{b}{a}\right|^u K_u(2|ab|)=\int_0^\infty \exp\left(-\left(a^2t+\frac{b^2}{t}\right)t^{u-1}\right)dt$$

([2] p. 85). K_u is called the modified Bessel function.

We consider ζ_0. Since

$$f_0(\omega, iy)=\theta_0(\omega)\sum_{m_0,m_2\in\boldsymbol{Z}}\exp(2\pi i\xi m_0m_2-\pi\eta(m_0^2/y^2+m_2^2y^2))$$

$$=\theta_0(\omega)\left\{1+\sum_{m_2\neq0}\exp(-\pi\eta m_2^2y^2)+\sum_{m_0\neq0}\exp(-\pi\eta m_0^2/y^2)+\sum_{m_0\neq0,m_2\neq0}\cdots\right\}$$

and

$$1+\sum_{m_0\neq0}\exp(-\pi\eta\, m_0^2/y^2)=\frac{y}{\eta^{\frac{1}{2}}}\sum_{m_0\in\boldsymbol{Z}}\exp(-\pi y^2m_0^2/\eta)$$

by (17), we have

$$(28') \qquad \zeta_0(\omega, s)=\int_0^\infty\Big\{f_0(\omega, iy)-\frac{y}{\eta^{\frac{1}{2}}}\theta_0(\omega)\Big\}y^{s-1}dy$$

$$=\theta_0(\omega)\Big\{\int_0^\infty\sum_{m_2\neq0}\exp(-\pi\eta\, m_2^2y^2)\,y^{s-1}dy$$

$$+\frac{1}{\eta^{\frac{1}{2}}}\int_0^\infty\sum_{m_0\neq0}\exp(-\pi y^2m_0^2/\eta)\,y^sdy$$

$$+\int_0^\infty\sum_{m_0\neq0, m_2\neq0}\exp(-\pi\eta(m_0^2/y^2+m_2^2y^2))\,y^{s-1}dy\Big\}.$$

We can change the order of $\int$ and Σ. Then the first integral gives

$$\sum_{m_2\neq0}\int_0^\infty\exp(-\pi\eta\, m_2^2y^2)y^{s-1}dy=\frac{\Gamma\left(\frac{1}{2}s\right)}{(\pi\eta)^{\frac{1}{2}s}}\zeta(s)$$

and the second

$$\sum_{m_0\neq0}\int_0^\infty\exp(-\pi y^2m_0^2/\eta)\,y^sdy=\left(\frac{\eta}{\pi}\right)^{\frac{1}{2}(s+1)}\Gamma\left(\frac{s+1}{2}\right)\zeta(s+1),$$

where $\zeta(s)$ means the Riemann zeta-function. By (28), we have

$$\int_0^\infty\exp(-\pi\eta(m_0^2/y^2+m_2^2y^2))\,y^{s-1}dy$$

$$=\frac{1}{2}\int_0^\infty\exp(-\pi\eta(m_0^2/y+m_2^2y))\,y^{\frac{1}{2}s-1}\,dy$$

$$=\left|\frac{m_0}{m_2}\right|^{\frac{1}{2}s}K_{\frac{1}{2}s}(2\pi\eta\,|m_0m_2|).$$

Summing up the above, we get the series expansion of ζ_0. Also we get the series expansion of ζ_1 if we replace m_1 by $m_1+\frac{1}{2}$.

THEOREM 4.

$$\zeta_i(\omega, s)=\theta_i(\omega)\left\{\frac{\Gamma\left(\frac{1}{2}s\right)}{(\pi\eta)^{\frac{1}{2}s}}\zeta(s)+\frac{\Gamma\left(\frac{1}{2}(s+1)\right)}{\eta^{\frac{1}{2}}}\left(\frac{\eta}{\pi}\right)^{\frac{1}{2}(s+1)}\zeta(s+1)\right\}$$

$$+\theta_i(\omega)\sum_{m_0\neq0, m_2\neq0}\exp 2\pi i\xi m_0m_2\left|\frac{m_0}{m_2}\right|^{\frac{1}{2}s}K_{\frac{1}{2}s}(2\pi\eta\,|m_0m_2|).$$

REMARK. (1) *On the functional equation.* Put

$$\varphi(s)=\pi^{-\frac{1}{2}s}\,\Gamma\left(\frac{1}{2}s\right)\zeta(s).$$

Then it is well-known that

$$\varphi(s)=\varphi(1-s)$$

and

$$\varphi(s+1)=\varphi(-s)\,.$$

Hence if we put

$$\begin{aligned}\Phi(s)&=\eta^{-\frac{1}{2}s}\,\varphi(s)+\eta^{\frac{1}{2}s}\,\varphi(s+1)\\&=\{\quad\}\text{-part in Theorem 4},\end{aligned}$$

then we have

$$(\sharp)\qquad \Phi(s)=\Phi(-s)\,.$$

Thus the right-hand side of the formula in Theorem 4 can be written as $\theta_i(\omega)(\Phi(s)+\Psi(s))$, where $\Psi(s)=\int_0^\infty \sum_{m_0\neq 0,m_2\neq 0}\cdots dy$ (Cf. the last term in (28′)). Here, the integrand $\sum_{m_0\neq 0,m_2\neq 0}\cdots$ is obviously invariant under $y\to 1/y$ and so $\Psi(s)$ is invariant under $s\to -s$. Hence, together with ($\sharp$), we have the functional equation of $\zeta_i(\omega, s)$ by the series expansion.

(2) *On residues and poles.* The pole $s=1$ of ζ_i comes from $\dfrac{\Gamma\left(\frac{1}{2}s\right)\zeta(s)}{(\pi\eta)^{\frac{1}{2}s}}$. Since we have $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$ and $\operatorname*{Res}_{s=1}\zeta(s)=1$, we get the residue $\theta_i(\omega)/\eta^{\frac{1}{2}}$ of ζ_i at $s=1$. The pole $s=-1$ comes from $\dfrac{\Gamma\left(\frac{1}{2}(s+1)\right)}{\eta^{\frac{1}{2}}}\left(\dfrac{\eta}{\pi}\right)^{\frac{1}{2}(s+1)}\zeta(s+1)$ and so we get the residue $-\theta_i(\omega)/\eta^{\frac{1}{2}}$ of ζ_i at $s=-1$. $s=0$ looks like a pole of ζ_i at first sight, but the residues of $\dfrac{\Gamma\left(\frac{1}{2}s\right)}{(\pi\eta)^{\frac{1}{2}s}}\zeta(s)$ and $\dfrac{\Gamma\left(\frac{1}{2}(s+1)\right)}{\eta^{\frac{1}{2}}}\left(\dfrac{\eta}{\pi}\right)^{\frac{1}{2}(s+1)}\zeta(s+1)$ at $s=0$ cancel with each other. Here we used $\operatorname*{Res}_{s=0}\Gamma\left(\frac{1}{2}s\right)=2$, $\zeta(0)=-\frac{1}{2}$, $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$ and $\operatorname*{Res}_{s=1}\zeta(s)=1$.

4.3. *Transformation formula.*

Put

$$f(\omega, z)=\begin{pmatrix}f_0(\omega, z)\\ f_1(\omega, z)\end{pmatrix}.$$

We quoted Siegel's results in § 2 and proved the transformation formula for $f(\omega, z)$ in Proposition 6. In our case ($\nu=2$), it is of the following form;

$$(29)\qquad (c\omega+d)^{-\frac{1}{2}}\,(c\bar{\omega}+d)^{-1}f(\sigma(\omega), z)=\Lambda(\sigma)f(\omega, z)$$

for $\sigma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma$ with $c>0$ or $c=0$ and $d=1$. In (29), by (8)~(12), (15), (16), we have

$$(30)\quad \begin{cases} \Lambda(\sigma)=\left(\exp\frac{1}{4}\pi i\right)2^{-\frac{1}{2}}c^{-\frac{3}{2}}(\lambda_{ab}(\sigma)) & \text{if}\quad c\neq 0\,, \\ \quad =\begin{pmatrix}1 & 0\\ 0 & \exp\left(-\frac{1}{2}\pi bi\right)\end{pmatrix} & \text{if}\quad c=0\,. \end{cases}$$

In particular for $\sigma_1=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$, $\sigma_2=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, (30) takes the following form

$$(31)\quad \begin{cases} \Lambda(\sigma_1)=2^{-\frac{1}{2}}\left(\exp\frac{1}{4}\pi i\right)\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}, \\ \Lambda(\sigma_2)=\begin{pmatrix}1 & 0\\ 0 & \exp\left(-\frac{1}{2}\pi i\right)\end{pmatrix}. \end{cases}$$

Now we shall prove that $\mathcal{Z}(\omega, s)$ satisfies the same transformation formula as that for $f(\omega, z)$. (Therefore we call $\mathcal{Z}(\omega, s)$ the zeta-theta function.) Since σ_1, σ_2 generate Γ and we know (12′), it is sufficient to prove that the transformation formulas of $\mathcal{Z}(\omega, s)$ for σ_1, σ_2 are given by (29) with (31). Thus our job is to show the following:

$$(32)\qquad \mathcal{Z}(-1/\omega, s)=2^{-\frac{1}{2}}\left(\exp\frac{1}{4}\pi i\right)\omega^{\frac{1}{2}}\,\bar{\omega}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\mathcal{Z}(\omega, s)\,,$$

$$(33)\qquad \mathcal{Z}(\omega+1, s)=\begin{pmatrix}1 & 0\\ 0 & \exp\left(-\frac{1}{2}\pi i\right)\end{pmatrix}\mathcal{Z}(\omega, s)\,.$$

First we prove

LEMMA 5.

(i) $\theta_0(-1/\omega)=\left(\frac{1}{2}i\bar{\omega}\right)^{\frac{1}{2}}(\theta_0(\omega)+\theta_1(\omega))$,

(ii) $\theta_1(-1/\omega)=\left(\frac{1}{2}i\bar{\omega}\right)^{\frac{1}{2}}(\theta_0(\omega)-\theta_1(\omega))$.

PROOF. By the theta-inversion formula (17), we have

$$\sum_{m_1\in \boldsymbol{Z}}\exp(-2\pi i(-1/\bar{\omega})m_1^2)=\left(\frac{1}{2}i\bar{\omega}\right)^{\frac{1}{2}}\sum_{m_1\in \boldsymbol{Z}}\exp\left(-\frac{1}{2}\pi i\bar{\omega}m_1^2\right).$$

Then

$$\sum_{m_1}\exp\left(-\frac{1}{2}\pi i\bar{\omega}m_1^2\right)=\sum_{a \bmod 2}\sum_{m}\exp\left(-\frac{1}{2}\pi i\bar{\omega}(2m+a)^2\right)$$
$$=\sum_{m}\exp(-2\pi i\bar{\omega}m^2)+\sum_{m}\exp\left(-2\pi i\bar{\omega}\left(m+\frac{1}{2}\right)^2\right).$$

Hence (i) holds. The proof of (ii) will go in the same way.

Now

$$(34)\qquad \mathcal{Z}(-1/\omega, s)=\begin{pmatrix}\zeta_0(-1/\omega, s)\\ \zeta_1(-1/\omega, s)\end{pmatrix}$$

$$=\int_0^\infty\left\{\begin{pmatrix}f_0(-1/\omega, iy)\\ f_1(-1/\omega, iy)\end{pmatrix}-\frac{y}{\operatorname{Im}(-1/\omega)^{\frac{1}{2}}}\begin{pmatrix}\theta_0(-1/\omega)\\ \theta_1(-1/\omega)\end{pmatrix}\right\}y^{s-1}dy\,.$$

The first integrand is equal to

$$2^{-\frac{1}{2}}\left(\exp\frac{1}{4}\pi i\right)\omega^{\frac{1}{2}}\,\bar{\omega}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}f_0(\omega, iy)\\ f_1(\omega, iy)\end{pmatrix}$$

by (29), (30), (31). The second integrand is equal to, by Lemma 5,

$$\omega^{\frac{1}{2}}\,\bar{\omega}^{\frac{1}{2}}\left(\frac{1}{2}i\bar{\omega}\right)^{\frac{1}{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\theta_0(\omega)\\ \theta_1(\omega)\end{pmatrix}=2^{-\frac{1}{2}}\left(\exp\frac{1}{4}\pi i\right)\omega^{\frac{1}{2}}\,\bar{\omega}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\theta_0(\omega)\\ \theta_1(\omega)\end{pmatrix}.$$

Observing that $\operatorname{Im}(-1/\omega)=\eta\omega^{-1}\bar{\omega}^{-1}$, we have

$$(34)=2^{-\frac{1}{2}}\left(\exp\frac{1}{4}\pi i\right)\omega^{\frac{1}{2}}\,\bar{\omega}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\int_0^\infty\left\{\begin{pmatrix}f_0(\omega, iy)\\ f_1(\omega, iy)\end{pmatrix}-\frac{y}{\eta^{\frac{1}{2}}}\begin{pmatrix}\theta_0(\omega)\\ \theta_1(\omega)\end{pmatrix}\right\}y^{s-1}dy$$

$$=2^{-\frac{1}{2}}\left(\exp\frac{1}{4}\pi i\right)\omega^{\frac{1}{2}}\,\bar{\omega}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\mathcal{Z}(\omega, s)\,.$$

Hence we have proved (32). Next we shall prove (33). By definition,

$$\mathcal{Z}(\omega+1, s)=\int_0^\infty\left\{\begin{pmatrix}f_0(\omega+1, iy)\\ f_1(\omega+1, iy)\end{pmatrix}-\frac{y}{\eta^{\frac{1}{2}}}\begin{pmatrix}\theta_0(\omega+1)\\ \theta_1(\omega+1)\end{pmatrix}\right\}y^{s-1}dy\,.$$

By (29), (30), (31), we see

$$\begin{pmatrix}f_0(\omega+1, iy)\\ f_1(\omega+1, iy)\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & \exp\left(-\frac{1}{2}\pi i\right)\end{pmatrix}\begin{pmatrix}f_0(\omega, iy)\\ f_1(\omega, iy)\end{pmatrix}.$$

Also by the definition (27) of θ_i, we have

$$\theta_0(\omega+1)=\theta_0(\omega)$$

and

$$\theta_1(\omega+1)=\sum\exp -2\pi i(\bar{\omega}+1)\left(m_1+\frac{1}{2}\right)^2$$

$$=\left(\exp\left(-\frac{1}{2}\pi i\right)\right)\theta_1(\omega+1)\,.$$

Therefore,

$$\mathcal{Z}(\omega+1, s)=\begin{pmatrix}1 & 0\\ 1 & \exp\left(-\frac{1}{2}\pi i\right)\end{pmatrix}\mathcal{Z}(\omega, s)\,.$$

Finally we get the following

THEOREM 5. *The zeta-theta function* $\mathcal{Z}(\omega, s)$ *satisfies the transformation formula;*

$$(c\omega+d)^{-\frac{1}{2}}(c\bar{\omega}+d)^{-1}\mathcal{Z}(\sigma(\omega), s)=\Lambda(\sigma)\mathcal{Z}(\omega, s) \qquad \textit{for} \quad \sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma .$$

REMARK. We considered only the case $\nu=2$ in this section. For $\nu\geqq 4$, we put

$$\mathcal{Z}(\nu\,;\,\omega, s)=\begin{pmatrix} \zeta_{\mathfrak{a}_1}(\nu\,;\,\omega, s) \\ \vdots \\ \zeta_{\mathfrak{a}_l}(\nu\,;\,\omega, s) \end{pmatrix}$$

with $\zeta_{\mathfrak{a}_i}$ in 3.1. It is hopeless to see whether this $\mathcal{Z}(\nu\,;\,\omega, s)$ satisfies the same transformation formula as that of $f(\nu\,;\,\omega, z)$. Hence we propose the following

PROBLEM 2. Find the definition of zeta-function $\mathcal{Z}^*(\nu\,;\,\omega, s)$ which will be "theta" function of ω at the same time and determine the transformation formula for $\mathcal{Z}^*(\nu\,;\,\omega, s)$ by means of "Gaussian sums".

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