J. Math. Soc. Japan
Vol. 23, No. 4, 1971

On the union of two Helson sets

By Sadahiro SAEKI

(Received March 2, 1971)

The purpose of this paper is to improve and generalize some results of N. Th. Varopoulos [8]. In particular, we shall show that the union of two Helson sets in a locally compact abelian group is a Helson set.

We begin with introducing some notations. Let K be any non-empty space, and let Fag(K) be the free abelian (additive) group generated by K with the discrete topology (cf. [3; p. 8]). For any positive integer $l \in Z^+$, we denote

$$K^{(l)} = \{\sum_{i=1}^{l} n_i x_i \,;\, n_i \in Z,\, x_i \in K,\, \sum_{i=1}^{l} |n_i| \leqq l\}\,,$$

which is a subset of Fag(K). Let also $F^*(K)$ be the multiplicative group consisting of all complex-valued functions f on K such that $|f(x)|=1$ for all $x \in K$. $F^*(K)$ is a metric abelian group under the metric

$$d(f, g) = \sup_{x \in K} |f(x)-g(x)| \qquad (f, g \in F^*(K))\,.$$

Then it is easy to see that every element x of Fag(K) defines a continuous character of $F^*(K)$ by

$$\langle f, x \rangle = \prod_{i=1}^{l} \{f(x_i)\}^{n_i} \qquad (f \in F^*(K))\,,$$

where $n_i \in Z$ and $x_i \in K$ are such that $x = \sum_{i=1}^{l} n_i x_i$. This fact allows us to identify $F^*(K)$ with a subgroup of $F^*(\mathrm{Fag}(K))$.

Suppose now that $D = \{K_j\}_1^N$ be any finite partition of K into pairwise disjoint, non-empty subsets. We denote by $F^*_D = F^*_D(K)$ the closed subgroup of $F^*(K)$ consisting of those functions of $F^*(K)$ that are constant on each set K_j $(j=1, 2, \cdots, N)$. It is trivial that F^*_D is topologically isomorphic to the N-dimensional torus $T^N = \{z\,;\, |z|=1\}^N$. Let now p be a given, continuous, positive-definite function on F^*_D, and let $\{x_j \in K_j\}_1^N$ be any choice of points. We can identify the subgroup of Fag(K)

$$G_p(\{x_j\}_1^N) = \{\sum_{j=1}^{N} n_j x_j \,;\, n_j \in Z,\, j=1, 2, \cdots, N\}$$

with the dual of F^*_D in a trivial way. It follows from the classical Bochner theorem [5] that there exists a non-negative measure $\lambda \in M(\mathrm{Fag}(K))$ such that

$$\lambda[\mathrm{Fag}(K)\backslash G_p(\{x_j\}_1{}^N)]=0$$

and

$$p(f)=\int_{\mathrm{Fag}(K)}\langle f, x\rangle d\lambda(x) \qquad (f\in F^*{}_D).$$

We call any such measure λ a representing measure of p, which, of course, depends on the choice $\{x_j \in K_j\}_1{}^N$ of points.

LEMMA 1. *Let H^* be a subgroup of $F^*(K)$, and p a continuous, positive-definite function on H^*. Then, for any $\varepsilon>0$, there exists a positive integer l_ε with the following property; if D is a finite partition of K such that $F^*{}_D\subset H^*$ and if λ_D is a representing measure of p_D (=the restriction of p to $F^*{}_D$), then we have*

$$\lambda_D[\mathrm{Fag}(K)\backslash K^{(l_\varepsilon)}]<\varepsilon.$$

PROOF. The proof is essentially identical with that of Lemma 2.3 in [8], and we omit the details.

THEOREM 1. *Let G be a locally compact abelian group, K a compact subset of G, and $B^*(K)$ the closed subgroup of $F^*(K)$ consisting of all Borel functions in $F^*(K)$. Then, for every continuous, positive-definite function p on $B^*(K)$, there exists a unique non-negative Radon measure $\mu\in M(G)$ such that*

$$\text{(i)} \qquad \mu[G\backslash G_p(K)]=0,$$

and

$$\text{(ii)} \qquad p(\gamma|_K)=\int_G \gamma(x)d\mu(x) \qquad (\gamma\in\hat{G}),$$

where $\hat{G}$ denotes the dual of G.

PROOF. The uniqueness of μ is trivial. Let $\mathcal{D}$ be the directed family consisting of all finite partitions of K into pairwise disjoint, non-empty, Borel subsets. To each partition $D\in\mathcal{D}$, we associate any representing measure $\lambda_D\in M^+(\mathrm{Fag}(K))$ of p_D (=the restriction of p to $F^*{}_D$).

We now consider the identity mapping

$$K\longrightarrow K\subset G,$$

and extend it to the natural group homomorphism

$$\theta:\mathrm{Fag}(K)\longrightarrow G_p(K)\subset G.$$

For each $D\in\mathcal{D}$, let us define a discrete measure $\mu_D\in M(G)$ by setting

$$\text{(1)} \qquad \mu_D(\{x\})=\lambda_D(\theta^{-1}(x)) \qquad (x\in G).$$

Then we have

(2) $$\int_G \gamma(x)d\mu_D(x)=\int_{\mathrm{Fag}(K)}\langle\gamma|_K, x\rangle d\lambda_D(x) \qquad (\gamma\in\hat{G}),$$

and

(3) $$\mu_D\geqq 0\,;\ \|\mu_D\|=\|\lambda_D\|=p(1)$$

for all $D\in\mathscr{D}$. It also follows from (1) and Lemma 1 that, for each $\varepsilon>0$, there exists a positive integer l_ε such that

(4) $$\mu_D[G\backslash K_{(l_\varepsilon)}]<\varepsilon \qquad (D\in\mathscr{D}),$$

where

$$K_{(1)}=K\cup(-K)\,;\ K_{(n)}=K_{(n-1)}+K_{(1)} \qquad (n=2, 3, \cdots).$$

We shall now prove that

(5) $$p(\gamma|_K)=\lim_{D\in\mathscr{D}}\int_G\gamma(x)d\mu_D(x) \qquad (\gamma\in\hat{G}).$$

To do this, take any $\gamma\in\hat{G}$ and any $\varepsilon>0$. By Lemma 1, we can choose a positive integer $l=l(\varepsilon)$ so that

(6) $$2\lambda_D[\mathrm{Fag}\,(K)\backslash K^{(l)}]<\varepsilon \qquad (D\in\mathscr{D}).$$

Using the continuity of p and the definition of the set $K^{(l)}$, it is easy to find a partition $D_0\in\mathscr{D}$ and an element $f_0\in F^*{}_{D_0}$ such that

(7) $$\max\{|p(\gamma|_K)-p(f_0)|,\ \sup_{x\in K^{(l)}}|\langle f_0, x\rangle-\langle\gamma|_K, x\rangle|\}<\varepsilon.$$

Then, for all $D\in\mathscr{D}$ with $D\succ D_0$, we have

$$\begin{aligned}&\left|p(\gamma|_K)-\int_G\gamma(x)d\mu_D(x)\right|\\ &\qquad\leqq|p(\gamma|_K)-p(f_0)|+\left|p_D(f_0)-\int_G\gamma(x)d\mu_D(x)\right|,\end{aligned}$$

which together with (2), (3), (6), and (7) yields

$$\begin{aligned}\left|p(\gamma|_K)-\int_G\gamma(x)d\mu_D(x)\right|&\leqq\varepsilon+\int_{\mathrm{Fag}(K)}|\langle f_0, x\rangle-\langle\gamma|_K, x\rangle|\,d\lambda_D(x)\\ &\leqq\varepsilon+\sup_{x\in K^{(l)}}|\langle f_0, x\rangle-\langle\gamma|_K, x\rangle|\cdot\|\lambda_D\|+2\lambda_D[\mathrm{Fag}\,(K)\backslash K^{(l)}]\\ &<(2+p(1))\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ was arbitrary, we obtain (5). But (3), (4), and (5) guarantee that the net μ_D of measures converges to some measure $\mu\in M^+(G)$ in the weak-star topology of $M(G)$, and that μ satisfies (i) and (ii) in our theorem (cf. [**4**; Chapter IV, § 11 and § 12]). This completes the proof.

THEOREM 2 (cf. [8; Theorem 2.1]). *Let G be a locally compact abelian* *group, K a totally disconnected, compact subset of G, and $C^*(K)$ the closed subgroup of $F^*(K)$ consisting of all continuous functions in $F^*(K)$. Then, for every continuous, positive-definite function p on $C^*(K)$, there exists a unique non-negative Radon measure $\mu \in M^+(G)$ such that*

(i) $$\mu[G\backslash G_p(K)]=0\,,$$

and

(ii) $$p(\gamma|_K)=\int_G \gamma(x)d\mu(x) \qquad (\gamma \in \hat{G})\,.$$

PROOF. If we use finite partitions of K into clopen subsets (instead of Borel subsets), then the proof of Theorem 1 is still valid in this case.

COROLLARY 2.1 (due to Varopoulos [7], and [8; Theorem 1.1]). *Let K be a totally disconnected, compact space, and let $C^*(K)$ be as in Theorem 2. Then, for every continuous character χ of $C^*(K)$, there exists a unique element x of* Fag(K) *such that*

$$\chi(f)=\langle f, x\rangle \qquad (f\in C^*(K))\,.$$

PROOF. Let G_K be the compact dual of $\hat{G}_K$, the group $C^*(K)$ endowed with the discrete topology. Then K can be regarded as a compact subset of G_K such that

$$C^*(K)=\{\gamma|_K\,;\,\gamma\in \hat{G}_K\}\,.$$

Since χ is a character of $\hat{G}_K$, it follows that there exists a point $x\in G_K$ such that

(1) $$\chi(\gamma|_K)=\gamma(x) \qquad (\gamma\in\hat{G}_K)\,.$$

But, since χ is a continuous, positive-definite function on $C^*(K)$, and since K is totally disconnected, it follows from Theorem 2 that there exists a unique measure $\mu\in M^+(G_K)$ such that

(2) $$\mu[G_K\backslash G_p(K)]=0$$

and

(3) $$\chi(\gamma|_K)=\int_{G_K}\gamma(x)d\mu(x) \qquad (\gamma\in\hat{G}_K)\,.$$

Comparison of (1) and (3) implies that μ is a dirac measure at x, and then (2) shows $x\in G_p(K)$. Since $G_p(K)$ and Fag(K) are algebraically isomorphic, this yields the desired conclusion.

Let now $(\Omega, \mathcal{B}, \nu)$ be a finite (positive) measure space, and let $S^*(\Omega\,;\nu)$ be the topological group defined as in § 3 of [8]. We characterize compact subgroups of $S^*(\Omega\,;\nu)$ as follows.

THEOREM 3 (cf. [8; Proposition 3.3]). *Let G be a compact abelian group, and let $h: G\to S^*(\Omega\,;\nu)$ be a continuous group homomorphism. Then there* *exists a unique (up to ν-null equivalence) measurable function $b: \Omega \to \hat{G}$ such that*

(i) *The range of b is countable;*

(ii) $$h(x) = \langle b, x \rangle \qquad (x \in G).$$

Conversely, every measurable function $b: \Omega \to \hat{G}$ that satisfies (i) *determines by* (ii) *a continuous group homomorphism $h: G \to S^*(\Omega; \nu)$.*

PROOF. We first prove the uniqueness of b. To do this, suppose that b_1 and b_2 satisfy (i) and (ii). Then we have two countable partitions of Ω:

(1) $$\Omega = \cup \{b_1^{-1}(\gamma); \gamma \in L\} = \cup \{b_2^{-1}(\gamma); \gamma \in L\},$$

where L is some countable subset of $\hat{G}$. Take any $\gamma_1 \in L$ and suppose that

$$\nu[b_1^{-1}(\gamma_1) \backslash b_2^{-1}(\gamma_1)] > 0.$$

Then we have by (1)

(2) $$\nu[b_1^{-1}(\gamma_1) \cap b_2^{-1}(\gamma_2)] > 0$$

for some $\gamma_2 \in L$ different from γ_1. But (ii) implies that

$$h(x) = \gamma_j(x) \quad \text{a. e. on } b_j^{-1}(\gamma_j) \qquad (j = 1, 2)$$

for all $x \in G$. Therefore (2) yields

$$\gamma_1(x) = \gamma_2(x) \qquad (x \in G),$$

that is, $\gamma_1 = \gamma_2$, a contradiction. Thus we have

$$b_1^{-1}(\gamma_1) \subset b_2^{-1}(\gamma_1) \qquad (\gamma_1 \in L)$$

up to ν-null equivalence, which, combined with (1), implies $b_1 = b_2$ a. e. on Ω. This proves the uniqueness of b.

Suppose now that h is a continuous group homomorphism from G to $S^*(\Omega; \nu)$. We take any measurable set $E \in \mathscr{B}$, and observe that the function

$$x \longrightarrow \int_E (h(x))(\omega) d\nu(\omega)$$

is a continuous positive-definite function on G. It follows from Bochner's theorem [5] that we have

(3) $$\int_E h(x) d\nu = \sum_{\gamma \in \hat{G}} \alpha_\gamma(E) \gamma(x) \qquad (x \in G),$$

where

(4) $$\alpha_\gamma(E) \geqq 0 \quad (\gamma \in \hat{G}); \quad \sum_{\gamma \in \hat{G}} \alpha_\gamma(E) = \nu(E).$$

It is also easy to see that, for every $\gamma \in \hat{G}$, $\alpha_\gamma(\cdot)$ is a countably additive set-function on $\mathscr{B}$. Let us put

$$L=\{\gamma\in\hat{G}\,;\ \alpha_\gamma(\Omega)\neq 0\}\,,$$

which is a countable subset of $\hat{G}$ by (4). Radon-Nikodym's theorem [**3**] and (4) assure that there exist measurable functions β_γ on Ω $(\gamma\in L)$ such that

$$\beta_\gamma(\omega)\geqq 0 \quad (\omega\in\Omega,\ \gamma\in L)\,;\ \ \sum_{\gamma\in L}\beta_\gamma(\omega)=1 \quad (\omega\in\Omega) \tag{5}$$

and

$$\alpha_\gamma(E)=\int_E\beta_\gamma(\omega)d\nu(\omega) \qquad (\gamma\in L,\ E\in\mathcal{B})\,. \tag{6}$$

Substituting (6) into (3), and using (5), we see

$$\int_E h(x)d\nu=\int_E\sum_{\gamma\in L}\beta_\gamma(\omega)\gamma(x)d\nu(\omega)$$

for all $E\in\mathcal{B}$ and all $x\in G$, and hence

$$(h(x))(\omega)=\sum_{\gamma\in L}\beta_\gamma(\omega)\gamma(x) \qquad (\text{a. a. } \omega\in\Omega) \tag{7}$$

for all $x\in G$. Therefore, the fact that $|h(x)|=1$ (a. e.) for all $x\in G$ and Fubini's theorem give

$$\begin{aligned}\nu(E)&=\int_G dx\int_E|\sum_{\gamma\in L}\beta_\gamma(\omega)\gamma(x)|^2d\nu(\omega)\\&=\int_E d\nu(\omega)\int_G|\sum_{\gamma\in L}\beta_\gamma(\omega)\gamma(x)|^2dx \qquad (E\in\mathcal{B})\,,\end{aligned}$$

where dx denotes the normalized Haar measure on G. Thus, by Plancherel's theorem [**5**], we have

$$\nu(E)=\int_E\sum_{\gamma\in L}\{\beta_\gamma(\omega)\}^2d\nu(\omega) \qquad (E\in\mathcal{B})\,,$$

and hence

$$\sum_{\gamma\in L}\{\beta_\gamma(\omega)\}^2=1 \qquad (\text{a. a. } \omega\in\Omega)\,,$$

which, combined with (5), implies

$$\beta_\gamma(\omega)=0 \quad \text{or} \quad 1 \qquad (\text{a. a. } \omega\in\Omega,\ \gamma\in L)\,.$$

Changing the values of β_γ on a ν-null set so that

$$\sum_{\gamma\in L}\beta_\gamma(\omega)=\sum_{\gamma\in L}\{\beta_\gamma(\omega)\}^2=1 \qquad (\omega\in\Omega)$$

we now define a measurable function $b:\Omega\to\hat{G}$ by setting

$$b(\omega)=\gamma \qquad (\omega\in\beta_\gamma^{-1}(1)\,;\ \gamma\in L)\,.$$

Then (7) implies (ii).

Finally, the converse statement in our theorem is trivial, and this completes the proof.

Let us now suppose that G is a locally compact abelian group, and that K is a compact H_1 subset of G. We fix two non-negative measures $\mu, \nu \in M^+(G)$ such that

$$\text{(I)} \qquad \nu(K) = 0 = \mu(G \backslash K),$$

and construct $\Theta(K; \mu, \nu)$ as in § 5 of [8], which is a weakly closed subset of $L^\infty(G; \nu)$. Suppose, in addition, that

$$\text{(II)} \qquad \Theta(K; \mu, \nu) = \{1\}.$$

Then there exists a unique continuous group homomorphism

$$\Gamma: S^*(K; \mu) \longrightarrow S^*(G; \nu)$$

such that

$$\text{(III)} \qquad \Gamma(c\gamma|_K) = c\gamma \qquad (c \in T, \ \gamma \in \hat{G}).$$

(See [8; Proposition 4.3].)

LEMMA 2 (cf. [8; Proposition 5.2]). *Under the hypothesis* (II), *we have*

$$\nu[G \backslash G_p(K)] = 0.$$

PROOF. Let $B^*(K)$ be the closed subgroup of $F^*(K)$ as in Theorem 1. If we define

$$p(f) = \int_G \Gamma(f) d\nu \qquad (f \in B^*(K)),$$

then it is trivial that p is a continuous, positive-definite function on $B^*(K)$. It follows from Theorem 1 that there exists a unique measure $\lambda \in M^+(G)$ such that

$$\lambda[G \backslash G_p(K)] = 0$$

and

$$\int_G \Gamma(\gamma|_K) d\nu = \int_G \gamma d\lambda \qquad (\gamma \in \hat{G}).$$

Thus (II) gives the desired conclusion.

Let us now regard Γ as a continuous group homomorphism from $B^*(K)$ to $S^*(G; \nu)$ in a natural way, and denote by $\mathcal{D}$ the directed family consisting of all finite partitions of K into pairwise disjoint, non-empty, Borel subsets. For each $D = \{K_j\}_1^N \in \mathcal{D}$, $F_D^* = F_D^*(K)$ is a compact abelian group, and hence Theorem 3 assures that there exists a Borel function $b_D: G \to (F_D^*)^\wedge$ such that

$$\Gamma(f) = \langle b_D, f \rangle \qquad (f \in F_D^*).$$

Choosing any points $\{x_j \in K_j\}_1^N$, we identify $\text{Fag}(\{x_j\}_1^N)$ with $(F_D^*)^\wedge$ in a trivial way, and set

$$E_D = b_D^{-1}(\{x_1, x_2, \cdots, x_N\}) \subset G.$$

LEMMA 3 (cf. [8; Lemma 5.1]). *Let K, μ, and ν satisfy* (I) *and* (II). *Then, for any $\varepsilon>0$, there exists $D\in\mathcal{D}$ such that $\nu(E_D)<\varepsilon$.*

PROOF. For each $D\in\mathcal{D}$, it is easy to find a non-negative discrete measure $\lambda_D\in M(K)$ such that

$$(1)\qquad \int_G \Gamma(f)\xi_D d\nu=\int_K f(x)d\lambda_D(x) \qquad (f\in F_D{}^*),$$

where ξ_D denotes the characteristic function of E_D. In particular, we have

$$(2)\qquad \|\lambda_D\|=\lambda_D(K)=\nu(E_D)\leqq\nu(G).$$

Let ξ_{D_j} be any subnet of the net ξ_D that converges to some $\varphi\in L^\infty(G;\nu)$ in the weak-star topology of L^∞. Then, by (III), we have

$$(3)\qquad \lim_j\int_K \gamma d\lambda_{D_j}=\lim_j\int_G \gamma\xi_{D_j}d\nu=\int_G\gamma\varphi d\nu \qquad (\gamma\in\hat{G})$$

(see the proof of Theorem 1). This combined with (2) implies that the net λ_{D_j} converges to some measure $\lambda\in M(K)$ in the weak-star topology of $M(K)$ such that

$$\int_K\gamma d\lambda=\int_G\gamma\varphi d\nu \qquad (\gamma\in\hat{G}).$$

But then we have $\lambda(K)=0$ by (I). It follows from (1) that

$$\lim_j\nu(E_{D_j})=\lim_j\lambda_{D_j}(K)=\lambda(K)=0,$$

which completes the proof.

LEMMA 4 (cf. [1] and [8; Lemma 5.2]). *Let H be a compact abelian group, X a finite independent (over Z) subset of H, and Y any closed subset of H such that $X\cap Y=\phi$. Then, for any ε with $0<\varepsilon<\frac{1}{2}$, there exists a function $P\in A(H)$ such that*

$$\|P\|_A<\varepsilon^{-1};\quad 0\leqq P(t)\leqq 1 \qquad (t\in H);$$
$$P(x)=1\quad(x\in X);\quad P(y)\leqq\varepsilon^2\quad(y\in Y).$$

PROOF. Without loss of generality, we may assume that $X\cup(-X)\subset Y^c$ and $X\cap(-X)=\phi$ (cf. [2]). Let $X=\{x_j\}_1{}^N$, and let $0<\varepsilon<\frac{1}{2}$ be given. For each $l=1, 2, \cdots, N$, we denote

$$(l)=\{\sum_{j=1}^N a_jx_j;\ a_j=0,\ \pm1,\ \sum_{j=1}^N|a_j|=l\}\subset H.$$

Letting $w=\pm1$, we define

$$(1)\qquad \mu_w=\delta_0+\sum_{l=1}^N(\varepsilon w)^l\sum_{x\in(l)}\delta_x\in M(H)$$

where, δ_x denotes the dirac measure at a point $x\in H$. It is then easy to see that

$$\hat{\mu}_w(\gamma)=\prod_{j=1}^{N}\{1+2\varepsilon w \operatorname{Re}\gamma(x_j)\}>0 \qquad (\gamma\in\hat{H};\ w=\pm 1). \tag{2}$$

Let us now choose any positive-definite function f in $A(H)$ such that $0\leqq f(t)\leqq f(0)=1$ $(t\in H)$ and its support is sufficiently near to $0\in H$; define

$$2\varepsilon P=\sum_w w\mu_w * f\in A(H).$$

We then have by (2)

$$2\varepsilon\|P\|_A=\|(\sum_w w\hat{\mu}_w)\cdot\hat{f}\|_{L^1(\hat{H})}<\sum_w\|\widehat{\mu w}\hat{f}\|_{L^1(\hat{H})}=\sum_w(\mu_w*f)(0)=2,$$

the last equality following from the facts that $f(0)=1$ and $\mu_w(\{0\})=1$ $(w=\pm 1)$, and that the support of f is sufficiently near to $0\in H$. We have also by (1)

$$2\varepsilon P(t)=2\sum_l{}'\varepsilon^l\sum_{x\in(l)}f(t-x) \qquad (t\in H),$$

where $\sum_l'$ denotes the sum over the odd integers l with $1\leqq l\leqq N$. Therefore it is easy to check that P has all the required properties. This completes the proof.

LEMMA 5 (cf. [8; Proposition 5.1]). *Let ε and η be two given real numbers such that $0<\varepsilon<1/2$ and $0<\eta<1$. Then, under the hypotheses* (I) *and* (II), *we can find a trigonometric polynomial Q on G such that*

(i) $\|Q\|_B<\varepsilon^{-1}$, *and* $0\leqq Q(t)\leqq 1$ $(t\in G)$;

(ii) $\mu[x\in K;\ |Q(x)-1|\geqq\eta]<\eta$;

(iii) $\nu[t\in G;\ Q(t)\geqq(1+\eta)\varepsilon^2]<\eta$.

PROOF. Use Lemma 4. (See the proof of Proposition 5.1 in [8].)

Using Lemma 2 and Lemma 5, we can prove the following Theorem 4, which we state without proof. The proof is almost identical with those of Theorem 1 and Theorem 2 in [8].

THEOREM 4 (cf. [8; Theorem 1 and Theorem 2]). *Let G be a locally compact abelian group, K a compact H_1 subset of G, and E a closed subset of G such that*

$$K\cap E=\phi \quad (\textit{resp. } G_p(K)\cap E=\phi).$$

Then, for any real numbers ε, $\eta\in\left(0,\ \frac{1}{2}\right)$, we can find a function f in $A(G)$ such that:

(i) $\|f\|_A<\varepsilon^{-1}$ (*resp.* $\|f\|_A<1$);

(ii) $|f(x)-1|<\eta$, $x\in K$ (*resp.* $|f(x)-1|<\varepsilon$, $x\in K$);

(iii) $|f(y)|<(1+\eta)\varepsilon^2, \quad y \in E \qquad (resp.\ |f(y)|<\varepsilon, \quad y \in E)$.

This theorem can be improved as follows.

THEOREM 5 (cf. [8; Theorem 4]). *Let G be a locally compact abelian group, K a compact H_α subset of G ($0<\alpha \leqq 1$), and E a closed subset of G such that*

$$K \cap E = \phi \qquad (resp.\ G_p(K) \cap E = \phi).$$

Then, for any real numbers $\varepsilon, \eta \in \left(0, \frac{1}{2}\right)$, we can find a function f in $A(G)$ such that:

(i) $\|f\|_A < 1/(\alpha^2\varepsilon) \qquad (resp.\ \|f\|_A < 1/\alpha^2)$;

(ii) $|f(x)-1|<\eta, \quad x \in K \qquad (resp.\ |f(x)-1|<\varepsilon, \quad x \in K)$;

(iii) $|f(y)|<(1+\eta)\varepsilon^2/\alpha^2, \quad y \in E \qquad (resp.\ |f(y)|<\varepsilon, \quad y \in E)$.

PROOF. We give only the proof in the case $K \cap E = \phi$. Let $\varepsilon, \eta \in \left(0, \frac{1}{2}\right)$ be given, and let $C^* = C^*(K)$ be as in Theorem 2. We set

$$T(h) = T \quad (h \in C^*), \quad \text{and} \quad T^{C^*} = \prod_{h \in C^*} T(h).$$

Then, it is trivial that the set

$$\tilde{K} = \{(x, \langle h(x) \rangle_{h \in C^*}) \in G \times T^{C^*} : x \in K\}$$

is a Kronecker subset of $G \times T^{C^*}$ homeomorphic to K (cf. [6; Theorem 2]). It follows from Theorem 4 that there exists a function $\varphi \in A(G \times T^{C^*})$ such that

$$\|\varphi\|_A < \varepsilon^{-1}; \ |\varphi(\tilde{x})-1| < \eta^3 \qquad (\tilde{x} \in \tilde{K});$$

$$|\varphi(y, z)| < (1+\eta^3)\varepsilon^2 \qquad (y \in E, \ z \in T^{C^*}).$$

For each subset L of C^*, let m_L be the normalized Haar measure of the compact subgroup

$$\{O_G\} \times \prod_{l \in L} \{O_l\} \times \prod_{h \in L^C} T(h) \subset G \times T^{C^*},$$

and set $\varphi_L = \varphi * m_L$, which we will regard as a function in $A(G \times T^L)$. Setting $\psi = \varphi_L$ for some sufficiently large finite subset $L = \{h_j\}_1^N$ of C^*, we see

(1) $$\|\psi\|_A \leqq \|\varphi\|_A < \varepsilon^{-1};$$

(2) $$|\psi(x, h_1(x), \cdots, h_N(x)) - 1| < \eta^2 \qquad (x \in K);$$

(3) $$|\psi(y, z)| < (1+\eta^2)\varepsilon^2 \qquad (y \in E, \ z \in T^L = T^N).$$

Note then that there exist $g_n \in L^1(\hat{G})$, $n \in Z^N$, such that

(4) $$\sum_{n \in Z^N} \|g_n\|_1 = \|\psi\|_A < \varepsilon^{-1}$$

and

(5) $$\phi(t, z)=\sum_{n\in Z^N}\int_{\hat{G}} g_n(\gamma)\gamma(t)d\gamma\cdot\langle n, z\rangle \qquad (t\in G,\ z\in T^N),$$

where

$$\langle n, z\rangle=\prod_{j=1}^{N} z_j^{n_j} \qquad (n=(n_j)_1^N\in Z^N,\ z=(z_j)_1^N\in T^N).$$

For given $\delta>0$, there exist $f_n\in A(G)$, $n\in Z^N$, such that

(6) $$\|f_n\|_A<(1+\delta)/\alpha, \quad\text{and}\quad f_n(x)=\prod_{j=1}^{N}\{h_j(x)\}^{n_j} \qquad (n\in Z^N,\ x\in K),$$

since K is an H_α subset of G. We take any finite subset M_0 of Z^N so that

(7) $$\sum_{n\in Z^N\setminus M_0}\|g_n\|_1<\delta.$$

There is a finite subset M of Z^N such that

(8) $$(\text{Card } M)^{-1}\sum_{m\in M}\xi_M(n-m)>1-\delta \qquad (n\in M_0),$$

where ξ_M denotes the characteristic function of M; set

(9) $$\phi_n(t)=(\text{Card } M)^{-1}\sum_{m\in M}\xi_M(n-m)f_{n-m}(t)f_m(t) \qquad (n\in Z^N,\ t\in G).$$

Then we have

(10) $$\|\phi_n\|_A<(1+\delta)^2/\alpha^2 \qquad (n\in Z^N)$$

by (6), and

(11) $$|f_n(x)-\phi_n(x)|=|f_n(x)|\{1-(\text{Card } M)^{-1}\sum_{m\in M}\xi_M(n-m)\}<\delta \qquad (n\in M_0,\ x\in K)$$

by (6) and (8). Furthermore, we see from (6) and (9) that there exist measures $\mu_t\in M(T^N)$, $t\in G$, such that

(12) $$\|\mu_t\|<(1+\delta)^2/\alpha^2 \quad\text{and}\quad \phi_n(t)=\int_{T^N}\langle n, z\rangle d\mu_t(z) \qquad (t\in G,\ n\in Z^N).$$

We set

(13) $$f(t)=\sum_{n\in Z^N}\int_{\hat{G}} g_n(\gamma)\gamma(t)d\gamma\cdot\phi_n(t) \qquad (t\in G),$$

and prove that f has all the required properties if δ is sufficiently small. In fact, we have

(i)′ $$\|f\|_A\leqq\sum_{n\in Z^N}\|g_n\|_1\cdot\|\phi_n\|_A\leqq\|\phi\|_A(1+\delta)^2/\alpha^2$$

by (4) and (10). But, if $x\in K$, we also have by (2), (5), (6), and (13)

$$|f(x)-1|<|f(x)-\phi(x, h_1(x), \cdots, h_N(x))|+\eta^2$$

$$\leqq \sum_{n\in Z^N} \|g_n\|_1\cdot|\phi_n(x)-f_n(x)|+\eta^2$$

$$=\sum_{n\in M_0}+\sum_{n\in Z^N\setminus M_0}+\eta^2$$

which, combined with (6), (7), (9), and (11), yields

(ii)′ $$|f(x)-1|<\delta(\|\phi\|_A+2)+\eta^2 \qquad (x\in K).$$

It also follows from (5), (12), and (13) that

$$f(y)=\sum_{n\in Z^N}\Big(\int_{\hat{G}} g_n(\gamma)\gamma(y)d\gamma\Big)\Big(\int_{T^N}\langle n, z\rangle d\mu_y(z)\Big)$$

$$=\int_{T^N}\phi(y, z)d\mu_y(z) \qquad (y\in E).$$

Therefore, by (3) and (12), we have

(iii)′ $$|f(y)|\leqq(1+\eta^2)\varepsilon^2\cdot(1+\delta)^2/\alpha^2.$$

This establishes our theorem.

COROLLARY 5.1. (a) *The union of two Helson sets in a locally compact abelian group is a Helson set.* (b) *The union of two SH-sets in a locally compact abelian group is an SH-set.*

PROOF. Statement (a) is an easy consequence of Theorem 5, and Statement (b) follows from (a) and [**6**; Theorem 4].

REMARKS (Added March 26, 1971). (a) By examining our arguments in detail, we have the following: The function f in Theorem 4 can be so chosen as to be

(0) $$0\leqq f(t)\leqq 1 \qquad (t\in G).$$

Furthermore, Condition (ii) in Theorem 4 and 5 (in the case $K\cap E=\phi$) can be strengthened to be

(ii)′ $$f(x)=1 \qquad (x\in K).$$

(b) By a different method, F. Lust [**9**] had our Theorem 5 in the case that G is compact, although his result is slightly weaker than ours. J. D. Stegemen [**10**] had also our Theorem 4 under a certain additional assumption.

Tokyo Metropolitan University

References

[1] S. W. Drury, Sur les ensembles de type $\mathcal{K}$ dans les groupes compacts abéliens metrisables d'ordre borné, C. R. Acad. Sci. Paris, **268** (1969), 775–777.

[2] S. W. Drury, Sur les ensembles de Sidon, C. R. Acad. Sci. Paris, **271** (1970), 162–163.

[3] E. Hewitt and K. A. Ross, Abstract Harmonic Analysis, Vol. I, Springer-Verlag, Heidelberg, 1963.

[4] M. Loève, Probability theory, Van Nostrand, 1955.

[5] W. Rudin, Fourier analysis on groups, Interscience, New York, 1962.

[6] S. Saeki, Spectral synthesis for the Kronecker sets, J. Math. Soc. Japan, **21** (1969), 549–563.

[7] N. Th. Varopoulos, Sur les ensembles de Kronecker, C. R. Acad. Sci. Paris, **268** (1969), 954–957.

[8] N. Th. Varopoulos, Groups of continuous functions in harmonic analysis, Acta Math., **125** (1970), 109–154.

[9] F. Lust, Sur la réunion de deux ensembles de Helson, C. R. Acad. Sci. Paris, **272** (1971), 720–723.

[10] J. D. Stegemen, On unions of Helson sets, Indag. Math., **32** (1970), 456–462.